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THE DYNAMICS OF RECURRENT STATISTICAL EXPERIMENTS WITH PERSISTENT NON–LINEAR REGRESSION AND EQUILIBRIUM

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Резюме. Вивчається послідовність бінарних статистичних експериментів з наполегливою нелінійною регресією типу Райта-Фішера, яка задається кубічної параболою, що має три дійсних корені. Будується стохастична апроксимація послідовності статистичних експериментів процесом авторегресії з нормальними збуреннями, а також стохастична апроксимація послідовності експоненційних статистичних експериментів процесом авторегресії, який задається процесом геометричного броунівського руху.

Abstract. We study a sequence of binary statistical experiments with persistent non–linear regression with Wright–Fisher normalization [1], which is given by a cubic parabola, which has three real roots. We construct stochastic approximation of recurrent statistical experiments by autoregression process with normal disturbances, as well as stochastic approximation of exponential statistical experiments by exponential autoregression process with normal disturbances.

1. Introduction

In our previous paper [1] there has been searched a limit behavior of recurrent *statistical experiments* (SE) with *persistent linear regression* by increasing sample volume $N \to \infty$. An important role in the analysis of SE with persistent linear regression plays the control parameter a of the regression function which provides a steady state with equilibrium point and, at the same time , gives a possibility of approximating the original recurrent SE by normal autoregression process, which statistical analysis is significantly easier.

In this paper, we study a similar problem for a sequence of SE with persistent regression with an additional term which determines the non-linear regression. The initial assumptions about the binary nature [1], as well as non-linear regression model with Wright–Fisher normalization [2] make a natural choice for the regression's nonlinear component as a cubic parabola, which has three real roots in the value interval $-1 \leq s \leq +1$ of the results of statistical experiments. It is natural to assume that the non-linear component of the regression takes the value 0 at the ends of the interval $s = \pm 1$, as well as at the equilibrium point ρ of the linear regression.

Key words. Binary statistical experiment, persistent regression, stabilization, stochastic approximation, exponential statistical experiment, exponential autoregression process.

These considerations lead to the clear conclusion that the non-linear component of the regression is as follows:

$$C_0(s) = -g(1-s^2)(s-\rho)\ ,\quad |s| \le 1\ ,\quad g>0. \tag{1}$$

2. Steady state regime

In this paper we consider the sequence of the SE

$$S_N(k) = \frac{1}{N}\sum_{r=1}^{N} \delta_r(k),\ \ k \ge 0. \tag{2}$$

with persistent non-linear regression:

$$E[S_N(k+1)|S_N(k)=s] = C(s)\ ,\quad C(s) = s + C_0(s). \tag{3}$$

The parameter g of non-linear regression significantly changes the dynamics of the recurrent SE.

Remark 1. *Setting the SE using the regression (2) - (3) means that the probability sample values are given by:*

$$P\{\delta_r(k+1) = \pm 1 | S_N(k) = s\} = \frac{1}{2}[1 \pm C(s)]. \tag{4}$$

At the same time, there exist control parameters g and ρ such that the condition (4) is correctly defined.

The specificity of the binary SE is, in particular, that the conditional variance SE is simply calculated

$$D[S_N(k+1)|S_N(k)=s] = B(s)/N\ ,\quad B(s) := 1 - C^2(s). \tag{5}$$

Now it is possible to verify the existence of the steady state (see [1, Theorem 1]).

Theorem 1. *Provided the initial condition (convergence with probability 1)*

$$S_N(0) \Rightarrow \rho\ ,\quad N \to \infty, \tag{6}$$

there is the convergence with probability 1

$$S_N(k) \Rightarrow \rho\ ,\quad N \to \infty, \tag{7}$$

for each finite $k > 0$.

