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STABILITY PRESERVING STRUCTURAL TRANSFORMATIONS OF SYSTEMS OF LINEAR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Резюме. В даній статті розроблено теорію структурний перетворень систем звичайних диференціальних рівнянь (ЗДР) другого порядку, які зберігають властивість стійкості за Ляпуновим. Особливу увагу приділено таким перетворенням, які приводять задану систему ЗДР до спеціального симетричного виду, більш зручного при дослідженні стійкості нульового розв'зку таких систем. Окремо розглянуто випадки автономних та неавтономних систем. Основна теорема даної статті суттєво узагальнює всі відомі результати в області структурних перетворень систем ЗДР та може розглядатися як аналог відомої теореми Єругіна про системи першого порядку. Зазначена теорема дозволила нам узагальнити 3-тю та 4-ту теореми Кельвіна-Тета-Четаєва. Одержані теоретичні результати були успішно застосовані при дослідженні стійкості обертального руху жорсткого тіла підвішеного на струні.

Abstract. In the paper we have developed a theory of stability preserving structural transformations (SPST) of systems of second-order ordinary differential equations (ODEs), i.e., the transformations which preserve the property of Lyapunov stability. Special attention is paid to those SPST, which can reduce a given system of ODEs to several special symmetric forms that are more approachable for the existing methods of stability investigation than the "nonsymmetric" ones. The autonomous and non-autonomous cases were discussed separately. The main Theorem proved in the paper essentially generalizes all the known theoretical results related to the SPST of systems of second-order ODEs and can be viewed as an analogous of the Erugin's theorem for the systems of second-order ODEs. The Theorem allowed us to generalize the 3-rd and 4-th Kelvin – Tait – Chetayev theorems. The obtained theoretical results were successfully applied to the stability investigation of the rotary motion of a rigid body suspended on a string.

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1. Introduction

It is well known that a great number of dynamical systems can be approximately described by the following system of second-order ordinary differential equations (ODEs):

$$J(t)\ddot{\mathbf{x}}(t) + (D(t) + G(t))\dot{\mathbf{x}}(t) + (P(t) + \Pi(t))\mathbf{x}(t) = F(t, \mathbf{x}(t)),$$
$$t \in [t_0, \infty) \tag{1}$$

where $\mathbf{x}(t) = col\,[x_1(t), x_2(t), \ldots, x_m(t)]$ is an unknown vector-function. From the physical point of view the matrix $J(t) = J^T(t) > 0$ (the upper index T denotes the operation of transposition) describes the inertia characteristics of a dynamical system; the matrices $D(t) = D^T(t), G(t) = -G^T(t), P(t) = -P^T(t)$ and $\Pi(t) = \Pi^T(t)$ represent a dissipative, gyroscopic, non-conservative positional and potential forces respectively; the vector-function $F(t, \mathbf{x}(t))^1$ represents an external forces acting on the system.

Let us consider a linear one-to-one mapping $L^{-1}(t)$, which maps the unknown vector-function $\mathbf{x}(t)$ to some other m-dimensional unknown vector-function $\xi(t)$. If the inverse mapping, $L(t)$, is sufficiently smooth then it can be viewed as a *structural transformation*[2] of system (1). Indeed, substituting vector $\mathbf{x}(t)$ in system (1) with expression $L(t)\xi(t)$ we can get a transformed second-order system of ODEs with respect to the unknown vector-function $\xi(t)$. In the present paper we confine ourselves to study only those transformations (or mappings) L, which do not change the stability properties of the null solution of system (1). We will call them the *stability preserving structural transformations*.

By "stability properties" of a null solution we mean the properties of being *unstable*, *stable* or *asymptotically stable* in the sense of Lyapunov.

Definition 1. The null solution, $\mathbf{x}_0(t) \equiv 0$, of system (1) is called **stable** (in the sense of Lyapunov) if $\forall \varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that $\|\mathbf{x}_\delta(t)\| < \varepsilon$, $\forall t \in [t_0, +\infty)$, where $\mathbf{x}_\delta(t)$ represents any solution of system (1) which satisfies initial condition $\|\mathbf{x}_\delta(t_0)\| < \delta$.

The null solution is called **asymptotically stable** (in the sense of Lyapunov) if it is **stable** and $\lim_{t \to +\infty} \|\mathbf{x}_\delta(t)\| = 0$.

The null solution is called **unstable** (in the sense of Lyapunov) if it is not **stable**.

As an example of stability preserving structural transformations we can mention the *Lyapunov transformations* (see [1, p. 117]) though those are not the only stability preserving transformations considered in the present paper (see section 2 for more detail).

Definition 2 (see [1], p. 117). A square matrix $L(t)$ will be called a Lyapunov matrix if it satisfies the following conditions :

1. $L(t)$ has continuous derivative $\frac{dL(t)}{dt}$ on some interval $[t_0, +\infty)$;
2. matrices $L(t)$ and $\frac{dL(t)}{dt}$ are bounded on the interval $[t_0, +\infty)$;
3. there exists a constant η such that $0 < \eta < |\det(L(t))|$, $\forall t \in [t_0, \infty)$.

A transformation $x(t) = L(t)\xi(t)$ will be called a Lyapunov transformation if the matrix $L(t)$ is a Lyapunov matrix.

The above classic definition is useful only for the case of first-order systems. To adapt it for the case of second-order systems we will additionally require that the Lyapunov matrix $L(t)$ has continuous second derivative on $[t_0, +\infty)$.

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[1]We assume that $\forall t_1 \in [t_0, +\infty)$, $\lim_{\|\mathbf{x}\| \to 0} \|F(t_1, \mathbf{x})\|/\|\mathbf{x}\| = 0$.

[2]In the next section we will give a thorough definition of "structural transformation" and here we are going to give a general idea.

As it was pointed out by V. N. Koshlyakov in [2], when the gyroscopic terms in system (1) are periodic in t with some period $\tau > 0$ then the formal application of the averaging method to the system could result in the discarding of gyroscopic structures though this structures had some stabilizing effect on the system before averaging. Thus, it is very desirable to have a theoretical framework which allows us to transform the initial system into a system that possess the same stability properties and contain no gyroscopic structures. In [2] the author has considered a case when $J(t) = a_0 E,^3$ $a_0 > 0$. He showed that the transformation $x(t) = L(t)\xi(t)$, with matrix $L(t)$ that satisfies the Cauchy problem $\dot{L}(t) = -(2a_0)^{-1}G(t)L(t)$, $L(t_0) = E$, will reduce system (1) to a system which does not contain gyroscopic structures. He also proved that the transformation mentioned above is a *Lyapunov transformation,* i.e., it preserves stability properties of system (1). Furthermore, V. N. Koshlyakov showed that in some real physical cases the reduced system could be so simple that its general solution can be easily found explicitly.

However, V. N. Koshlyakov was not the first one who pointed out the practical benefits of using stability preserving transformations. Almost twenty five years earlier D. L. Mingori did this.

In [3] D. L. Mingori has considered a case when system (1) is autonomous, that is,

$$J\ddot{x}(t) + (D + G)\,\dot{x}(t) + (P + \Pi)\,x(t) = 0^4, \tag{2}$$

where $x = col\,[x_1\,(t)\,, x_2\,(t)\,, \ldots, x_m\,(t)]$ is an unknown vector; here again the matrix $J = J^T > 0$ describes the inertia characteristics of the dynamical system and matrices $D = D^T$, $G = -G^T$, $\Pi = \Pi^T$, $P = -P^T$ represent a dissipative, gyroscopic, non-conservative positional and potential forces respectively. D. L. Mingori proved that under some restrictions imposed on the matrix coefficients of system (2) there exists a Lyapunov transformation

$$x(t) = J^{-\frac{1}{2}}L(t)\xi(t) \tag{3}$$

that can reduce autonomous system (2) to another autonomous system

$$\ddot{\xi}(t) + V\dot{\xi}(t) + W\xi(t) = 0 \tag{4}$$

which does not contain nonconservative positional structures, i.e, $W = W^T$. Since the matrix $L(t)$ is a Lyapunov matrix, the null solutions of systems (2) and (4) are stable, asymptotically stable or unstable simultaneously. On the other hand, because of the symmetrical properties of system (4) the stability investigation of its null solution is an easier task than the stability investigation of the null solution of system (2).

Thereby in [3] D. L. Mingori had suggested an approach to the stability investigation of the second-order systems of ODEs (2) which consists of reducing the initial problem to the problem of stability investigation of the corresponding equivalent (in the sense of Lyapunov, see [1, p. 118]) symmetric system (4). He has shown that such approach can be very useful and fruitful for the

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[3] E denotes the identity matrix of corresponding order.

[4] For the sake of simplisity the vector function of external forces was not taken into account.

stability investigation in analytical mechanics. However, in [3] the author has considered the case when $D > 0$ only. Though in [4] the results of D. L. Mingori were extended on the case when $D \geq 0$, the necessary and sufficient conditions providing that a given non-symmetric second-order system of ODEs is equivalent in the sense of Lyapunov to some symmetric second-order system remains unknown: both papers [3] and [4] contain the sufficient conditions only.

Later the necessary and sufficient conditions providing that the autonomous system (2) can be reduced to some other autonomous system (4) with $W_1 = W_1^T$ via substitution (3) were found in papers [5] and [6]. However, the results of these papers were obtained under additional assumptions that

$$G = H\hat{G},$$

$$\frac{dL(t)}{dH} = 0, \forall t \geq 0, \tag{5}$$

$$D > 0, \quad \det(G) \neq 0,$$

where H denotes a positive numerical parameter.

In some cases the parameter H can be a part of matrix Π. This will be the case when equation (2) describes a perturbed motion of a gyroscopic systems installed on the platform which rotates around the vertical with the angular velocity ω. Using assumptions (5) and assuming that $\Pi = \Pi^{(0)} + H\Pi^{(H)}$, where matrices $\Pi^{(0)}, \Pi^{(H)}$ are independent on H, the necessary and sufficient conditions providing the reducibility of system (2) to some other system (4) with $W_1 = W_1^T$ where obtained in [7].

One of the common features of the series of works [3, 4, 5, 6, 7, 8, 9] is that their authors included the commutation of the matrices D and P into a set of conditions which provide the reducibility of system (2) to system (4) with $W_1 = W_1^T$. As it is show in Section 4 (see Example 1) this commutativity condition is not the necessary one[5] and in the present paper we finally got rid of it.

In the present paper without any additional assumptions we have obtained the necessary and sufficient conditions (in terms of the matrix coefficients) providing that a given system of second-order ODEs is equivalent in the sense of Lyapunov to some other system of second-order ODEs with symmetric matrix coefficients. We have considered both the autonomous and non-autonomous cases. In the case when the initial system is autonomous we require that the reduced system be autonomous too.

The paper is organized as follows.

In Section 2 we introduce the notion of the *structural transformation* of a system of second-order ODEs and give the definition of the L_k-equivalent systems of second-order ODEs. Using the notion of the L_k-equivalence we formulate two symmetrization problems for the non-autonomous systems of second-order ODEs: the *problem of Elimination of Gyroscopic Structures* (EGS problem) and the *problem of Elimination of Non-conservative Positional Structures* (ENPS

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[5]We mean the case when there are no additional restrictions as, for example, those that were introduced in [5], [6], [7].

problem). In the section the necessary and sufficient conditions providing the solvability of the both problems were obtained.

In Section 3 we reformulate the EGS and ENPS problems for the case of the autonomous systems and introduce the notion of the L-equivalence of two autonomous systems of second-order ODEs. Theorem 6 proved in the section can be considered as an analogous of the Erugin's theorem (see [1, p. 121]) for the autonomous systems of second-order ODEs. Some useful consequences from Theorem 6 are stated in Section 5. Among them there is a theorem which generalizes the theorems of Mingori (see [3]) and Müller (see [4]).

In Section 5 we discuss the question of the interconnection between the notions of the L_k-equivalence and equivalence in the sense of Lyapunov.

In Section 6 we demonstrate how the using of structural transformations can facilitate the stability investigation of the null solution of the autonomous second-order system of ODEs describing the rotary motion of a rigid body suspended on a string.

Section 7 contains several conclusions about the theoretical results presented in the paper.

2. Stability preserving structural transformations of the non-autonomous systems of second-order ODEs

Let us consider the following system of second-order ordinary differential equations:

$$\ddot{\mathbf{x}} + A\left(t\right)\dot{\mathbf{x}} + B\left(t\right)\mathbf{x} = 0, \tag{6}$$

where $\mathbf{x} = \overrightarrow{x}\left(t\right) = \left[x_1\left(t\right),\ldots,x_m\left(t\right)\right]^T$ is an unknown vector-function. By default, we assume that $A\left(t\right),B\left(t\right)$ are square matrices of order m whose elements are continuous on $[t_0,+\infty)$ functions, i.e., $A\left(t\right),B\left(t\right)\in M_m\left(C\left[t_0,+\infty\right)\right)$. Also we will use the notation $M_m\left(C^i\left[t_0,+\infty\right)\right),\ i=1,2$ to denote the linear spaces of square matrices of order m whose elements belong to the functional space $C^i\left[t_0,+\infty\right),\ i=1,2$, and the notation $M_{m,n}(\mathbb{R})$ will be used to denote the space of constant real matrices of dimension $m\times n$.

Definition 3. The structural transformation of the second-order system of ordinary differential equations (6) is the transformation of unknown vector $\mathbf{x}$ which can be expressed in the form

$$\mathbf{x} = L\left(t\right)\xi, \tag{7}$$

where $\xi = \left[\xi_1\left(t\right),\ldots,\xi_m\left(t\right)\right]^T$ is a new unknown vector-function,

$$L\left(t\right)\in M_m\left(C^2\left[t_0,+\infty\right)\right), \quad \det\left(L\left(t\right)\right)\neq 0,\ \forall t\in\left[t_0,+\infty\right).$$

Applying transformation (7) to system (6) we obtain the following system of second-order ordinary differential equations:

$$L\left(t\right)\ddot{\xi}\left(t\right) + \left(2\dot{L}\left(t\right) + A\left(t\right)L\left(t\right)\right)\dot{\xi}\left(t\right) + \tag{8}$$
$$+ \left(\ddot{L}\left(t\right) + A\left(t\right)\dot{L}\left(t\right) + B\left(t\right)L\left(t\right)\right)\xi\left(t\right) = 0,$$

or , in more convenient form,

$$\ddot{\xi} + V\left(t\right)\dot{\xi} + W\left(t\right)\xi = 0, \tag{9}$$

where

$$V\left(t\right) = L^{-1}\left(t\right)\left(2\dot{L}\left(t\right) + A\left(t\right)L\left(t\right)\right),$$

$$W\left(t\right) = L^{-1}\left(t\right)\left(\ddot{L}\left(t\right) + A\left(t\right)\dot{L}\left(t\right) + B\left(t\right)L\left(t\right)\right). \tag{10}$$

Apparently we have that $V\left(t\right), W\left(t\right) \in M_m\left(C\left[t_0, +\infty\right)\right)$. Therefore, applying transformation (7) to system (6) with continuous on $[t_0, +\infty)$ matrix coefficients, we arrive at system (9) that also possesses continuous on $[t_0, +\infty)$ matrix coefficients.