Proof of Theorem 1. We introduce a martingale as the sum of martingale differences:

$$\mu_N(n) := \sum_{k=0}^{n} [S_N(k+1) - E[S_N(k+1)\,|\,S_N(k)]] \tag{8}$$

or another, in view of the properties of persistent regression (3)

$$\mu_N(n) = \sum_{k=0}^{n} [S_N(k+1) - C(S_N(k)]. \tag{9}$$

The quadratic characteristic of the martingale (8), considering (9), is given by the sum:

$$\langle \mu_N \rangle_n := \sum_{k=0}^{n} D[S_N(k+1) \,|\, S_N(k)] = \frac{1}{N} \sum_{k=0}^{n} B(S_N(k)). \tag{10}$$

Hence for any fixed $n \geq 0$ the following convergence takes place (with probability 1):

$$\langle \mu_N \rangle_n \Rightarrow 0, \;\; N \to \infty. \tag{11}$$

This implies the convergence with probability 1 of the martingales (8) for each finite $n \geq 0$

$$\mu_N(n) \Rightarrow 0, \;\; N \to \infty, \;\; n \geq 0. \tag{12}$$

In particular when $n = 0$ we have

$$\mu_N(0) = S_N(1) - C(S_N(0)) = S_N(1) - \rho - [S_N(0) - \rho] - C_0(S_N(0))$$

In this case, by the condition of Theorem 1

$$C_0(S_N(0)) \Rightarrow C_0(\rho) = 0, \;\; N \to \infty.$$

So there is the convergence with probability 1:

$$S_N(1) - \rho \Rightarrow 0, \;\; N \to \infty.$$

By induction, we deduce that for every $k \geq 1$ the convergence (7) takes place. □

3. Stochastic approximation of statistical experiments

As in previous work [1] appears the problem of simplified description of the recurrent SE dynamics by increasing sample volume $N \to \infty$. The nonlinear component of the regression function, which has the factor $(s - \rho)$, preserves the possibility of approximating SE by normal autoregression process.

Theorem 2. *Under the conditions of Theorem 1 there takes place the limit relation (in probability):*

$$\sqrt{N}[S_N(k+1) - C(S_N(k)] \Rightarrow \sigma W(k+1) \;, \quad N \to \infty \tag{13}$$

for each finite $k \geq 0$.

The sequence of independent, normally distributed random variables $W(k)$, $k \geq 1$ satisfies the normalization conditions :

$$EW(k) = 0 \;, \quad DW(k) = 1 \;, \quad k \geq 1 \;, \quad \sigma^2 = 1 - \rho^2. \tag{14}$$

Proposition 1. *The limit relation* (13) *is the basis to use the normal process of autoregression*

$$\begin{aligned} \widetilde{S}_N(k+1) &= C(\widetilde{S}_N(k)) + \frac{\sigma}{\sqrt{N}} W(k+1) \;, \quad k \geq 0, \\ &= \widetilde{S}_N(k) + C_0(\widetilde{S}_N(k)) + \frac{\sigma}{\sqrt{N}} W(k+1) \;, \quad k \geq 0, \end{aligned} \tag{15}$$

as an approximation of the original SE (2)–(3) *with nonlinear regression function* (3).

Remark 2. *It is clear that the stochastic approximation in* (15) *is considerably simpler than the original model* (2)–(5) *and at the same time, preserves the condition of persistent regression:*

$$E[\widetilde{S}_N(k+1)\,|\,\widetilde{S}_N(k)] = C(\widetilde{S}_N(k)),\ k \geq 0. \tag{16}$$

Proof of Theorem 2. We introduce a martingale as the sum of martingale differences:

$$\mu_N(n) := \sqrt{N}\sum_{k=0}^{n}[S_N(k+1) - C(S_N(k))],\ n \geq 0. \tag{17}$$

Using the equilibrium state ρ (см. Теорему 1), and the relations (2), (3) and (5), we get the following result.