Definition 4. We say that the system of second-order ODEs (6) is L_k-equivalent to system (9) $(k \in \{0, 1, 2\})$ if there exists a matrix $L\left(t\right) \in M_m\left(C^2\left[t_0, +\infty\right)\right)$ satisfying conditions

1. $\left|\det\left(L\left(t\right)\right)\right| > \eta > 0, \ \forall t \in [t_0, +\infty),$

2. $\displaystyle\sup_{t \in [t_0, +\infty)} \left\|\frac{d^i}{dt^i}L\left(t\right)\right\| < +\infty, \ \forall i \in \overline{0, k},$

together with equalities (10). A matrix $L\left(t\right) \in M_m\left(C^2\left[t_0, +\infty\right)\right)$ which satisfies conditions 1, 2 for some $k \in \{0, 1, 2\}$ is called an L_k-matrix.

According to the definition given in [10, p. 353], a matrix $L\left(t\right) \in M_m(C^1[t_0, +\infty))$ which satisfies conditions 1, 2 for $k = 0$, is called a *regular on $[t_0, +\infty)$ matrix*. Transformation (7), where $L\left(t\right)$ is an L_2-matrix can also be referenced to as a *Lyapunov transformation of system of second-order ODEs* (compare with the definition of a *Lyapunov transformation* form [1, p. 117]).

Let us consider the following symmetrization problems for the given system of second-order ODEs (6):

1. the *problem of Elimination of Gyroscopic Structures* (EGS problem) which consists in finding an L_k-matrix $L\left(t\right)$ $(k = 0, 1, 2)$ together with matrices $V\left(t\right), W\left(t\right) \in M_m\left(\mathbb{C}\left[t_0, +\infty\right)\right), \ V\left(t\right) = V^T\left(t\right),$ such that equalities (10) hold true $\forall t \in [t_0, +\infty);$

2. the *problem of Elimination of Non-conservative Positional Structures* (ENPS problem) which consists in finding an L_k-matrix $L\left(t\right)$ $(k = 0, 1, 2)$ together with matrices $V\left(t\right), W\left(t\right) \in M_m\left(\mathbb{C}\left[t_0, +\infty\right)\right), \ W\left(t\right) = W^T\left(t\right),$ such that equalities (10) hold true $\forall t \in [t_0, +\infty).$

If the matrices $L\left(t\right), V\left(t\right), W\left(t\right)$ mentioned in items 1 and/or 2 exist then we say that the EGS and/or ENPS problems for system (6) can be solved by means of L_k-transformation.

Both symmetrization problems can be stated in terms of the L_k-equivalence in the following way:

1. to find a system (9) with $V\left(t\right) = V^T\left(t\right)$ which is L_k-equivalent to the given system (6) (EGS problem);

2. to find a system (9) with $W\left(t\right) = W^T\left(t\right)$ which is L_k-equivalent to the given system (6) (ENPS problem).

Let us find the necessary and sufficient conditions (in terms of matrices $A(t), B(t)$) providing the solvability of the EGS and/or ENPS problems for the given system (6), or, in other words, the necessary and sufficient conditions providing that system (6) is L_k-equivalent to some system (9) with $V(t) = V^T(t)$ and/or $W(t) = W^T(t)$ for some $k = 0, 1, 2$.

Supposing that the matrix coefficient in front of the vector-function $\dot{\xi}$ in system (9) is symmetric (i.e., there is no gyroscopic structures), we arrive at the following matrix differential equation with respect to the unknown L_k-matrix $L(t)$:

$$2\left(\dot{L}(t)L^T(t) - L(t)\dot{L}^T(t)\right) + A(t)L(t)L^T(t) - L(t)L^T(t)A^T(t) = 0. \quad (11)$$

Similarly to that, assuming that the matrix coefficient in front of the vector-function ξ in system (9) is symmetric (i.e., there is no non-conservative positional structures) we arrive at the equation

$$\ddot{L}(t)L^T(t) - L(t)\ddot{L}^T(t) + A(t)\dot{L}(t)L^T(t) -$$
$$-L(t)\dot{L}^T(t)A^T(t) + B(t)L(t)L^T(t) - L(t)L^T(t)B^T(t) = 0. \quad (12)$$

It is easy to verify that there exists a unique pair of matrices $K(t)$, $S(t)$, such that

$$\dot{L}L^T = \dot{L}(t)L^T(t) = K(t) + S(t), \quad K(t) = -K^T(t), \quad S(t) = S^T(t). \quad (13)$$

If matrix $L(t)$ is an L_k-matrix ($k = 0, 1, 2$) then matrices $K(t)$ and $S(t)$ (13) belongs to $M_m(C^1[t_0, +\infty))$. It is easy to see that

$$\frac{d}{dt}\left(L(t)L^T(t)\right) = \dot{L}(t)L^T(t) + L(t)\dot{L}^T(t) = 2S(t), \quad (14)$$

and

$$L(t)L^T(t) = 2\int\limits_{t_0}^{t} S(\nu)\,d\nu + S_0, \quad L(t_0)L^T(t_0) = S_0 = S_0^T > 0. \quad (15)$$

Taking into account equalities (13), (15), we can rewrite equations (11) and (12) in the form of

$$4K(t) + A(t)\left(2\int\limits_{t_0}^{t} S(\nu)\,d\nu + S_0\right) - \left(2\int\limits_{t_0}^{t} S(\nu)\,d\nu + S_0\right)A^T(t) = 0 \quad (16)$$

and

$$2\dot{K}(t) + A(t)\left(S(t) + K(t)\right) - \left(S(t) - K(t)\right)A^T(t) +$$
$$+B(t)\left(\int\limits_{t_0}^{t} 2S(\nu)\,d\nu + S_0\right) - \left(\int\limits_{t_0}^{t} 2S(\nu)\,d\nu + S_0\right)B^T(t) = 0 \quad (17)$$

respectively. What are the necessary and sufficient requirements which have to be imposed on the matrices $K(t)$ and $S(t)$ to provide the existence of an L_k-matrix $L(t)$ which satisfies equality (13)? The answer to this question is given by the following theorem.

Theorem 1. *A regular on $[t_0, +\infty)$ matrix $L(t)$ which satisfies equality (13) exists if and only if the matrices $K(t), S(t)$ belong to $M_m(C[t_0, +\infty))$ and satisfy the following inequalities:*

$$\left| 2 \int_{t_0}^{t} Tr(S(\nu)) \, d\nu + Tr(S_0) \right| \le \mu^2, \forall t \in [t_0, +\infty), \qquad (18)$$

$$\det \left(2 \int_{t_0}^{t} S(\nu) \, d\nu + S_0 \right) \ge \eta^2, \forall t \in [t_0, +\infty) \qquad (19)$$

for some constants $\mu > 0$, $\eta > 0$ and real valued positive definite symmetric matrix $S_0 \in M_m(\mathbb{R})$.

Proof. Necessity. Suppose that there exists a matrix $L(t)$ which belongs to $M_m(C^1[t_0, +\infty))$ and satisfies equality (13) together with inequalities

$$\|L(t)\|_F \le \mu, \quad \forall t \in [t_0, +\infty)^6, \qquad (20)$$

$$|\det(L(t))| \ge \eta, \quad \forall t \in [t_0, +\infty), \qquad (21)$$

for some constants $\mu > 0$, $\eta > 0$. It easy to see that the matrices $K(t), S(t)$ appearing in (13) belong to $M_m(C[t_0, +\infty))$, and the necessity of conditions (18), (19) immediately follows from (15). The necessity in the theorem is proved.

Sufficiency . Suppose that $K(t), S(t) \in M_m(C[t_0, +\infty))$, $K(t) = -K^T(t)$, $S(t) = S^T(t)$ and inequalities (18), (19) hold true for some constants $\mu > 0$, $\eta > 0$ and some positive definite symmetric matrix S_0. Assuming that the matrix $L = L(t)$ satisfies equality (13) $\forall t \in [t_0, +\infty)$ together with the initial condition

$$L(t_0) = L_0, \quad L_0 L_0^T = S_0, \qquad (22)$$

we arrive at the conclusion that equality (15) together with inequality (18) imply inequality (20) as well as inequality (19) implies inequality (21).

Let us prove that the solution $L = L(t)$ to the Cauchy problem (13), (22) supplemented with conditions (18), (19) exists and is unique on $[t_0, T]$ for any arbitrary $T > t_0$. If we denote by λ_i, $i = 1, 2, \ldots, m$ the ascending ordered eigenvalues of matrix S_0, that is, $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_m$, then inequality (18) implies that

$$\lambda_m \le \sum_{i=1}^{m} \lambda_i = Tr(S_0) \le \mu^2. \qquad (23)$$

Taking into account inequality (23) we can obtain from inequality (19) the estimate

$$\lambda_1 = \frac{\det(S_0)}{\lambda_2 \ldots \lambda_m} \ge \frac{\eta^2}{\mu^{2(m-1)}}$$

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[6] Here $\|A\|_F$ denotes the Frobenius norm of matrix A, that is, $\|A\|_F = \sqrt{Tr(AA^T)}$.

which leads us to the inequality

$$\left\|L_0^{-1}\right\|_E^2 = Tr\left(S_0^{-1}\right) \leq \frac{m}{\lambda_1} \leq \frac{m\mu^{2(m-1)}}{\eta^2} = \kappa^2.$$

Since $\det(L(t)) \neq 0$, $\forall t \in [t_0, +\infty)$, equality (13) can be rewritten in the form of

$$\dot{L} = F(t, L) = \left(S(t) + K(t)\right)\left(L^{-1}\right)^T. \tag{24}$$

Now we intend to show that the matrix-valued function $F(t, L)$ satisfies conditions of the Picard-–Lindelöf theorem (see, for example, [11, p. 8]) in the rectangle

$$\mathfrak{P} = \left\{ (t, L) \in \mathbb{R} \times M_m(\mathbb{R}) : t_0 \leq t \leq T, \quad \left\|L - L_0\right\|_E \leq \frac{\delta}{\kappa},\ 0 < \delta < 1 \right\}. \tag{25}$$

Taking into account that the elements of matrix-functions $S(t)$ and $K(t)$ are continuous on $[t_0, +\infty)$, it remains only to show that the matrix-valued function $F(t, L)$ is Lipschitz-continuous on $\mathfrak{P}$ (25) with respect to its second argument L. This fact follows from the following inequalities, which are valid for any matrices $L_i \in M_m(\mathbb{R})$, $i = 1, 2$, such that $\|L_i - L_0\|_E \leq \frac{\delta}{\kappa}$:

$$\left\|L_1^{-1} - L_2^{-1}\right\|_E = \left\|(L_1 - L_0 + L_0)^{-1} - (L_2 - L_0 + L_0)^{-1}\right\|_E =$$

$$= \left\|L_0^{-1}\left((L_1 - L_0)L_0^{-1} + E\right)^{-1} - L_0^{-1}\left((L_2 - L_0)L_0^{-1} + E\right)^{-1}\right\|_E =$$

$$= \left\|L_0^{-1}\left(\sum_{i=0}^{\infty}(-1)^i\left((L_1 - L_0)L_0^{-1}\right)^i - \sum_{i=0}^{\infty}(-1)^i\left((L_2 - L_0)L_0^{-1}\right)^i\right)\right\|_E =$$

$$= \left\|L_0^{-1}\left(\sum_{i=1}^{\infty}(-1)^i\left(\left((L_1 - L_0)L_0^{-1}\right)^i - \left((L_2 - L_0)L_0^{-1}\right)^i\right)\right)\right\|_E \leq$$

$$\leq \left\|L_0^{-1}\right\|_E \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}\|L_1 - L_0\|_E^{i-j}\|L_1 - L_2\|_E\|L_2 - L_0\|_E^{j-1}\left\|L_0^{-1}\right\|_E^i\right) =$$

$$= \left\|L_0^{-1}\right\|_E^2\|L_1 - L_2\|_E \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}\|L_1 - L_0\|_E^{i-j}\|L_0 - L_2\|_E^{j-1}\left\|L_0^{-1}\right\|_E^{i-1}\right) \leq$$

$$\leq \|L_1 - L_2\|_E\,\kappa^2 \sum_{i=1}^{\infty} i\delta^{i-1} = \frac{\kappa^2}{(1-\delta)^2}\|L_1 - L_2\|_E. \tag{26}$$

In the above formula we have used the equality (see, for example, [12, p. 113])

$$(A + E)^{-1} = \sum_{i=0}^{\infty}(-1)^i A^i, \quad \forall A \in M_m(\mathbb{R}),\ \|A\| < 1,$$

and the evident identity

$$A^n - B^n = \sum_{i=1}^{n} A^{n-i}(A - B)B^{i-1}, \quad \forall A, B \in M_m(\mathbb{R}),\ n = 1, 2, \ldots.$$

Using (26) we can estimate the norm of $F(t, L)$ on the rectangle $\mathfrak{P}$ (25) in the following way:

$$\max_{\|L-L_0\|_E \leq \frac{\delta}{\kappa}} \left\|\left(L^{-1}\right)^T\right\|_E = \max_{\|L-L_0\|_E \leq \frac{\delta}{\kappa}} \left\|L^{-1} - L_0^{-1} + L_0^{-1}\right\|_E \leq$$

$$\leq \frac{\kappa^2}{(1-\delta)^2} \max_{\|L-L_0\|_E \leq \frac{\delta}{\kappa}} \|L - L_0\|_E + \left\|L_0^{-1}\right\|_E \leq \frac{\kappa\delta}{(1-\delta)^2} + \kappa,$$

$$\max_{(t,L)\in\Pi} \|F(t,L)\|_E \leq \max_{t\in[t_0,T]} \|K(t) + S(t)\|_E \max_{\|L-L_0\|_E \leq \frac{\delta}{\kappa}} \left\|\left(L^{-1}\right)^T\right\|_E = F_{\mathfrak{P}}.$$

Thus, the conditions of the Picard—Lindelöf theorem are satisfied and the solution of the Cauchy problem (22), (24) exists at least on the interval $I_h = [t_0, h]$, where $h = \min\left\{T, \dfrac{\delta}{\kappa F_{\mathfrak{P}}}\right\}$. If $h = T$ then the theorem is proved. Otherwise, if $h < T$ then, applying the same reasoning as above to equation (24) with the initial condition $L_h = L(h)$, we arrive at the conclusion that the solution to the Cauchy problem (22), (24) exists at least on the interval $[t_0, 2h]$. Apparently, after a finite number of iterations we will prove that the solution exists on $[t_0, T]$. From the arbitrariness of T it follows that the solution to the Cauchy problem (22), (24) exists on $[t_0, +\infty)$. The theorem is proved.

It is not hard to verify that the matrix $K(t) + S(t)$ where $K(t) = -K^T(t)$, $S(t) = S^T(t)$ is bounded on $[t_0, +\infty]$ and/or belongs to $M_m(C^k[t_0, +\infty))$ if and only if both of the two matrices $K(t)$ and $S(t)$ are bounded on $[t_0, +\infty)$ and/or belong to $M_m(C^k[t_0, +\infty))$. Taking this fact into account and using Theorem 1 we can make several conclusions stated below.

Corollary 3. *An L_k-matrix $L(t)$ $(k = 1, 2)$ satisfying equality (13) exists if and only if $K(t), S(t) \in M_m\left(C^1[t_0, +\infty)\right)$ and the following conditions hold true:*

    1. *there exist constants $\mu > 0, \eta > 0$ and matrix $S_0 \in M_m(\mathbb{R})$, $S_0 = S_0^T > 0$ satisfying inequalities (18), (19);*

    2. $\displaystyle\sup_{t\in[t_0,+\infty)} \left\|\frac{d^i}{dt^i}K(t)\right\| + \sup_{t\in[t_0,+\infty)} \left\|\frac{d^i}{dt^i}S(t)\right\| < +\infty, \forall i \in \overline{0, k-1}.$

Equation (16) and Corollary 3 imply the following theorem.

Theorem 2. *The given system of second-order ODEs (6) with $A(t) \in M_m\left(C^1[t_0, +\infty)\right)$ is L_k-equivalent $(k = 0, 1, 2)$ to some system (9) with $V(t) = V^T(t)$ if and only if there exist the symmetric matrices $S(t) \in M_m\left(C^1[t_0, +\infty)\right)$, $S_0 \in M_m(\mathbb{R})$, $S_0 > 0$ which define the skew-symmetric matrix $K(t)$*

$$4K(t) = \Lambda(t)A^T(t) - A(t)\Lambda(t), \quad \Lambda(t) = 2\int_{t_0}^{t} S(\nu)\,d\nu + S_0, \qquad (27)$$

and satisfy conditions

    1. *(18), (19) for some constants $\mu > 0$, $\eta > 0$;*

2. $\displaystyle\sup_{t\in[t_0,+\infty)}\left\|\frac{d^i}{dt^i}K\left(t\right)\right\|+\sup_{t\in[t_0,+\infty)}\left\|\frac{d^i}{dt^i}S\left(t\right)\right\|<+\infty,\ \forall i\in\overline{0,k-1},\ (k\neq 0)^7.$

From Theorem 2 we obtain the following corollary.

Corollary 4. *The given system of second-order ODEs* (6) *with* $A\left(t\right)\in M_m\left(C^1\left[t_0,+\infty\right)\right)$ *is always* L_0-*equivalent to some system* (9) *with* $V\left(t\right)=V^T\left(t\right).$

From equation (17) and Corollary 3 we can easily obtain the theorem which gives the necessary and sufficient conditions for solvability of the ENPS problem.

Theorem 3. *The given system of second-order ODEs* (6) *is* L_k-*equivalent* ($k=0,1,2$) *to some system* (9) *with* $W\left(t\right)=W^T\left(t\right)$ *if and only if there exist the symmetric matrices* $S\left(t\right)\in M_m\left(C^1\left[t_0,+\infty\right)\right),\ S_0\in M_m\left(\mathbb{R}\right),\ S_0>0$ *and the skew-symmetric matrix* $K\left(t\right)$ *which satisfies the matrix differential equation*

$$2\dot{K}\left(t\right)+A\left(t\right)K\left(t\right)+K\left(t\right)A^T\left(t\right)+A\left(t\right)S\left(t\right)$$

$$-S\left(t\right)A^T\left(t\right)+B\left(t\right)\Lambda\left(t\right)-\Lambda\left(t\right)B^T\left(t\right)=0,$$

$$\Lambda\left(t\right)=2\int\limits_{t_0}^{t}S\left(\nu\right)d\nu+S_0,$$

(28)

and conditions 1, 2 of Theorem 2.

It is worth to emphasize that for any initial condition $K\left(t_0\right)=K_0=-K_0^T\in M_m\left(\mathbb{R}\right)$ the solution $K\left(t\right)$ to the matrix differential equation (28) is a skew-symmetric matrix. Indeed, if we sum up equation (28) with the transposed equation (28) we obtain the Cauchy problem

$$2\dot{N}\left(t\right)+A\left(t\right)N\left(t\right)+N\left(t\right)A^T\left(t\right)=0,$$

$$N\left(t\right)=K\left(t\right)+K^T\left(t\right),\ N\left(0\right)=0.$$

(29)

It is easy to see that the conditions of the Picard--Lindelöf theorem for the Cauchy problem (29) are fulfilled and its solution $N(t)$ exists and is unique on $[t_0,+\infty)$. Therefore, the problem has the trivial solution only, that is, $N\left(t\right)=0,\ \forall t\in[t_0,+\infty)$ and $K\left(t\right)=-K^T\left(t\right),\ \forall t\in[t_0,+\infty).$ Such conclusion can also be obtained from the analysis of the analytical expression for the general solution $K\left(t\right)$ of equation (28) (see, for example, [13, p. 188]).

From Theorem 3 we can easily obtain the corollary.

Corollary 5. *The given system of second-order ODEs* (6) *is always* L_0-*equivalent to some other system* (9) *with* $W\left(t\right)=W^T\left(t\right).$

Combining Theorems 4 and 2 we arrive at the following one.

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[7]In the case when $k=0$ condition 2 should be neglected.

Theorem 4. *The given system of second-order ODEs* (6) *with* $A(t) \in M_m\left(C^1[t_0, +\infty)\right)$ *is* L_k-*equivalent* $(k = 0, 1, 2)$ *to some system* (9) *with* $V(t) = V^T(t)$, $W(t) = W^T(t)$ *if and only if there exist the symmetric matrices* $S(t) \in M_m\left(C^1[t_0, +\infty)\right)$, $S_0 \in M_m(\mathbb{R})$, $S_0 > 0$ *which define the skew-symmetric matrix* $K(t)$ (27) *and satisfy conditions 1, 2 of Theorem 2 together with equality*

$$\Lambda(t) M^T(t) = M(t) \Lambda(t), \quad \forall t \in [t_0, +\infty),$$

$$M(t) = \frac{1}{2}\frac{d}{dt}A(t) + \frac{1}{4}A^2(t) - B(t), \quad \Lambda(t) = 2\int_{t_0}^{t} S(\nu)\, d\nu + S_0. \tag{30}$$

Condition (30) can be obtained as a result of substitution of the matrix $K(t)$ from equation (28) by its expression from (27).

Remark 1. *Suppose that the conditions of at least one of the Theorems 2, 3 or 4 are fulfilled. Then each suitable* L_k-*matrix* $L(t)$ *can be found as the solution to the matrix differential equation* (13) *supplemented with an initial condition* $L(t_0) = L_0$ *where* L_0 *is an arbitrary matrix form* $M_m(\mathbb{R})$, *such that* $L_0 L_0^T = S_0$. *Additionally to that, the matrix coefficients of the respective symmetrized system* (9) *can be found via formulas* (10).

3. Stability preserving structural transformations of the autonomous systems of second-order ODEs

Let us consider the two systems of second-order ordinary differential equations

$$\ddot{\mathbf{x}} + A\dot{\mathbf{x}} + B\mathbf{x} = 0, \quad A, B \in M_m(\mathbb{R}), \tag{31}$$

$$\ddot{\xi} + V\dot{\xi} + W\xi = 0, \quad V, W \in M_m(\mathbb{R}). \tag{32}$$

Definition 5. We say that the given autonomous system (31) is L-equivalent to system (32) if there exists a regular on $[0, +\infty)$ matrix $L(t)$ (see definition on p. 40) which satisfies equalities [8]

$$V = L^{-1}(t)\left(2\dot{L}(t) + AL(t)\right),$$

$$W = L^{-1}(t)\left(\ddot{L}(t) + A\dot{L}(t) + BL(t)\right), \quad \forall t \in [0, +\infty). \tag{33}$$

From the first equality of (33) we can easily obtain

$$L(2t) = \exp(-At)\, C \exp(Vt), \quad C \in M_m(\mathbb{R}). \tag{34}$$

It is easy to see that if the matrix $L(t)$ (34) is regular on $[0, +\infty)$ then it is an L_k-matrix for $k = 0, 1, 2$. Hence, we can see that the notion of the L_k-equivalence $(k = 0, 1, 2)$ for two autonomous systems according to definition 4 is tantamount to the notion of the L-equivalence according to definition 5.

In this section we consider the following symmetrization problems for the autonomous systems of second-order ODEs (31):

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[8]Without loss of generality and for the sake of simplicity, in this section we consider the segment $[0, +\infty)$ instead of $[t_0, +\infty)$.

1. to find an autonomous system (32) with $V = V^T$ which is L-equivalent to the given system (31) (compare with the EGS problem);
2. to find an autonomous system (32) with $W = W^T$ which is L-equivalent to the given system (31) (compare with the ENPS problem).

Let us find the necessary and sufficient requirements which have to be imposed on matrices A, B to provide the solvability of the EGS and/or ENPS problems for autonomous system (31).

To proceed with this task we have to introduce several convenient notations. We will use the notation $[A, B]$ to describe a commutator of two square matrices A and B, that is,

$$[A, B] = AB - BA.$$

Also, we will use the notation $\{A_1 A_2 \ldots A_n\}$ to describe a superposition of commutators, that is,

$$\{A_1 A_2\} = [A_1, A_2], \quad \{A_1 A_2 \ldots A_n\} = [\{A_1 A_2 \ldots A_{n-1}\}, A_n].$$

It is easy to ensure that the commutators obey the following properties:

$$[AB, C] = [A, C] B, \quad \forall A, B, C \in M_m(\mathbb{R}) : [B, C] = 0, \tag{35}$$

$$[[A, B], C] = [A, [B, C]], \quad \forall A, B, C \in M_m(\mathbb{R}) : [A, C] = 0. \tag{36}$$

It is a well known fact that every matrix $A \in M_m(\mathbb{R})$ can be expressed in the form of

$$A = T_A \, diag \left[\lambda_1(A) E^{(p_1)} + H^{(p_1)}, \ldots, \lambda_r(A) E^{(p_r)} + H^{(p_r)} \right] T_A^{-1}, \tag{37}$$

where

$$\lambda_k(A) = \alpha_k(A) + i \, \beta_k(A), \quad \alpha_k(A), \beta_k(A) \in \mathbb{R}, \tag{38}$$

$k = 1, 2, \ldots, r$ denote the eigenvalues of matrix A; $E^{(p_k)}$ denotes the identity matrix; all the elements of square matrix $H^{(p_k)}$ are zero except those in the first superdiagonal which are equal to 1. The orders of square matrices $E^{(p_k)}$ and $H^{(p_k)}$ are equal to the power p_k of the k-th elementary devisor of matrix A. The matrix T_A denotes some nonsingular matrix from $M_m(\mathbb{R})$ (see, for example, [12, p. 152]).

According to formulas (37) and (38) we define

$$A_R = T_A \, diag \left[\alpha_1(A) E^{(p_1)} + H^{(p_1)}, \ldots, \alpha_r(A) E^{(p_r)} + H^{(p_r)} \right] T_A^{-1},$$

$$A_I = T_A \, diag \left[i \, \beta_1(A) E^{(p_1)}, \ldots, i \, \beta_r(A) E^{(p_r)} \right] T_A^{-1}, \tag{39}$$

then

$$A = A_R + A_I, \ A_R A_I = A_I A_R. \tag{40}$$

Using the notion of real Jordan canonical form of a real matrix (see [14, p. 184]) it is not hard to prove that if $A \in M_m(\mathbb{R})$ then $A_R, A_I \in M_m(\mathbb{R})$.

Let us consider a Jordan matrix (see, for example, [14, p. 150])

$$J_R = diag \left[J_1(\lambda_1), \ldots, J_s(\lambda_s) \right], \tag{41}$$

where $J_i(\lambda_i)$ denotes a Jordan block of size m_i corresponding to the eigenvalue $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, s$. For the definiteness we will use the assumption that

$$\lambda_i > \lambda_j,\; i < j,\; \sum_{i=1}^{s} m_i = m. \tag{42}$$

In the above formula m_i denotes an algebraic multiplicity of the eigenvalue λ_i of matrix J_R (see [14, p. 58]). The following lemma holds true.

Lemma 1. *Suppose that the matrix $L(t)$ is defined by the formula*

$$L(t) = \exp\left(-J_R t\right) Q \exp\left(J_R t\right),\; t \geq 0, \tag{43}$$

where $Q \in M_m(\mathbb{R})$. Matrix $L(t)$ (43) is a regular on $[0, +\infty)$ matrix if and only if the matrix Q possesses the following structure:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & \ldots & Q_{1s} \\ O_{21} & Q_{22} & \ldots & Q_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ O_{s1} & O_{s2} & \ldots & Q_{ss} \end{bmatrix}, \tag{44}$$

where matrices $Q_{ij} \in M_{m_i m_j}(\mathbb{R})$ satisfy the conditions

$$\det\left(Q_{ii}\right) \neq 0,\; \left[J_i^{(R)}, Q_{ii}\right] = 0 \tag{45}$$

and O_{ij} denotes a zero-matrix of dimension $m_i \times m_j$, $i, j = 1, 2, \ldots, s$.