Lemma 1. *The martingale* (17) *has the following asymptotic representation:*

$$\mu_N(n) = \sum_{k=0}^{n}[\zeta_N(k+1) - b_0\zeta_N(k)] + \frac{1}{\sqrt{N}}\sum_{k=0}^{n}\zeta_N^2(k)R(S_N(k)) \ , \quad n \geq 0. \tag{18}$$

Here

$$\begin{aligned} &\zeta_N(k) := \sqrt{N}[S_N(k) - \rho],\ k \geq 0, \\ &R(s) = g(s+\rho) \ , \quad b_0 := 1 - g\sigma^2 \ , \quad \sigma^2 = 1 - \rho^2. \end{aligned} \tag{19}$$

According to Theorem 1 and relation (19), the nonlinear term in (19) converges (in probability) to zero as $N \to \infty$ for each finite $n \geq 0$. Now the normal approximation of martingale (17) - (19) is realized in the same manner as in [1]. First, we compute the quadratic characteristic of martingale (17)

$$\langle \mu_N \rangle_n = \sum_{k=0}^{n} B(S_N(k)) \ , \quad B(S) := 1 - C^2(s) \ , \quad n \geq 0. \tag{20}$$

Then, according to Theorem 1, there is a limit (with probability 1)

$$\langle \mu_N \rangle_n \Rightarrow \ (n+1)\sigma^2 \ , \quad N \to \infty \ , \quad n \geq 0. \tag{21}$$

However, according to the central limit theorem, the primary (linear) martingale portion (18) converges (in probability) to the sum of normally distributed random variables.

The convergence of the quadratic characteristics (21) implies the convergence in probability of martingale-differences

$$\mu_N^0(n) := \sum_{k=0}^{n}[\zeta_N(k+1) - (1-g\sigma^2)\zeta_N(k)] \Rightarrow \sigma\sum_{k=0}^{n} W(k+1). \tag{22}$$

The limit normally distributed random variables $W(k)$, $k \geq 1$ are mutually independent with

$$EW(k) = 0 \quad , \quad EW^2(k) = 1 \quad , \quad k \geq 1$$

because the limit dispersion of martingale (21) is equal too sum of dispersions of martingale-differences.

The convergence of the original martingale (17) means that there is the convergence (in probability) :

$$\sqrt{N}[S_N(k+1) - C(S_N(k))] \Rightarrow \sigma W(k+1), \ N \to \infty \tag{23}$$

for each finite $k \geq 0$. The proof of Theorem 2 in complete. □

Corollary 1. *The convergence of the linear component of the martingale (18) implies the convergence (in probability)*

$$\zeta_N(k+1) - (1 - g\sigma^2)\zeta_N(k) \Rightarrow \sigma W(k+1) \ , \quad N \to \infty. \tag{24}$$

Proposition 2. *The convergence (24) serves as the basis to use approximation, in the neighborhood of the equilibrium point ρ, of the original statistical experiments with persistent regression (2) by the new process of normal autoregression with linear regression function:*

$$\widetilde{S_N^0}(k+1) - \rho = (1 - g\sigma^2)[\widetilde{S_N^0}(k) - \rho] + \frac{\sigma}{\sqrt{N}} W(k+1), \tag{25}$$

so that

$$\widetilde{S_N^0}(k+1) = (1 - g\sigma^2)\widetilde{S_N^0}(k) + g\sigma^2\rho + \frac{\sigma}{\sqrt{N}} W(k+1). \tag{26}$$

The stochastic approximation by the normal process of autoregression (25) - (26) (Proposition 2) for the linear regression function

$$\widetilde{C}(s) = s - g\sigma^2(s - \rho) \tag{27}$$

has a *stationary distribution*, which is given by the density of the normal distribution (see [2, item 5])

$$\rho(s) = \frac{1}{(\widetilde{\sigma}^2/N)\sqrt{2\pi}} \exp[-(s-\rho)^2/2\widetilde{\sigma}^2/N]. \tag{28}$$

$$\widetilde{\sigma}^2 = \sigma^2/(1 - g\sigma^2). \tag{29}$$

4. Exponential statistical experiments: steady-state behavior

In many applications in biology [2, 6] and economics [7] important role is played *symmetric exponential statistics*

$$\Pi_N(\lambda, k) := \prod_{r=1}^{N} [1 + \lambda\delta_r(k)] \ , \quad k \geq 0. \tag{30}$$