Proof. Without loss of generality, we consider the case when $s = 2$, that is, when the matrix J_R has only two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > \lambda_2$ of the algebraic multiplicities $m_1 \geq 0$ and $m_2 \geq 0$ respectively, $m_1 + m_2 = m$. Let us denote

$$G_1 = J_1(0),\; G_2 = J_2(0). \tag{46}$$

From formula (41), taking into account notation (46), we obtain (see [12, p. 157])

$$\begin{aligned} \exp\left(J_R t\right) = &diag\left[e^{\lambda_1 t}\sum_{i=0}^{m_1}\frac{1}{i!}t^i G_1^i,\; e^{\lambda_2 t}\sum_{i=0}^{m_2}\frac{1}{i!}t^i G_2^i\right], \\ \exp\left(-J_R t\right) = &\left(\exp\left(J_R t\right)\right)^{-1} = \\ = &diag\left[e^{-\lambda_1 t}\sum_{i=0}^{m_1}\frac{(-t)^i}{i!}G_1^i,\; e^{-\lambda_2 t}\sum_{i=0}^{m_2}\frac{(-t)^i}{i!}G_2^i\right]. \end{aligned} \tag{47}$$

Necessity. Assume that the matrix $L(t)$ (43) is a regular on $[0, +\infty)$ matrix. Taking into account formulas (47) we obtain

$$L(t) = \exp(-J_R t) \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \exp(J_R t) = \begin{bmatrix} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{bmatrix},$$

$$L_{11}(t) = \left(\sum_{i=0}^{m_1} \frac{(-t)^i}{i!} G_1^i\right) Q_{11} \left(\sum_{i=0}^{m_1} \frac{t^i}{i!} G_1^i\right),$$

$$L_{12}(t) = e^{(\lambda_2 - \lambda_1)t} \left(\sum_{i=0}^{m_1} \frac{(-t)^i}{i!} G_1^i\right) Q_{12} \left(\sum_{i=0}^{m_2} \frac{t^i}{i!} G_2^i\right), \tag{48}$$

$$L_{21}(t) = e^{(\lambda_1 - \lambda_2)t} \left(\sum_{i=0}^{m_2} \frac{(-t)^i}{i!} G_2^i\right) Q_{21} \left(\sum_{i=0}^{m_1} \frac{t^i}{i!} G_1^i\right),$$

$$L_{22}(t) = \left(\sum_{i=0}^{m_2} \frac{(-t)^i}{i!} G_2^i\right) Q_{22} \left(\sum_{i=0}^{m_2} \frac{t^i}{i!} G_2^i\right).$$

Since the matrices $\sum_{i=0}^{m_j} \frac{(\pm t)^i}{i!} G_j^i$, $j = 1, 2$ are nonsingular, it is easy to see that the matrix $L(t)$ (43) has unbounded norm on $[0, \infty)$ unless $Q_{21} = O_{21}$ and matrices $L_{11}(t)$, $L_{22}(t)$, whose elements are polynomials of t, are constant. The latter fact implies that

$$L_{11}(t) = Q_{11}, \quad L_{22}(t) = Q_{22}. \tag{49}$$

Particulary, from equalities (49) it follows that $\det(Q_{jj}) \neq 0$, $j = 1, 2$. Taking into account the equalities

$$\left(\sum_{i=0}^{m_j} \frac{(-t)^i}{i!} G_j^i\right) = \left(\sum_{i=0}^{m_j} \frac{(t)^i}{i!} G_j^i\right)^{-1}, \; j = 1, 2,$$

from (48) and (49) we obtain

$$Q_{jj} \left(\sum_{i=0}^{m_j} \frac{(t)^i}{i!} G_j^i\right) = \left(\sum_{i=0}^{m_j} \frac{(t)^i}{i!} G_j^i\right) Q_{jj}, \; j = 1, 2, \quad \forall t \geq 0. \tag{50}$$

Equalities (50) imply that $[G_j, Q_{jj}] = 0$, $j = 1, 2$, and we immediately arrive at the conclusion about necessity of commutativity equalities in (45). The proof of the necessity is complete.

Sufficiency. Assume that the matrix Q has a structure described in (44), that is,

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ O_{21} & Q_{22} \end{bmatrix},$$

and conditions (45) holds true. Then, taking into account equalities (47), we get

$$L(t) = \exp(-J_R t) \begin{bmatrix} Q_{11} & Q_{12} \\ O_{21} & Q_{22} \end{bmatrix} \exp(J_R t) = \begin{bmatrix} Q_{11} & L_{12}(t) \\ O_{21} & Q_{22} \end{bmatrix},$$

$$L_{12}(t) = e^{(\lambda_2 - \lambda_1)t} \left(\sum_{i=0}^{m_1} \frac{(-t)^i}{i!} G_1^i\right) Q_{12} \left(\sum_{i=0}^{m_2} \frac{t^i}{i!} G_2^i\right). \tag{51}$$

Conditions (45) together with assumption (42) imply that the matrix $L(t)$ (51) is regular on $[0, +\infty)$. Hence, the sufficiency is proved and the Theorem is proved.

Lemma 2. *Suppose that A, V, $C \in M_m(\mathbb{R})$. If the matrix $L(t)$, defined by formula*

$$L(t) = \exp(-At)\, C \exp(Vt), t \geq 0, \tag{52}$$

is regular on $[0, +\infty)$ then the spectra of matrices A and V has the same real part (see definition in [15, p. 145]), that is, there exists a nonsingular matrix $C_1 \in M_m(\mathbb{R})$, such that

$$V_R = C_1^{-1} A_R C_1.$$

Proof. From the commutativity of matrices A_I and A_R (40) it follows that

$$L(t) = \exp(-A_I t) \exp(-A_R t)\, C \exp(V_R t) \exp(V_I t). \tag{53}$$

Taking into account the definitions of matrices A_I and V_I and equality (53) we arrive at the conclusion that the matrix $L(t)$ is regular on $[0, +\infty)$ if and only if the matrix

$$L_1(t) = \exp(-A_R t)\, C \exp(V_R t)$$

is regular on $[0, +\infty)$. On the other hand, it is easy to see that the matrix $L_1(t)$ represents the general solution to the matrix differential equation (supposing that C represents an arbitrary matrix from space $M_m(\mathbb{R})$)

$$\frac{d}{dt} L_1(t) = L_1(t) V_R - A_R L_1(t). \tag{54}$$

In [12, pp. 121–125] it was proved that equation (54) possesses a solution $L_1(t)$ that is a regular on $[0, +\infty)$ matrix if and only if the matrices A_R and V_R has the same set of elementary devisors. It is known (see [14, p. 185]) that if the matrices $A_R, V_R \in M_m(\mathbb{R})$ has the same set of elementary devisors then they are similar, furthermore, the similarity matrix C_1 can be chosen from the space $M_m(\mathbb{R})$. This completes the proof of the Theorem.

Lemma 3. *Suppose that $A, V, C, Z \in M_m(\mathbb{R})$, the matrix*

$$L(t) = \exp(-At)\, C \exp(Vt) \tag{55}$$

is regular on $[0, +\infty)$ and

$$\left[Z, L(t) C^{-1} \right] = 0, \ \forall t \geq 0. \tag{56}$$

Then there exists a nonsingular matrix $C_1 \in M_m(\mathbb{R})$, such that

$$V_R = C_1^{-1} A_R C_1, \tag{57}$$

$$C^{-1} Z C = C_1^{-1} Z C_1 \tag{58}$$

and the matrix

$$L_1(t) = \exp(-At)\, C_1 \exp(Vt) \tag{59}$$

is a regular on $[0, +\infty)$ matrix satisfying the identity

$$\left[Z, L_1(t) C_1^{-1} \right] = 0, \ \forall t \geq 0. \tag{60}$$

Proof. Suppose that the conditions of the Lemma are fulfilled. Then, according to Lemma 2, the spectra of matrices A and V has the same real part. Thus, there exist nonsingular matrices $T_A, T_V \in M_m (\mathbb{R})$, such that

$$A = T_A (J_R + I_A) T_A^{-1}, \; V = T_V (J_R + I_V) T_V^{-1},$$

$$I_A = T_A^{-1} A_I T_A, \; I_V = T_V^{-1} V_I T_V, \; [J_R, I_A] = [J_R, I_V] = 0, \tag{61}$$

where J_R is the Jordan matrix defined in (41).

Let us consider the matrix $L(t)$ (55). Using notation (61), we can rewrite it as following

$$L(t) = T_A \exp\left(-(J_R + I_A)t\right) T_A^{-1} C T_V \exp\left((J_R + I_V)t\right) T_V^{-1} =$$

$$= T_A \exp\left(-I_A t\right) \exp\left(-J_R t\right) \left(T_A^{-1} C T_V\right) \exp\left(J_R t\right) \exp\left(I_V t\right) T_V^{-1}. \tag{62}$$

From Lemma 1 it follows that $T_A^{-1} C T_V = Q$, where $Q \in M_m (\mathbb{R})$ is the matrix defined in (44).

Formula (62) leads us to the equality

$$T_A^{-1} L(t) C^{-1} T_A = \exp\left(-I_A t\right) \times$$

$$\times diag\left[\exp\left(-J_1 (\lambda_1) t\right), \ldots, \exp\left(-J_s (\lambda_s) t\right)\right] Q \times \tag{63}$$

$$\times diag\left[\exp\left(J_1 (\lambda_1) t\right), \ldots, \exp\left(J_s (\lambda_s) t\right)\right] \exp\left(I_V t\right) Q^{-1}.$$

From equality (63), owing to the commutation properties (45), we get

$$T_A^{-1} L(t) C^{-1} T_A = \exp\left(-I_A t\right) Q_D \exp\left(I_V t\right) Q^{-1} + \mathbf{E}_1(t) = \mathbf{E}_0(t) + \mathbf{E}_1(t), \tag{64}$$

where

$$Q_D = diag\left[Q_{11}, \ldots, Q_{ss}\right].$$

It is easy to see that identity (56) can be rewritten in the form of

$$\left[T_A^{-1} Z T_A, T_A^{-1} L(t) C^{-1} T_A\right] = \left[T_A^{-1} Z T_A, \mathbf{E}_0(t) + \mathbf{E}_1(t)\right] = 0, \; \forall t \geq 0. \tag{65}$$

It is not hard to verify that the elements of matrix $\mathbf{E}_0(t)$ (64) can be expressed as linear combinations of functions of type

$$\sin(\alpha\, t) \pm \cos(\alpha\, t), \; \alpha, \in \mathbb{R}. \tag{66}$$

On the other hand, the elements of matrix $\mathbf{E}_1(t)$ (64) can be expressed as linear combinations of functions of type

$$t^p e^{\rho t} \left(\cos(\alpha t) \pm \sin(\alpha t)\right), \; \rho, \alpha \in \mathbb{R}, \rho < 0, \tag{67}$$

$p \in \mathbb{N} \bigcup \{0\}, \; p < m.$

If the matrix $\mathbf{E}_0(t) + \mathbf{E}_1(t)$ commutates with the constant matrix $T_A^{-1} Z T_A$ for all $t \geq 0$ (see (65)) then the same remains true for each of the summands $\mathbf{E}_0(t)$ and $\mathbf{E}_1(t)$ separately. Indeed, assume to the contrary that there exists a value $t_1 \geq 0$, such that $\left[\mathbf{E}_0(t_1), T_A^{-1} Z T_A\right] \neq 0$. It is obvious that in this

case $\left[\mathbf{E}_1\left(t_1\right), T_A^{-1}ZT_A\right] \neq 0$. Taking into account the continuity of elements of matrices $\mathbf{E}_0\left(t\right), \mathbf{E}_1\left(t\right)$, we obtain

$$\left[\mathbf{E}_0\left(t\right), T_A^{-1}ZT_A\right] \neq 0, \quad \left[\mathbf{E}_1\left(t\right), T_A^{-1}ZT_A\right] \neq 0, \quad \forall t \in [t_1 - \delta, t_1 + \delta], \quad (68)$$

for some sufficiently small positive real number δ.

It is easy to see that each element of the matrix $\left[\mathbf{E}_0\left(t\right), T_A^{-1}ZT_A\right]$ can be expressed as a linear combination of functions of type (66) and each element of the matrix $\left[\mathbf{E}_1\left(t\right), T_A^{-1}ZT_A\right]$ can be expressed as a linear combination of functions of type (67). Since the functions of types (66) and (67) are linearly independent, we conclude that

$$\left[\mathbf{E}_0\left(t\right) + \mathbf{E}_1\left(t\right), T_A^{-1}ZT_A\right] = \left[\mathbf{E}_0\left(t\right), T_A^{-1}ZT_A\right] + \left[\mathbf{E}_1\left(t\right), T_A^{-1}ZT_A\right] \neq 0$$

for some $t \in [t_1 - \delta, t_1 + \delta]$. Thus, we get a contradictions to condition (65). This contradiction proves the incorrectness of our assumption. Therefore we proved the identity

$$\left[\mathbf{E}_0\left(t\right), T_A^{-1}ZT_A\right] = 0, \forall t \geq 0. \tag{69}$$

Setting $t = 0$ in (69) we obtain

$$\left[Q_D Q^{-1}, T_A^{-1}ZT_A\right] = 0. \tag{70}$$

Let us construct a matrix $C_1 \in M_m\left(\mathbb{R}\right)$, $\det\left(C_1\right) \neq 0$ satisfying equality (58). Using equality (70) we get

$$C^{-1}ZC = C^{-1}T_A\left(T_A^{-1}ZT_A\right)T_A^{-1}C = T_V\left(T_V^{-1}C^{-1}T_A\right)\left(T_A^{-1}ZT_A\right) \times$$

$$\times \left(T_A^{-1}CT_V\right)T_V^{-1} = T_V Q^{-1}\left(T_A^{-1}ZT_A\right)QT_V^{-1} =$$

$$= T_V Q_D^{-1}\left(Q_D Q^{-1}\right)\left(T_A^{-1}ZT_A\right)\left(QQ_D^{-1}\right)Q_D T_V^{-1} = \tag{71}$$

$$= T_V Q_D^{-1}\left(T_A^{-1}ZT_A\right)Q_D T_V^{-1} = \left(T_V Q_D^{-1}T_A^{-1}\right)Z\left(T_A Q_D T_V^{-1}\right).$$

From equalities (71) it follows that the matrix C_1 satisfying condition (58) can be chosen in the following way

$$C_1 = T_A Q_D T_V^{-1} \in M_m\left(\mathbb{R}\right). \tag{72}$$

Equality (57) can be obtained from the following chain of equalities

$$C_1^{-1}A_R C_1 = \left(T_A Q_D T_V^{-1}\right)^{-1}\left(T_A J_R T_A^{-1}\right)T_A Q_D T_V^{-1} =$$

$$= T_V Q_D^{-1}T_A^{-1}\left(T_A J_R T_A^{-1}\right)T_A Q_D T_V^{-1} = T_V J_R T_V^{-1} = V_R.$$

Let us prove that the matrix $L_1\left(t\right)$ (59) is regular on $[0, +\infty)$. Taking into account equality (57) and executing several elementary transformations, we

get

$$L_1\left(t\right) = \exp\left(-At\right)C_1\exp\left(Vt\right) =$$

$$= \exp\left(-A_It\right)\exp\left(-A_Rt\right)C_1\exp\left(V_Rt\right)\exp\left(V_It\right) =$$

$$= \exp\left(-A_It\right)C_1C_1^{-1}\exp\left(-A_Rt\right)C_1\exp\left(V_Rt\right)\exp\left(V_It\right) =$$

$$= \exp\left(-A_It\right)C_1\exp\left(-C_1^{-1}A_RC_1t\right)\exp\left(V_Rt\right)\exp\left(V_It\right) =$$

$$= \exp\left(-A_It\right)C_1\exp\left(-V_Rt\right)\exp\left(V_Rt\right)\exp\left(V_It\right) =$$

$$= \exp\left(-A_It\right)C_1\exp\left(V_It\right).$$

It is easy to see that the matrix $\exp\left(-A_It\right)C_1\exp\left(V_It\right)$ is regular on $[0,+\infty)$.

Now we intend to prove identity (60). Equalities

$$T_A^{-1}L_1\left(t\right)C_1^{-1}T_A = T_A^{-1}\Big(\exp\left(-A_It\right)\left(T_AQ_DT_V^{-1}\right)\exp\left(V_It\right)\Big)\times$$

$$\times\left(T_AQ_DT_V^{-1}\right)^{-1}T_A = \exp\left(-I_At\right)Q_D\exp\left(I_Vt\right)Q^{-1}QQ_D^{-1} = \mathbf{E}_0\left(t\right)\left(QQ_D^{-1}\right)$$

together with commutation identities (69) and (70) immediately lead us to the equalities

$$\left[Z, L_1\left(t\right)C_1^{-1}\right] = \left[T_A^{-1}ZT_A, T_A^{-1}L_1\left(t\right)C_1^{-1}T_A\right] =$$

$$= \left[T_A^{-1}ZT_A, \mathbf{E}_0\left(t\right)\left(Q\left(Q_D\right)^{-1}\right)\right] = 0,$$

which are valid for all $t \geq 0$.

The Theorem is proved.

Let us denote by $\mathcal{X}_n$ a set containing all the solutions of the system of linear matrix equations

$$\left\{ZA^{(k)}X\right\} = 0, \ k = 0, 1, \ldots, n, \tag{73}$$

where Z, $A \in M_m\left(\mathbb{R}\right)$ are given matrices and X is the unknown square matrix of order m.

Theorem 5. *There exists a positive integer number $n < m^2$, such that the set equalities*

$$\mathcal{X}_n = \mathcal{X}_k, \ k = n+1, n+2, \ldots. \tag{74}$$

hold true.

Proof. It is not hard to verify that the set $\mathcal{X}_n$ can be represented in the multi-parametric matrix form

$$\mathcal{X}_n = \left[\sum_{k=1}^{p_n}\chi_{k,i,j}^{(n)}c_k\right]_{i,j=1}^m, \tag{75}$$

where $\chi_{k,i,j}^{(n)}$ are constant real coefficients and c_k are the arbitrary parameters $k = 1, 2, \ldots, p_n$ (see, for example, [12, p. 221]), $0 < p_n \leq m^2$.

To begin with, we prove that if for some non-negative integer $n \in \mathbb{N} \bigcup \{0\}$ the set equality

$$\mathcal{X}_n = \mathcal{X}_{n+1}, \tag{76}$$

holds true then equalities (74) hold as well. Indeed, equality (76) implies that

$$0 = \left\{ ZA^{(n+1)}\mathcal{X}_{n+1} \right\} = \left[\left[\left\{ ZA^{(n)} \right\}, A \right], \mathcal{X}_n \right] = \tag{77}$$
$$= \left[\left\{ ZA^{(n)} \right\}, [A, \mathcal{X}_n] \right] = \left\{ ZA^{(n)} [A, \mathcal{X}_n] \right\}.$$

From here and below by equality of type (77) we mean the equality for every element of set $\mathcal{X}_n$. From (77) it follows that

$$[A, \mathcal{X}_n] \subseteq \mathcal{X}_n. \tag{78}$$

From equality (76) and inclusion (78) it follows that

$$\left\{ ZA^{(n+2)}\mathcal{X}_n \right\} = \left[\left[\left\{ ZA^{(n+1)} \right\}, A \right], \mathcal{X}_n \right] = \tag{79}$$
$$= \left[\left\{ ZA^{(n+1)} \right\}, [A, \mathcal{X}_n] \right] = \left\{ ZA^{(n+1)} [A, \mathcal{X}_n] \right\} = 0,$$

i.e., $\mathcal{X}_n \subseteq \mathcal{X}_{n+2}$. On the other hand, from the definition of the set $\mathcal{X}_n$ it follows that $\mathcal{X}_n \supseteq \mathcal{X}_{n+1} \supseteq \mathcal{X}_{n+2}$. Therefore, we have

$$\mathcal{X}_n = \mathcal{X}_{n+1} = \mathcal{X}_{n+2}. \tag{80}$$

Using reasoning similar to that used above and the method of mathematical induction it is not hard to prove that equality (76) implies equalities (74).

Now let us prove that the non-negative integer $n \in \mathbb{N} \bigcup \{0\}$, mentioned in the Theorem, exists and is less then m^2. For this purpose we consider the system of matrix equations

$$\left\{ ZA^{(n)}X \right\} = 0, \; n = 0, 1, \ldots, m^2 - 1 \tag{81}$$

with respect to unknown matrix $X \in M_m(\mathbb{R})$. If we would show that every solution X of system (81) satisfies equalities

$$\left\{ ZA^{(n)}X \right\} = 0, \; n = m^2, m^2 + 1, \ldots, \tag{82}$$

then we will prove the Theorem.

Let us consider the process of solving of system (81). Suppose that $n = 0$. There are only two possible cases (see representation (75)):

a)
$$\{ ZA\mathcal{X}_0 \} = 0, \; \forall c_k \in \mathbb{R}, \; k = 1, \ldots, p_0,$$

that is, we already have found a non-negative integer $n = 0$, such that equalities (76) hold true. Therefore, as it was proved above, equalities (81), (82) hold true for all $X \in \mathcal{X}_0$, the process is completed and the Theorem is proved;

b) equality
$$\{ ZA\mathcal{X}_0 \} = 0 \tag{83}$$

does not hold true for all possible values of the parameters $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, p_0$. This means that $m^2 > p_0 \geq 2$, because the assumption that $p_0 = 1$ or $p_0 = m^2$ immediately leads us to the equalities $\mathcal{X}_0 = c_0 E$ or $Z = aE$, $a \in \mathbb{R}$

respectively and we arrive at the case a). Thus, equality (83) can be viewed as a system of m^2 linear homogeneous equations with respect to the arbitrary parameters c_k, $k = 1, 2, \ldots, p_0$. Since this system possesses a non-zero solution, the rank r_1 of its matrix satisfies the two-sided inequality $0 < r_1 < p_0$. If we would solve the given system we will arrive at the matrix (set) $\mathcal{X}_1$ (75). In addition to that (see [16, p. 40–41]) $p_1 = p_0 - r_1$. Therefore, $p_1 < p_0$, that is, the number of the arbitrary parameters has decreased. Again, there are only two possible cases

a)

$$\left\{ Z A^{(2)} \mathcal{X}_1 \right\} = 0, \forall c_k \in \mathbb{R}, \ k = 1, \ldots, p_1,$$

i.e., the process is completed and the Theorem is proved;

b) equality

$$\left\{ Z A^{(2)} \mathcal{X}_1 \right\} = 0 \tag{84}$$

does not hold true for all possible values of the parameters $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, p_1$. It means that $m^2 > p_0 \geq p_1 + 1 \geq 3$. Equality (84) can be viewed as a system of m^2 linear homogeneous equations with respect to the parameters c_k, $k = 1, 2, \ldots, p_1$. If we would solve this new system we will arrive at the matrix (set) $\mathcal{X}_2$ (75). It is obvious that in this case $p_2 < p_1$, that is, the number of the arbitrary parameters has decreased again. And so on.

This process could not contain more than $p_0 < m^2$ steps. The Theorem is proved.

Lemma 4. *Suppose that $A, V, Z, C \in M_m(\mathbb{R})$ and $\det(C) \neq 0$. Then the commutation identity*

$$\left[Z, L(t) C^{-1} \right] = 0, \quad \forall t \geq 0, \tag{85}$$

where

$$L(2t) = \exp(-At) C \exp(Vt) \tag{86}$$

holds true if and only if the infinite system of matrix equalities

$$\left\{ Z A^{(n)} \left(C V C^{-1} - A \right) \right\} = 0, \ n = 0, 1, \ldots. \tag{87}$$

holds true.

Proof. Necessity. The matrix-valued function $L(t)$ (86) satisfies the equalities

$$\left[A, L^{(n)}(t) C^{-1} \right] = A L^{(n)}(t) C^{-1} - L^{(n)}(t) C^{-1} A =$$

$$= A L^{(n)}(t) C^{-1} - L^{(n)}(t) V C^{-1} + L^{(n)}(t) V C^{-1} - L^{(n)}(t) C^{-1} A =$$

$$= -2 L^{(n+1)}(t) C^{-1} + L^{(n)}(t) C^{-1} \left(C V C^{-1} - A \right) = \tag{88}$$

$$= -2 L^{(n+1)}(t) C^{-1} + 2 L^{(n)}(t) C^{-1} L^{(1)}(0) C^{-1}, \quad L^{(n)}(t) \overset{\text{def}}{=} \frac{d^n}{dt^n} L(t)$$

$\forall n \in \mathbb{N} \bigcup \{0\}$ or, that is the same,

$$L^{(n+1)}(t) C^{-1} = L^{(n)}(t) C^{-1} L^{(1)}(0) C^{-1} - \frac{1}{2} \left[A, L^{(n)}(t) C^{-1} \right]. \qquad (89)$$

Suppose that the commutativity identity (85) holds. Let us prove that it implies the identities

$$\left\{ Z A^{(n)} \left(L(t) C^{-1} \right) \right\} = 0, \quad \forall t \geq 0, \ \forall n \in \mathbb{N} \cup \{0\}. \qquad (90)$$

In order to prove this, we will use the method of mathematical induction with respect to n. If $n = 0$ then identity (90) coincides with (85). If $n = 1$ then form identity (85), using (89) and the properties of commutators (36), we get

$$0 = \left[Z, L^{(1)}(t) C^{-1} \right] = \left[Z, L(t) C^{-1} L^{(1)}(0) C^{-1} \right] -$$
$$- \frac{1}{2} \left[Z, \left[A, L(t) C^{-1} \right] \right] = -\frac{1}{2} \left[\left[Z, A \right], L(t) C^{-1} \right] = \qquad (91)$$
$$= -\frac{1}{2} \left\{ Z A \left(L(t) C^{-1} \right) \right\}.$$

Equality (91) proves identity (90) with $n = 1$. Let us assume that identity (90) is proved for $n = k \geq 2$ and let us prove it for $n = k + 1$. Using equality (89) and the properties of commutators (36), from the latter assumption we obtain

$$0 = \left[\left\{ Z A^{(k)} \right\}, L^{(1)}(t) C^{-1} \right] = \left[\left\{ Z A^{(k)} \right\}, L(t) C^{-1} L^{(1)}(0) C^{-1} \right] -$$
$$- \frac{1}{2} \left[\left\{ Z A^{(k)} \right\}, \left[A, L(t) C^{-1} \right] \right] = -\frac{1}{2} \left[\left[\left\{ Z A^{(k)} \right\}, A \right], L(t) C^{-1} \right] = \qquad (92)$$
$$= -\frac{1}{2} \left\{ Z A^{(k+1)} \left(L(t) C^{-1} \right) \right\}.$$

Therefore, according to the principle of mathematical induction, we have that identity (90) holds for all $n \in \mathbb{N} \cup \{0\}$.

Taking into account the arbitrariness of $n \in \mathbb{N} \cup \{0\}$ in formula (90), we can obtain equalities (87) via differentiation of identity (90) with respect to t and subsequent substitution $t = 0$.

Sufficiency. Suppose that equalities (87) hold. Let us prove that they imply identity (85). If $n = 0$ then from (87) we get

$$\left[Z, \left(CVC^{-1} - A \right) \right] = 2 \left[Z, L^{(1)}(0) C^{-1} \right] = 0. \qquad (93)$$

If $n = 1$ then from (87), taking into account (36), (88) and (93), we obtain

$$0 = \left[[Z, A], \left(CVC^{-1} - A \right) \right] = 2 \left[Z, \left[A, L^{(1)}(0) C^{-1} \right] \right] =$$
$$= -4 \left[Z, L^{(2)}(0) C^{-1} \right] + 4 \left[Z, \left(L^{(1)}(0) C^{-1} \right)^2 \right] = \qquad (94)$$
$$- (-2)^2 \left[Z, L^{(2)}(0) C^{-1} \right].$$

Let us assume that we already have proved equalities

$$\left[Z, L^{(n)}(0) C^{-1}\right] = 0, \ n = 1, 2, \dots, k \tag{95}$$

for some positive integer $k \geq 2$.

From equalities (87), assumption (95), properties of commutators (36) and equality (88) we get

$$
\begin{aligned}
0 &= \left\{ ZA^{(k)} \left(CVC^{-1} - A\right) \right\} = \left[\left[\left\{ ZA^{(k-1)} \right\}, A\right], \left(CVC^{-1} - A\right)\right] = \\
&= \left[\left\{ ZA^{(k-1)} \right\}, \left[A, CVC^{-1} - A\right]\right] = 2\left[\left\{ ZA^{(k-1)} \right\}, \left[A, L^{(1)}(0) C^{-1}\right]\right] = \\
&= -4\left[\left\{ ZA^{(k-1)} \right\}, L^{(2)}(0) C^{-1}\right] + 4\left[\left\{ ZA^{(k-1)} \right\}, \left(L^{(1)}(0) C^{-1}\right)^2\right] = \\
&= -4\left[\left[ZA^{(k-1)} \right\}, L^{(2)}(0) C^{-1}\right] = -4\left[\left[\left\{ ZA^{(k-2)} \right\}, A\right], L^{(2)}(0) C^{-1}\right] = \\
&= -4\left[\left\{ ZA^{(k-2)} \right\}, \left[A, L^{(0)}(0) C^{-1}\right]\right] = 8\left[\left\{ ZA^{(k-2)} \right\}, L^{(3)}(0) C^{-1}\right] - \\
&- 8\left[\left\{ ZA^{(k-2)} \right\}, L^{(2)}(0) C^{-1} L^{(1)}(0) C^{-1}\right] = \dots \\
\dots &= -(-2)^k \left[\left[Z, A\right], L^{(k)}(0) C^{-1}\right] = -(-2)^k \left[Z, \left[A, L^{(k)}(0) C^{-1}\right]\right] = \\
&= -(-2)^{k+1} \left[Z, L^{(k+1)}(0) C^{-1}\right]. \tag{96}
\end{aligned}
$$

Thus, according to the principle of mathematical induction, we have that equalities (95) hold for every non-negative integer $n \in \mathbb{N} \cup \{0\}$.