For example, if the sample values $\delta_r(k)$, $1 \leq r \leq N$, $k \geq 0$ define success rates $\delta_r(k)) = +1$ or failure ones $\delta_r(k)) = -1$, then (31) sets the total value of the interest rate in the k-th experiment. The parameter $\lambda > 0$ can be considered as a discount factor.

We consider *exponential statistical experiments* (ESE) (30) in the series scheme with increasing sample size $N \to \infty$.

The property of persistent regression (3) is converted to the following form:

$$E[\Pi_N(\lambda, k+1)|S_N(k)] = [1 + \lambda C(S_N(k))]^N. \tag{31}$$

Now we introduce the exponential martingale

$$\mu_N^e(\lambda, k+1) = \Pi_N(\lambda, k+1)/\overline{\Pi_N}(\lambda, k)\ , \quad k \geq 0. \tag{32}$$

$$\overline{\Pi_N}(\lambda, k) := [1 + \lambda C(S_N(k))]^N\ , \quad k \geq 0. \tag{33}$$

Its martingale property is obvious:

$$E[\mu_N^e(\lambda, k+1)|S_N(k)] = 1\ , \quad k \geq 0.$$

Steady state of SE is established by the following

Theorem 3. *By the condition of convergence with probability 1 of the SE initial values*

$$S_N(0) \Rightarrow \rho = p/(1-a)\ , \quad N \to \infty, \tag{34}$$

there is the convergence in probability of ESE (31)

$$P \cdot \lim_{N\to\infty} \Pi_N(\lambda/N, k) = \exp(\lambda\rho)\ , \quad k \geq 0, \tag{35}$$

and also the convergence in probability of conditional expectations (32), (34)

$$P \cdot \lim_{N\to\infty} \overline{\Pi_N}(\lambda/N, k) = \exp(\lambda\rho)\ , \quad k \geq 0. \tag{36}$$

Corollary 2. *Under the condition (35) the following convergence takes place:*

$$P \cdot \lim_{N\to\infty} \mu_N^e(\lambda/N, k) = 1. \tag{37}$$

Proof of Theorem 3. We use the approximation formula of Le Cam in the following form:

Lemma 2. *(cp. [5, Lemma 6.3.1]) Assume that the convergence in probability takes place:*

$$\max_{1\leq r\leq N} |\delta_r(k+1)| \Rightarrow 0\ ,\ N \to \infty\ , \quad k \geq 0. \tag{38}$$

Then there takes place the convergence in probability

$$P \cdot \lim_{N\to\infty} \{\sum_{r=1}^N \ln[1 + \lambda\delta_r(k+1)/N] - \lambda S_N(k+1)\} = 0. \tag{39}$$

The condition (38) is obviously satisfied for binary random variables $\delta_r(k+1),\ 1 \leq r \leq N,\ k \geq 0$, taking two values ± 1. In addition, by Theorem 1

$$S_N(k+1) \Rightarrow \rho\ , \quad N \to \infty.$$

Hence the convergence (40) is equivalent to the convergence:

$$P \cdot \lim_{N\to\infty} \{\sum_{r=1}^N \ln[1 + \lambda\delta_r(k+1)/N] = \lambda\rho. \tag{40}$$

We now use the obvious identity

$$\prod = \exp\ln\prod.$$

The convergence (36) is equivalent to the convergence in (40). Even easier to establish the convergence (37) using the relation

$$C(S_N(k)) \Rightarrow \rho\ ,\quad N \to \infty,\quad k \geq 0.$$

Theorem 3 is proved. The Corollary 1 is obvious.