From (86) it follows that the matrix series

$$\sum_{n=0}^{\infty} L^{(n)}(0) C^{-1} \frac{t^n}{n!}$$

is dominated by the number series

$$\sum_{n=0}^{\infty} \frac{\left(\|A\| + \|CVC^{-1}\|\right)^n \left(\frac{t}{2}\right)^n}{n!} = \exp\left(\left(\|A\| + \|CVC^{-1}\|\right) \frac{t}{2}\right).$$

Thus, the matrix series is uniformly convergent on $[0, +\infty)$ and its sum coincides with the matrix $L(t) C^{-1}$. This fact together with equalities (95) immediately lead us to the commutativity identity (85). This completes the proof of the Theorem.

Now we are in position to prove the main theorem of the paper. It is stated below.

Theorem 6 (An analogue of the Erugin's theorem). *Suppose that $A, B, V, W \in M_m(\mathbb{R})$. The two systems of second-order differential equations*

$$\ddot{x}(t) + A\, \dot{x}(t) + B\, x(t) = 0, \tag{97}$$

$$\ddot{\xi}(t) + V\, \dot{\xi}(t) + W\, \xi(t) = 0 \tag{98}$$

are L-equivalent if and only if there exists a nonsingular matrix $C \in M_m(\mathbb{R})$ satisfying conditions

$$V_R = C^{-1} A_R C, \tag{99}$$

$$4W = V^2 + C^{-1}\left(4B - A^2\right)C, \tag{100}$$

$$\left\{\left(4B - A^2\right) A^{(n)}\left(CVC^{-1} - A\right)\right\} = 0, \; n = 0, 1, \ldots, m^2 - 1. \tag{101}$$

Proof. Sufficiency. Suppose that for some nonsingular matrix $C \in M_m(\mathbb{R})$ conditions (99), (100) and (101) are fulfilled. It is easy to see that the matrix

$$L(2t) = \exp(-At) C \exp(Vt) = \exp(-A_I t) C \exp(V_I t) \tag{102}$$

is regular on $[0, +\infty)$. Substituting the matrix $L(t)$ (102) into the first equality of (33) we obtain the identity

$$L^{-1}(t)\left(2\dot{L}(t) + AL(t)\right) = L^{-1}(t)\left(-AL(t) + L(t)V + AL(t)\right) = V. \tag{103}$$

From the second equality of (33) we get

$$\begin{aligned} 4L^{-1}(t)\left(\ddot{L}(t) + A\dot{L}(t) + BL(t)\right) \\ = L^{-1}(t)\left(4B - A^2\right)L(t) + V^2 = 4W. \end{aligned} \tag{104}$$

Here we have taken into account that equalities (101), according to Theorem 5 and Lemma 4, are equivalent to the commutativity identity

$$\left[4B - A^2, L(t)C^{-1}\right] = 0, \quad \forall t \geq 0. \tag{105}$$

Since the regular on $[0, +\infty)$ matrix $L(t)$ (102) satisfies conditions (33), systems (97) and (98) are L-equivalent. The sufficiency is proved.

Necessity. Suppose that systems (97) and (98) are L-equivalent. Then, according to the definition of the L-equivalence, there exists a regular on $[0, +\infty)$ matrix $L(t)$, such that

$$L^{-1}(t)\left(2\dot{L}(t) + AL(t)\right) = V, \tag{106}$$

$$L^{-1}(t)\left(\ddot{L}(t) + A\dot{L}(t) + BL(t)\right) = W, \; \forall t \in [0, +\infty). \tag{107}$$

From (106) we obtain that

$$L(2t) = \exp(-At) C \exp(Vt), \tag{108}$$

where $C \in M_m(\mathbb{R})$, $\det(C) \neq 0$. Then from (107), using formula (108) and setting $t = 0$, we obtain equality (100) and commutativity identity (105).

Since the conditions of Lemma 3 are fulfilled, we can assume that the matrix C is chosen in such a way that identity (105), equality (100) and condition (99) hold and in addition to that matrix (108) is regular on $[0, +\infty)$. From identity (105), according to Lemma (4), we get equalities (101). The necessity is proved and the proof of the Theorem is completed.

Remark 2. *Suppose that* $A, V, C \in M_m(\mathbb{R})$ *and* $\det(C) \neq 0$. *If for some non-negative integer* $n \in \mathbb{N} \cup \{0\}$ *the spectrum of the matrix* $Z_n = \left\{ \left(4B - A^2\right) A^{(n)} \right\}$ *is simple, i.e., all the eigenvalues of matrix* Z_n *are different, then conditions* (101) *are equivalent to the equalities*

$$\left[B, CVC^{-1} - A\right] = 0, \tag{109}$$

$$\left[A, CVC^{-1}\right] = 0. \tag{110}$$

Proof. It is almost obvious that conditions (109) and (110) imply conditions (101).

Suppose that conditions (101) are fulfilled and for some non-negative integer n the spectrum of matrix Z_n is simple. Then there exists a nonsingular matrix T, such that the matrix $T^{-1} Z_n T$ is diagonal with pairwise different diagonal elements. Thus, (see [12, p. 221]) we have that

$$T^{-1} \left(CVC^{-1} - A\right) T = diag\left[\sigma_1, \ldots, \sigma_m\right], \ \sigma_i \in \mathbb{R}, \ i \in \overline{1, m}. \tag{111}$$

Using Theorem 5 and equalities (101) we obtain

$$0 = \left[\left[Z_n, A\right], CVC^{-1} - A\right] = \left[Z_n, \left[A, CVC^{-1} - A\right]\right].$$

Applying the same reasoning as above to the latter equalities we arrive at the following representation, which is similar to (111):

$$T^{-1} \left[A, CVC^{-1} - A\right] T = T^{-1} A T T^{-1} \left(CVC^{-1} - A\right) T-$$

$$-T^{-1} \left(CVC^{-1} - A\right) T T^{-1} A T = T A T^{-1} diag\left[\sigma_1, \ldots, \sigma_m\right]-$$

$$-diag\left[\sigma_1, \ldots, \sigma_m\right] T A T^{-1} = diag\left[\tau_1, \ldots, \tau_m\right], \ \ \tau_i \in \mathbb{R}, \ i \in \overline{1, m}. \tag{112}$$

It is easy to see that all the diagonal elements of matrix $\left[T A T^{-1}, diag\left[\sigma_1, \ldots, \sigma_m\right]\right]$ are equal to 0. On the other hand, from (112) it follows that all the elements of matrix $\left[T A T^{-1}, diag\left[\sigma_1, \ldots, \sigma_m\right]\right]$ except for the diagonal are equal to 0. Therefore we get the equality $\left[A, CVC^{-1} - A\right] = 0$ which implies equality (110). Additionally to that equality (109) obviously follows from (101) and (110). The proof is completed.

Remark 3. *Conditions* (109) *and* (110) *imply conditions* (101). *The reverse implication is true only when the spectrum of the matrix* $Z_n = \left\{ \left(4B - A^2\right) A^{(n)} \right\}$ *is simple for some non-negative integer* n.

Though Theorem 6 gives us the necessary and sufficient conditions providing that systems (97) and (98) are equivalent (L-equivalent, to be precise), conditions (99), (100) and (101) of the Theorem do not possess the property of symmetry, which is one of the main properties of an equivalence relation. However, this is only the matter of the wording. In that form the theorem about L-equivalence will be useful in the further sections of the paper. Theorem 6 can be reformulated in the "symmetric" form presented below.

Theorem 7 (An analogue of the Erugin's theorem in the "symmetric" form). *Suppose that $A, B, V, W \in M_m(\mathbb{R})$. The two systems of second-order differential equations (97) and (98) are L-equivalent if and only if there exists a nonsingular matrix $C \in M_m(\mathbb{R})$, such that*

$$CV_R = A_R C, \tag{113}$$

$$C\left(4W - V^2\right) = \left(4B - A^2\right) C, \tag{114}$$

$$\left\{\left(4B - A^2\right) A^{(n)}\right\}(CV - AC) = (CV - AC)\left\{\left(4W - V^2\right) V^{(n)}\right\}, \tag{115}$$

$$n = 0, 1, 2, \ldots, m^2 - 1.$$

Proof. To prove the Theorem it is enough to show that conditions (99) − (101) are equivalent to conditions (113) − (115). It is easy to see that condition (99) is equivalent to condition (113), as well as condition (100) is equivalent to condition (114).

Taking into account (114), from equalities (115) with $n = 0$ we obtain the equalities

$$\left(4B - A^2\right)\left(CVC^{-1} - A\right) = \left(CVC^{-1} - A\right) C\left(4W - V^2\right) C^{-1} = \tag{116}$$
$$= \left(CVC^{-1} - A\right)\left(4B - A^2\right)$$

which lead us to condition (101) with $n = 0$. Multiplying equality (116) on C^{-1} from the left and on C from the right and rearranging the summands, we get

$$[C^{-1}\left(4B - A^2\right) C, V] = [\left(4W - V^2\right), V] = C^{-1}\left[\left(4B - A^2\right), A\right] C. \tag{117}$$

From equalities (115) with $n = 1$, taking into account (114) and (117), we obtain the equalities

$$\left\{\left(4B - A^2\right) A\right\}\left(CVC^{-1} - A\right) =$$
$$= \left(CVC^{-1} - A\right) C\left\{\left(4W - V^2\right) V\right\} C^{-1} = \tag{118}$$
$$= \left(CVC^{-1} - A\right)\left\{\left(4B - A^2\right) V\right\}$$

which lead us to condition (101) with $n = 1$. Multiplying equality (118) on C^{-1} from the left and on C from the right and rearranging the summands, we get

$$\left(C^{-1}\left\{\left(4B - A^2\right) A\right\} C\right) V - V\left(C^{-1}\left\{\left(4B - A^2\right) A\right\} C\right) =$$
$$= C^{-1}\left\{\left(4B - A^2\right) A^{(2)}\right\} C.$$

Combining the latter equality with (117) we obtain

$$\left\{\left(4W - V^2\right) V^{(2)}\right\} = C^{-1}\left\{\left(4B - A^2\right) A^{(2)}\right\} C. \tag{119}$$

Therefore we have proved that the first two equalities of (101) (with $n = 0, 1$) are equivalent to the first two equalities of (115) (with $n = 0, 1$) respectively. Besides that we have proved the auxiliary equalities (117) and (119). Let us assume that for some positive integer k, $2 < k < m^2 - 1$ we have proved that the first k equalities of (101) are equivalent to the first k equalities of (115)

(with $n = 0, 1, \ldots, k - 1$) respectively and the auxiliary equality (similar to (119))

$$\left\{\left(4W - V^2\right) V^{(k)}\right\} = C^{-1} \left\{\left(4B - A^2\right) A^{(k)}\right\} C \qquad (120)$$

holds. Then from equalities (115) with $n = k$, taking into account (120), we obtain equalities

$$\left\{\left(4B - A^2\right) A^{(k)}\right\} \left(CVC^{-1} - A\right) =$$
$$= \left(CVC^{-1} - A\right) C \left\{\left(4W - V^2\right) V^{(k)}\right\} C^{-1} = \qquad (121)$$
$$= \left(CVC^{-1} - A\right) \left\{\left(4B - A^2\right) V^{(k)}\right\}$$

which lead us to condition (101) with $n = k$. In addition to that, multiplying equality (121) on C^{-1} from the left and on C from the right and rearranging the summands, we obtain

$$\left(C^{-1} \left\{\left(4B - A^2\right) A^{(k)}\right\} C\right) V - V \left(C^{-1} \left\{\left(4B - A^2\right) A^{(k)}\right\} C\right) =$$
$$= C^{-1} \left\{\left(4B - A^2\right) A^{(k+1)}\right\} C.$$

Combining the latter equality with assumption (120) we get

$$\left\{\left(4W - V^2\right) V^{(k+1)}\right\} = C^{-1} \left\{\left(4B - A^2\right) A^{(k+1)}\right\} C. \qquad (122)$$

Therefore we have proved that the first $k + 1$ equalities of (101) are equivalent to the first $k + 1$ equalities of (115) (with $n = 0, 1, \ldots, k$) respectively. Also, we have proved the auxiliary equality (122). According to the principle of mathematical induction we can conclude that equalities (101) are equivalent to equalities (115), provided that condition (114) holds. This completes the proof of the Theorem.

4. Consequences from Theorem 6

Below we have stated several consequences from Theorem 6 that are related to the question of symmetrization of the matrix differential equation (or, in other words, the system of differential equations)

$$J\ddot{\mathbf{x}} + (D + G) \dot{\mathbf{x}} + (P + \Pi) \mathbf{x} = 0, \qquad (123)$$

where $J, D, G, P, \Pi \in M_m(\mathbb{R})$, $J = J^T > 0$, $D = D^T$, $\Pi = \Pi^T$, $G = -G^T$, $P = -P^T$. Let us denote

$$A = J^{-\frac{1}{2}} (D + G) J^{-\frac{1}{2}}, \quad B = J^{-\frac{1}{2}} (P + \Pi) J^{-\frac{1}{2}}. \qquad (124)$$

Corollary 6. *Suppose that there exist a symmetric matrix $V \in M_m(\mathbb{R})$ and a nonsingular matrix $C \in M_m(\mathbb{R})$ satisfying conditions*[9]

$$\left[A, \, CVC^{-1}\right] = ACVC^{-1} - CVC^{-1}A = 0, \qquad (125)$$

$$\left[B, \, A - CVC^{-1}\right] = B\left(A - CVC^{-1}\right) - \left(A - CVC^{-1}\right) B = 0, \qquad (126)$$

$$CV_R = A_R C. \qquad (127)$$

---

[9]See Remark 3.

Then the autonomous equation (123) *is L-equivalent to the autonomous equation*

$$\ddot{\xi} + V\dot{\xi} + W\xi = 0, \quad V, W \in M_m\left(\mathbb{R}\right), \tag{128}$$

$$W = \frac{1}{4}V^2 + C^{-1}\left(B - \frac{1}{4}A^2\right)C, \tag{129}$$

containing no gyroscopic structures $(V = V^T)$.

Corollary 7. *Suppose that there exist matrices* $V, C \in M_m\left(\mathbb{R}\right)$, $\det\left(C\right) \neq 0$ *satisfying conditions* (125)–(127) *and*

$$V^2 - \left(V^2\right)^T + C^{-1}ZC - C^TZ^T\left(C^{-1}\right)^T = 0 \tag{130}$$

where

$$Z = \left(B - \frac{1}{4}A^2\right).$$

Then the autonomous equation (123) *is L-equivalent to the autonomous equation* (128), (129), *containing no non-conservative positional structures* $(W = W^T)$.

Corollary 8. *Suppose that there exist a symmetric matrix* $V \in M_m\left(\mathbb{R}\right)$ *and a nonsingular matrix* $C \in M_m\left(\mathbb{R}\right)$ *satisfying conditions* (125)–(127) *together with the equality*

$$C^{-1}ZC - C^TZ^T\left(C^{-1}\right)^T = 0. \tag{131}$$

Then the autonomous equation (123) *is L-equivalent to the "symmetric" autonomous equation* (128), (129) $(W = W^T,\ V = V^T)$.

Corollary 9. *If for some non-negative integer* n *the spectrum of the matrix*

$$Z_n = \left\{\left(4B - A^2\right)A^{(n)}\right\}$$

is simple then the conditions of Corollaries 6–8 *are the necessary ones (not only sufficient!).*

Combining Theorem 6 with the theorems of Kelvin – Tait – Chetayev it is not hard to prove the following theorem that can be viewed as a generalization of the Mingori's [3] and Müller's [4] theorems.

Theorem 8. *Suppose that the matrices* $V, C \in M_m\left(\mathbb{R}\right)$, $\det\left(C\right) \neq 0$, $V + V^T > 0$ *satisfy conditions*

$$CV_R = A_R C, \tag{132}$$

$$\left\{\left(4B - A^2\right)A^{(n)}\left(CVC^{-1} - A\right)\right\} = 0,\ n = 0, 1, \ldots, m^2 - 1, \tag{133}$$

$$V^2 - \left(V^2\right)^T + C^{-1}\left(4B - A^2\right)C - C^T\left(4B - A^2\right)^T\left(C^{-1}\right)^T = 0. \tag{134}$$

If the symmetric matrix

$$W = \frac{1}{4}V^2 + C^{-1}\left(B - \frac{1}{4}A^2\right)C \tag{135}$$

is positive definite then the null solution of system (123) *is asymptotically stable (in the sense of Lyapunov) and if matrix* (135) *is nonsingular and has at least*

one negative eigenvalue then the null solution of system (123) *is unstable* (*in the sense of Lyapunov*).

It is easy to see that if $P = 0$ then conditions (132) − (134) can be satisfied once we take $V = A$, $C = E$. In this case we would have that $W = \Pi$. This means that Theorem 8 can be considered as a generalization of the 3-rd and 4-th theorems of Kelvin − Tait − Chetayev (see [17]).

It is not hard to verify that the conditions of the Mingori's [3] and Müller's [4] theorems implies conditions (132) − (134). However, the following example shows that the converse of above proposition is not correct in general.

Example 1. Assume that

$$A = diag\,[A_1, A_2]\,, \quad B = \begin{bmatrix} B_1 & b_5 E^{(2)} \\ b_5 E^{(2)} & B_2 \end{bmatrix},$$

$$A_1 = T \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix} T^{-1}, \; A_2 = T \begin{bmatrix} a_3 & a_4 \\ -a_4 & a_3 \end{bmatrix} T^{-1},$$

$$B_1 = T \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix} T^{-1}, \; B_2 = T \begin{bmatrix} b_3 & \frac{a_3 b_2}{a_1} \\ -\frac{a_3 b_2}{a_1} & b_3 \end{bmatrix} T^{-1}, \tag{136}$$

$$J = diag\,[1,1,1,1]\,, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then in terms of matrix coefficients of equation (123) we have

$$D = diag\,[D_1, D_2]\,, \quad G = diag\,[G_1, G_2]\,,$$

$$D_1 = \begin{bmatrix} a_1 - a_2 & \frac{a_2}{2} \\ \frac{a_2}{2} & a_2 + a_1 \end{bmatrix}, \; D_2 = \begin{bmatrix} a_3 - a_4 & \frac{a_4}{2} \\ \frac{a_4}{2} & a_4 + a_3 \end{bmatrix},$$

$$G_1 = -\frac{3a_2}{2} S^{(2)}, \quad G_2 = -\frac{3a_4}{2} S^{(2)}, \quad S^{(2)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{137}$$

$$\Pi = \begin{bmatrix} \Pi_1 & b_5 E^{(2)} \\ b_5 E^{(2)} & \Pi_2 \end{bmatrix}, \quad P = -diag\left[\frac{3b_2}{2} S^{(2)}, \frac{3a_3 b_2}{2a_1} S^{(2)}\right],$$

$$\Pi_1 = \begin{bmatrix} b_1 - b_2 & \frac{b_2}{2} \\ \frac{b_2}{2} & b_2 + b_1 \end{bmatrix}, \; \Pi_2 = \begin{bmatrix} \frac{b_3 a_1 - b_2 a_3}{a_1} & \frac{a_3 b_2}{2a_1} \\ \frac{a_3 b_2}{2a_1} & \frac{a_3 b_2 + b_3 a_1}{a_1} \end{bmatrix}.$$

Both, the Mingori's [3] and Müller's [4] theorems demand the commutativity of the matrices P and D. [10] However, it is easy to verify that for the matrices P and D (137) this condition is not fulfilled in general. Thus, we can't use the results of the mentioned theorems for the stability investigation of system (123), (137). On the other hand, the matrices

$$V = diag\,[V_1, V_2]\,, \quad C = diag\,[T, T]\,,$$

$$V_1 = a_1 E^{(2)} - \left(a_2 - \frac{2b_2}{a_1}\right) S^{(2)}, \quad V_2 = a_3 E^{(2)} - \left(a_4 - \frac{2b_2}{a_1}\right) S^{(2)}, \tag{138}$$

---

[10] The same is true for the results of papers [5, 6, 7, 8, 9]

satisfy conditions (125) – (127) of Theorem 8, according to which the matrix W (135) can be expressed in the form of

$$W = \left[\begin{array}{cc} w_1 E^{(2)} & b_5 E^{(2)} \\ b_5 E^{(2)} & w_2 E^{(2)} \end{array} \right],$$

$$w_1 = \frac{a_1 a_2 b_2 - b_2^2 + a_1^2 b_1}{a_1^2}, \quad w_2 = \frac{a_1 a_4 b_2 - b_2^2 + a_1^2 b_3}{a_1^2}. \tag{139}$$

The conditions of Sylvester's criterion (see [18, c. 99]), when applied to the matrix W (139), lead us to the inequalities

$$w_1 > 0, \quad w_2 > 0. \tag{140}$$

Inequalities (140) together with the conditions $a_1 > 0, a_3 > 0$ (which provide that the matrix V (138) is positive definite) describe the region of the asymptotical stability of the null solution of system (123), (137).

5. ON THE INTERCONNECTION BETWEEN THE NOTIONS OF THE L_k-EQUIVALENCE AND THE EQUIVALENCE IN THE SENSE OF LYAPUNOV

It is well known that systems (6) and (9) can be rewritten in the form of

$$\frac{d}{dt} \left[\begin{array}{c} \mathbf{x} \\ \dot{\mathbf{x}} \end{array} \right] = \mathbf{A}^* (t) \left[\begin{array}{c} \mathbf{x} \\ \dot{\mathbf{x}} \end{array} \right], \quad \mathbf{A}^* (t) = \left[\begin{array}{cc} O & E \\ -B(t) & -A(t) \end{array} \right], \tag{141}$$

and

$$\frac{d}{dt} \left[\begin{array}{c} \xi \\ \dot{\xi} \end{array} \right] = \mathbf{V}^* (t) \left[\begin{array}{c} \xi \\ \dot{\xi} \end{array} \right], \quad \mathbf{V}^* (t) = \left[\begin{array}{cc} O & E \\ -W(t) & -V(t) \end{array} \right], \tag{142}$$

respectively. Suppose that systems (141) and (142) are connected by the transformation

$$\left[\begin{array}{c} \mathbf{x} \\ \dot{\mathbf{x}} \end{array} \right] = \mathbb{L}(t) \left[\begin{array}{c} \xi \\ \dot{\xi} \end{array} \right], \quad \mathbb{L}(t) = \left[\begin{array}{cc} L_{11}(t) & L_{12}(t) \\ L_{21}(t) & L_{22}(t) \end{array} \right], t \in [t_0, +\infty), \tag{143}$$

$L_{ij}(t) \in M_m \left(C^1 [t_0, +\infty] \right)$. It is not hard to verify that this would be the case if and only if the equalities

$$\dot{L}_{11}(t) - L_{12}(t) W(t) - L_{21}(t) = 0,$$

$$\dot{L}_{12}(t) - L_{12}(t) V(t) + L_{11}(t) - L_{22}(t) = 0, \tag{144}$$

$$B(t) L_{11}(t) + A(t) L_{21}(t) = L_{22}(t) W(t) - \dot{L}_{21}(t),$$

$$B(t) L_{12}(t) + A(t) L_{22}(t) = -L_{21}(t) + L_{22}(t) V(t) - \dot{L}_{22}(t) \tag{145}$$

hold true $\forall t \in [t_0, +\infty)$.

In accordance with the definition of the equivalence in the sense of Lyapunov of two systems of first-order ODEs that was given in [1, p. 118] we can introduce the same notion for the case of second-order systems.

Definition 6. We say that the systems of second-order ODEs (6) and (9) are equivalent in the sense of Lyapunov if there exists a Lyapunov matrix (see definition in [1, p. 117]) $\mathbb{L}(t)$ (143) satisfying conditions (144), (145).

Let us assume that the matrix $\mathbb{L}(t)$ (143) satisfies conditions

$$L_{12}(t) = O, \quad L_{11}(t) = L(t) \in M_m\left(C^2\left[t_0, +\infty\right)\right),$$

$$\inf_{t \in [t_0, +\infty]} |\det(L(t))| > 0, \quad \sup_{t \in [t_0, +\infty)} \left\| \frac{d^k}{dt^k} L(t) \right\| < +\infty, \ \forall k \in \overline{0, 2}.$$

Then from equalities (144) we immediately obtain that $L_{21}(t) = \dot{L}_{11}(t)$, $L_{22}(t) = L(t)$; transformation (143) reduces to the form

$$\left[\begin{array}{c} \mathbf{x} \\ \dot{\mathbf{x}} \end{array} \right] = \left[\begin{array}{cc} L(t) & O \\ \dot{L}(t) & L(t) \end{array} \right] \left[\begin{array}{c} \xi \\ \dot{\xi} \end{array} \right] \tag{146}$$

and represents a Lyapunov transformation (see definition in [1, p. 117]); conditions (145) reduce to conditions (10), i.e.,

$$V(t) = L^{-1}(t)\left(2\dot{L}(t) + A(t)L(t)\right),$$

$$W(t) = L^{-1}(t)\left(\ddot{L}(t) + A(t)\dot{L}(t) + B(t)L(t)\right). \tag{147}$$

Thus, we can conclude that if systems (6) and (9) are L_2-equivalent according to Definition 4 then they are equivalent in the sense of Lyapunov according to Definition 6. However, it is almost obvious that the converse of above proposition is not correct in general. It is easy to see that the notion of the equivalence in the sense of Lyapunov includes the notions of the L_2-equivalence (see Definition 4) and the L-equivalence (see Definition 5) as partial cases. Therefore, when we consider the possibility of using structural transformations to aid the investigation of stability of the null solution of system (6), we inevitably arrive at the following *general problems of symmetrization*:

1. *for the given system (6), find a Lyapunov matrix $\mathbb{L}(t)$ (143) and matrices $V(t), W(t) \in M_m(C[t_0, +\infty))$ which satisfy the symmetry conditions $V(t) = V^T(t)$ and/or $W(t) = W^T(t)$ together with equalities (144), (145) $\forall t \in [t_0, +\infty)$;*

2. *for the given autonomous system (6), i.e, $A(t) = A \in M_m(\mathbb{R})$, $B(t) = B \in M_m(\mathbb{R})$, find a Lyapunov matrix $\mathbb{L}(t)$ (143) and matrices $V(t) = V \in M_m(\mathbb{R})$, $W(t) = W \in M_m(\mathbb{R})$ which satisfy the symmetry conditions $V = V^T$ and/or $W = W^T$ together with equalities (144), (145) $\forall t \in [0, +\infty)$.*

In the case when systems (6) and (9) are autonomous, i.e., $\mathbf{A}^*(t) = \mathbf{A}^* \in M_{2m}(\mathbb{R})$, $\mathbf{V}^*(t) = \mathbf{V}^* \in M_{2m}(\mathbb{R})$, the necessary and sufficient conditions providing that they are equivalent in the sense of Lyapunov were found by Erugin (see the Erugin's theorem in [15, p. 145]): *Two systems (141) and (142) ($\mathbf{A}^*$ and $\mathbf{V}^*$ are constant matrices of the same order) are equivalent in the sense of Lyapunov if and only if the matrices $\mathbf{A}^*$ and $\mathbf{V}^*$ have a single, similar real part of the spectrum or, in other words, there exists a nonsingular matrix $\mathbf{C} \in M_{2m}(\mathbb{R})$, satisfying equality*

$$\mathbf{A}_R^* = \mathbf{C}\mathbf{V}_R^*\mathbf{C}^{-1}.$$

Evidently, in general case, to check whether the conditions of the Erugin's theorem are fulfilled could be as difficult as to solve both systems (141) and (142) directly. However, Theorems 2 – 4, 6 indicate that in some cases the question about equivalence in the sense of Lyapunov of two systems (141) and (142) can be answered without necessity to solve them.