5. Exponential statistical experiments: approximation by the normal process of autoregression

The exponential statistical experiments (ESE) (31) with conditional expectation (32) are considered in the scheme of series with series parameter $\lambda_N = \lambda/\sqrt{N}$:

$$\Pi_N(\lambda/\sqrt{N}, k+1) = \prod_{r=1}^{N} [1 + \lambda\delta_r(k+1)/\sqrt{N}]\ ,\quad k \geq 0. \tag{41}$$

However, the averaging of ESE in given by the relation:

$$\overline{\Pi_N}(\lambda/\sqrt{N}, k) = [1 + \lambda C(S_N(k))/\sqrt{N}]^N\ ,\quad k \geq 0. \tag{42}$$

So that the corresponding exponential martingale has the form:

$$\mu_N^e(\lambda_N, k+1) := \Pi_N(\lambda_N, k+1)/\overline{\Pi_N}(\lambda_N, k)\ ,\quad k \geq 0. \tag{43}$$

The fundamental importance for the of ESE approximation has the following

Theorem 4. *(ESE approximation) Under the conditions of Theorem 3 we have the convergence in probability*

$$P \cdot \lim_{N\to\infty} \mu_N^e(\lambda/\sqrt{N}, k+1) = \exp[\lambda\sigma W(k+1) - \lambda^2\sigma^2/2]\ ,\quad k \geq 0, \tag{44}$$

Remark 3. *The exponential martingale in the series scheme* (44)*, given by ESE* (41)*, converges (as $N \to \infty$) to exponential normal martingale. It is obvious that*

$$E \exp[\lambda\sigma W(k+1) - \lambda^2\sigma^2/2] = 1\ ,\quad k \geq 0. \tag{45}$$

Proof of Theorem 4. As in the proof of Theorem 3, we use Lemma approximation of Le Cam and the obvious identity $\Pi = \exp \ln \Pi$.

Lemma 3. *(Le Cam approximation [5, Lemma 6.3.1]) Assume that the convergence in probability takes place*

$$\max_{1\leq r\leq N} |\delta_r(k+1)/N| \Rightarrow 0\ ,\ N \to \infty,$$

and also the sums

$$V_N(k) := \frac{1}{N}\sum_{r=1}^{N} (\delta_r(k))^2$$

are bounded in probability. Then there takes place the convergence in probability

$$P \cdot \lim_{N\to\infty} \sum_{r=1}^{N} \ln[1 + \lambda\delta_r(k)/\sqrt{N}] - \lambda\sqrt{N}S_N(k) + \lambda^2 V_N(k)/2 = 0. \tag{46}$$

Note that in our case $V_N(k) = 1$. So the convergence in Lemma 3 has the following form:

$$P \cdot \lim_{N\to\infty} \sum_{r=1}^{N} \ln[1 + \lambda\delta_r(k)/\sqrt{N}] - \lambda\sqrt{N} S_N(k) = -\lambda^2/2. \tag{47}$$

Note that ESE (42) is represented in the form:

$$\Pi_N(\lambda/\sqrt{N}, k+1) = \exp \sum_{r=1}^{N} \ln[1 + \lambda\delta_r(k+1)/\sqrt{N}] \ , \quad k \geq 0. \tag{48}$$

So that the convergence (48) means

$$P \cdot \lim_{N\to\infty} \Pi_N(\lambda/\sqrt{N}, k+1) \exp[-\lambda\sqrt{N} S_N(k+1)] = \exp[-\lambda^2/2] \ , \quad k \geq 0. \tag{49}$$

Similarly, the conditions of Lemma 3 provide the convergence in probability (by $N \to \infty$) of the averaged ESE:

$$P \cdot \lim_{N\to\infty} \overline{\Pi_N}(\lambda/\sqrt{N}, k) \exp[-\lambda\sqrt{N} C(S_N(k))] = \exp[-\lambda^2\rho^2/2] \ , \quad k \geq 0. \tag{50}$$