Let as suppose that $V(t), W(t) \in M_m\left(C^1\left[t_0, +\infty\right)\right)$. Then substituting the expressions for matrices $L_{21}(t)$ and $L_{22}(t)$ obtained from equations (144) into equation (145), we get the following system of second-order matrix differential equations with respect to the unknown matrices $L_{11}(t), L_{12}(t)$:

$$
\begin{aligned}
\frac{d}{dt}&\left(\dot{Z}(t) + Z(t)\mathbf{V}^*(t)\right) + \left(\dot{Z}(t) + Z(t)\mathbf{V}^*(t)\right)\mathbf{V}^*(t) + \\
&+ B(t)Z(t) + + A(t)\left(\dot{Z}(t) + Z(t)\mathbf{V}^*(t)\right) = 0, \\
&Z(t) = [L_{11}(t), L_{12}(t)].
\end{aligned}
\tag{148}
$$

Thus, we arrive at the conclusion that systems (6) and (9) are equivalent in the sense of Lyapunov if and only if system (148) possesses a solution $Z(t)$ satisfying conditions

$$
\sup_{t \in [t_0, +\infty)} \left\|\frac{d^k}{dt^k}\mathbb{L}(t)\right\| < +\infty, \ k = 0, 1, \quad \inf_{t \in [t_0, +\infty)}|\det(\mathbb{L}(t))| > 0,
$$
$$
L_{21}(t) = \dot{L}_{11}(t) - L_{12}(t)W(t),
\tag{149}
$$

$$
L_{22}(t) = \dot{L}_{12}(t) - L_{12}(t)V(t) + L_{11}(t).
$$

The *general problems of symmetrization* (GPS) stated above have not been studied in this paper. However, on our opinion, the problem of finding necessary and sufficient conditions for solvability of the GPS can be interesting from both practical and theoretical points of view. This problem is significantly more complicated then the problem of finding necessary and sufficient conditions for solvability of the EGS and/or ENPS problems (see definitions on pp. 40 and 46). The main reason for that is the significant complexity of conditions (148), (149) for finding the matrices $\mathbb{L}(t)$, $V(t)$ and $W(t)$. On the other hand, as it was mentioned above, in some cases to solve the GPS for the given system the one should be able to determine the Jordan canonical form of the system's matrix (see the conditions of the Erugin's theorem). Evidently, in this case the using of structural transformations can't facilitate the stability investigation of the null solution of the system.

6. APPLICATION OF THE STRUCTURAL TRANSFORMATIONS TO THE STABILITY INVESTIGATION OF DYNAMICAL SYSTEMS

The stability of rotary motion of a rigid body suspended on a string. Let us consider the symmetrization problem for the system of second-order differential equations describing the perturbed motion of a heavy, symmetric rigid body suspended to the stationary point O by the inextensible weightless string. We assume that the string is attached to the body at the point S lying on the body's symmetry axis. We denote the distance between

point S and the center of mass of the body by a, and the length of the string by b. It is known that the rotary motion of the body can be approximately described (assuming that the displacement and rotations are small enough) by the following equations (see equations (2.8) and (2.9) from [19]):

$$\begin{cases} J_1\ddot{x}_1 + \lambda\dot{x}_1 + cx_1 - (2J_1 - J_3)\omega\dot{x}_2 + (\lambda_1 - \lambda)\omega x_2 + mgax_3 = 0, \\ J_1\ddot{x}_2 + \lambda\dot{x}_2 + cx_2 + (2J_1 - J_3)\omega\dot{x}_1 - (\lambda_1 - \lambda)\omega x_1 + mgax_4 = 0, \\ mb^2\ddot{x}_3 + mb(g - b\omega^2)x_3 - 2mb^2\omega\dot{x}_4 + mgax_1 = 0, \\ mb^2\ddot{x}_4 + mb(g - b\omega^2)x_4 + 2mb^2\omega\dot{x}_3 + mgax_2 = 0, \end{cases} \quad (150)$$

where $c = mga\left(\varepsilon + 1\right) + (J_3 - J_1)\omega^2$, $a = b\varepsilon$, $\lambda = fD_1$, $\lambda_1 = fD_3$. In equations (150) by $\omega > 0$ we denote the angular velocity of rotation of the body, by m — the mass of the body, by g — the free fall acceleration, and by $J^* = diag\left[J_1, J_1, J_3\right]$ — the central tensor of inertia of the body. The authors of [19] assume that the body is effected by the dissipative moment $M_d = -fD\omega$, where $D = diag\left[D_1, D_1, D_3\right]$, $D_1 > 0, D_3 > 0, f > 0$. Additionally, we assume that $2J_1 - J_3 \neq 0$, $J_1 > 0$.

It is easy to see that system (150) can be represented in the form of (123) with

$$J = diag\left[J_1\,E^{(2)}, mb^2\,E^{(2)}\right], \quad D = diag\left[\lambda\,E^{(2)}, O^{(2)}\right],$$

$$G = diag\left[(2J_1 - J_3)\omega\,S^{(2)}, 2mb^2\omega\,S^{(2)}\right],$$

$$\Pi = \begin{bmatrix} c\,E^{(2)} & mga\,E^{(2)} \\ mga\,E^{(2)} & mb(g - b\omega^2)\,E^{(2)} \end{bmatrix},$$

$$P = diag\left[-(\lambda_1 - \lambda)\omega\,S^{(2)}, O^{(2)}\right],$$

$$E^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S^{(2)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $O^{(2)}$ denotes the square zero matrix of order 2. Furthermore, using notation (124) we get

$$A = J^{-\frac{1}{2}}\left(D + G\right)J^{-\frac{1}{2}} = D_1 + G_1 =$$

$$= diag\left[\frac{1}{J_1}\begin{bmatrix} \lambda & -(2J_1 - J_3)\omega \\ (2J_1 - J_3)\omega & \lambda \end{bmatrix}, 2\omega\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right], \quad (151)$$

$$B = J^{-\frac{1}{2}}\left(P + \Pi\right)J^{-\frac{1}{2}} = P_1 + \Pi_1 =$$

$$= \begin{bmatrix} cJ_1^{-1}E^{(2)} - J_1^{-1}(\lambda_1 - \lambda)\omega\,S^{(2)} & \frac{ga\sqrt{m}}{b\sqrt{J_1}}E^{(2)} \\ \frac{ga\sqrt{m}}{b\sqrt{J_1}}E^{(2)} & \left(gb^{-1} - \omega^2\right)E^{(2)} \end{bmatrix}. \quad (152)$$

Let us find the sufficient conditions in terms of the parameters of system (150) which provide that the system is equivalent to some other system that does not contain the gyroscopic structures and (or) non-conservative positional structures.

The elimination of the gyroscopic structures. It is easy to verify that the spectrum of matrix $Z_0 = \left(4B - A^2\right)$ (151), (152) is simple. Therefore, according to Corollary 9 the conditions of Corollary 6 are necessary and sufficient simultaneously. Let us check whether the conditions of Corollary 6 are fulfilled. The matrix CVC^{-1} satisfying condition (125) can be expressed in the form of

$$CVC^{-1} = diag \left[\begin{bmatrix} v_{11} & v_{12} \\ -v_{12} & v_{11} \end{bmatrix}, \begin{bmatrix} v_{33} & v_{34} \\ -v_{34} & v_{33} \end{bmatrix} \right]. \tag{153}$$

Taking into account representation (153) and the fact that matrix V is real we arrive at the conclusion that condition (127) can be satisfied if and only if

$$v_{11} = \frac{\lambda}{J_1}, \; v_{33} = 0. \tag{154}$$

Taking into account (153) and (154), from condition (126) we can find that

$$CVC^{-1} = diag \left[\begin{bmatrix} \lambda J_1^{-1} & v_{12} + \omega J_3 J_1^{-1} \\ -v_{12} - \omega J_3 J_1^{-1} & \lambda J_1^{-1} \end{bmatrix}, \begin{bmatrix} 0 & v_{12} \\ -v_{12} & 0 \end{bmatrix} \right]. \tag{155}$$

From formula (155) it follows that the matrix V is a symmetric matrix if and only if

$$J_3 = 0, \; v_{12} = 0. \tag{156}$$

Thus, the gyroscopic structures can be excluded from system (150) if and only if $J_3 = 0$.

We can assume that the condition $J_3 = 0$ is satisfied if the value of the inertia moment J_3 is fairly small in comparison with the value of $2J_1$. This can be the case when the body is heavy and has a shape of a cylinder with a very small transverse section.

Following to the Sommerfeld- Greenhill concept we can set $\lambda = \mu J_1, \quad \lambda_1 = \mu J_3$, where μ is a small constant coefficient depending on the environment characteristics. Returning to the case of a heavy cylinder with a very small transverse section, we can assume that $\lambda_1 = 0$.

The elimination of the non-conservative positional structures. As it was shown above, conditions (125) – (127) led us to representation (155). Let us take $C = E^{(4)}$. To satisfy condition (130) we take $v_{12} = -\frac{2\omega\lambda_1}{\lambda}$ and according to Corollary 7, whose conditions are fulfilled, obtain the matrix coefficients of equation (128)

$$V = diag \left[V^{(1)}, V^{(2)} \right],$$

$$V^{(1)} = \frac{1}{\lambda J_1} \begin{bmatrix} \lambda^2 & J_3\omega\lambda - 2\omega\lambda_1 J_1 \\ -J_3\omega\lambda + 2\omega\lambda_1 J_1 & \lambda^2 \end{bmatrix},$$

$$V^{(2)} = \frac{2\omega\lambda_1}{\lambda} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{157}$$

$$W = \begin{bmatrix} \left(\omega^2\frac{\lambda_1}{\lambda}\left(\frac{J_3}{J_1} - \frac{\lambda_1}{\lambda}\right) + \frac{mga}{J_1}(\varepsilon + 1)\right) E^{(2)} & \frac{\sqrt{mg}\varepsilon}{\sqrt{J_1}} E^{(2)} \\ \frac{\sqrt{mg}\varepsilon}{\sqrt{J_1}} E^{(2)} & \left(\frac{g}{b} - \frac{\lambda_1^2\omega^2}{\lambda^2}\right) E^{(2)} \end{bmatrix}.$$

We see that, according to Corollary 7, the elimination of the non-conservative positional structures is possible without any additional restrictions on the parameters of system (150).

Once the non-conservative positional structures are eliminated, we can try to find the region of the asymptotic stability of the null solution of system (150). Since systems (150) and (128), (157) are L-equivalent, that is, equivalent in the sense of Lyapunov, their regions of the asymptotic stability coincide. Let us find the region of the asymptotic stability of system (128), (157).

Unfortunately, the matrix V (157) is not a positive definite matrix, that is, the conditions of Theorem 8 are not fulfilled. However, this problem can be overcame. First of all let us emphasize the fact that if the parameters of system (150) are chosen in such a way that $\det(W) = 0$ then the null solution of system (150) is unstable. Thus, we can assume that the matrix W is nonsingular.

It is easy to verify that if

$$W > 0 \tag{158}$$

then function $V(\xi) = \dot{\xi}^T \dot{\xi} + \xi^T W \xi$ where $\xi = \xi(t)$ represents an arbitrary solution of system (128), (157), satisfies the conditions of the Krasovsky theorem on asymptotic stability (see, for example, [17, p. 42]). On the other hand if the symmetric matrix W (157) is nonsingular and has at least one negative eigenvalue then the function $-V(\xi)$ satisfies the conditions of the Krasovsky theorem on instability (see, for example, [17, p. 51]). Thus, we can conclude that condition (158) describes the required region of asymptotic stability.

The conditions of Sylvester's criterion, when applied to the matrix W (157), lead us to the following system of inequalities:

$$P > 0, \quad PS - R^2 > 0, \tag{159}$$

where

$$P = \left(\omega^2 \frac{\lambda_1}{\lambda} \left(\frac{J_3}{J_1} - \frac{\lambda_1}{\lambda} \right) + \frac{mga}{J_1}(\varepsilon + 1) \right), \quad S = \left(\frac{g}{b} - \frac{\lambda_1^2 \omega^2}{\lambda^2} \right), \ R = \frac{\sqrt{m} g \varepsilon}{\sqrt{J_1}}.$$

Returning to the case of a heavy cylinder with a very small transverse section and setting $\lambda_1 = 0$ we see that

$$P = \frac{mga}{J_1} \left(1 + \frac{a}{b} \right), \quad S = \frac{g}{b}, \quad R = \frac{ga\sqrt{m}}{b\sqrt{J_1}}.$$

Therefore, the first inequality of (159) is fulfilled and the second one reduces to the form

$$\frac{mg^2}{J_1} \frac{a}{b} > 0.$$

It is worth to emphasize that conditions (159) are in good agreement with the similar conditions obtained in [9]. On the other hand, a sophisticated method proposed in paper [19] for the stability investigation of the null solution of system (150) results in a set of inequalities which do not describe the region of asymptotic stability of the system (contrary to the expectations of the authors of paper [19]). The reason for that is an essential error introduced in [19] by the authors.

7. CONCLUSIONS

In the present paper we have extended and generalized the results of a series of papers devoted to the stability investigation of the null solution of systems of second-order ODEs via the stability preserving structural transformations. The series was started with D.L. Mingori [3] and then continued by Von P. C. Müller [4], V.N. Koshlyakov [8], V.N. Koshlyakov and V.L. Makarov [20, 5, 6, 7], V.N. Koshlyakov and V.A. Storozhenko [9].

We have found the necessary and sufficient conditions providing that a given autonomous (non-autonomous) system of second-order ODEs is equivalent in the sense of Lyapunov to some autonomous (non-autonomous) system of second-order ODEs which does not contain gyroscopic and/or non-conservative positional structures (see Theorems 2, 3, 4, 6).

Particularly, using Theorem 6 we managed to generalize the results of papers [3, 4, 5, 6, 7, 8, 9] which are related to the ENPS problem for the autonomous system (123). The results of the mentioned papers are applicable only when the matrix D commutate with P whereas in the present paper we got rid of this unnecessary constraint (see Example 1).

Theorem 8 proved in the paper generalizes the 3-rd and 4-th Kelvin – Tait – Chetayev theorems as well as the Mingori's [3] and Müller's [4] theorems.

In Section 5 we stated and briefly discussed the *general problems of symmetrization* (GPS). It was shown that the EGS and ENPS problems can be considered as particular cases of the GPS. The solution of the latter problems can provide us with essentially more powerful tools for the stability investigation of systems of ODEs than those obtained in the present paper. Due to the significant complexity of the GPS we left them for the subsequent publications.

In Section 6 it was shown that the theoretical results presented in the paper can be successfully applied to the stability investigation of real mechanical systems.

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