We should use the Theorem 1, according to which

$$C(S_N(k)) \Rightarrow C(\rho) = \rho = p/(1-a) \ , \quad N \to \infty \ , \quad k \geq 0.$$

Now we introduced the centered ESE:

$$\Pi_N^0(\lambda/\sqrt{N}, k+1) := \Pi_N(\lambda/\sqrt{N}, k+1) \exp[-\lambda\sqrt{N} S_N(k+1)] \ , \quad k \geq 0, \tag{51}$$

$$\overline{\Pi_N^0}(\lambda/\sqrt{N}, k) := \overline{\Pi_N}(\lambda/\sqrt{N}, k) \exp[-\lambda\sqrt{N} C(S_N(k))] \ , \quad k \geq 0. \tag{52}$$

By Theorem 2, there is convergence (in probability):

$$\sqrt{N}[S_N(k+1) - C(S_N(k))] \Rightarrow \sigma W(k+1), \ N \to \infty \ , \quad k \geq 0. \tag{53}$$

So that the exponential martingale (44) is represented in the following form:

$$\begin{aligned} \mu_N^e(\lambda/\sqrt{N}, k+1) = & \left[\Pi_N^0(\lambda/\sqrt{N}, k+1) \ / \ \overline{\Pi_N^0}(\lambda/\sqrt{N}, k)\right] \times \\ & \times \exp\{\lambda\sqrt{N}[S_N(k+1) - C(S_N(k))]\} \ , \quad k \geq 0. \end{aligned} \tag{54}$$

Using the the relations (50) - (53), taking into account the relations $\sigma^2 = 1 - \rho^2$, we get the assertion (45).

Theorem 4 is proved. □

We now rewrite the approximations (51) and (45) in the original series scheme with the series parameter $\lambda_N = \lambda/N$:

$$\overline{\Pi_N}(\lambda/N, k) \exp[-\lambda C(S_N(k))] = \exp(\lambda^2\rho^2/2N) e^{R_N} \ , \quad k \geq 0, \tag{55}$$

$$\mu_N^e(\lambda/N, k+1) = \exp[\lambda(\sigma/\sqrt{N}) W(k+1) - \lambda^2\sigma^2/2N] e^{R_N}. \tag{56}$$

Here the residual term $R_N = o(1/N), \ n \to \infty$.

Hence the normalized ESE (31) admit the following approximation:

$$\Pi_N(\lambda/N, k+1) = \exp[\lambda C(S_N(k)) - \lambda^2\rho^2/2N]\times \\ \times \exp[\lambda(\sigma/\sqrt{N})W(k+1) - \lambda^2\sigma^2/2N]e^{R_N}. \tag{57}$$

The approximation of the ESE (57) serves as a basis the following statement.

Proposition 3. *The exponential statistical experiments* (31) *can be approximated by an exponential process of autoregression*

$$\widetilde{\Pi_N}(\lambda/N, k+1) := \prod_{r=1}^{N}[1 + \lambda\widetilde{\delta_r}(k+1)/N] = \\ = \exp[\lambda C(\widetilde{S_N}(k)) - \lambda^2\rho^2/2N] \ \cdot \ \exp[\lambda(\sigma/\sqrt{N})W(k+1) - \lambda^2\sigma^2/2N], \tag{58}$$

Here by definition

$$\widetilde{S_N}(k) := \frac{1}{N}\sum_{r=1}^{N}\widetilde{\delta_r}(k), \ \ k \geq 0.$$

Remark 4. *An important basis for the application of approximation (58) is the fact that the conditional expectations asymptotically coincides with the regression function (conditional expectation) of the original ESE* (30), *namely (cf. (58)):*

$$E\left[\prod_{r=1}^{N}[1 + \lambda\widetilde{\delta_r}(k+1)/N] | \widetilde{S_N}(k)\right] = \exp[\lambda C(\widetilde{S_N}(k)) - \lambda^2\rho^2/2N]e^{R_N}. \tag{59}$$

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