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THE BOUNDARY EFFECT IN THE ERROR ESTIMATE OF THE FINITE-DIFFERENCE SCHEME FOR THE TWO-DIMENSIONAL HEAT EQUATION

Nataliya Mayko

Резюме. У роботі досліджено точність традиційної скінченно-різницевої схеми для початково-крайової задачі для рівняння теплопровідності з умовами Діріхле і Неймана на межі одиничного квадрата. Доведено апріорну оцінку з вагою для швидкості збіжності методу. З одержаної нерівності випливає, що точність схеми збільшується поблизу тих бічних граней просторово-часового паралелепіпеда, де задана крайова умова Діріхле.

Abstract. The accuracy of the conventional finite-difference scheme for the initial-boundary value problem for the parabolic equation in a unit square with Dirichlet's and Neumann's boundary value conditions is considered. The error estimate with the weight function is proved. This inequality shows that the accuracy order is higher near the three side faces of the space-time parallelepiped where the Dirichlet boundary condition is satisfied precisely than that is far from them.

1. Introduction

The grid method is widely used for solving numerically many problems of mathematical physics, and the theory of the method is profoundly developed (see [1]). It is obvious that the Dirichlet boundary condition is satisfied exactly and may therefore influence the order of the error estimate: the accuracy of the difference scheme is likely to be higher near the boundary of the domain than it is in the middle of that. Such supposition turned justified and were quantitatively estimated in [2] where the initial-boundary value problem with the Dirichlet boundary condition is investigated and the error estimate of the usual finite-difference scheme is proved. These ideas were further developed in [3] where the one- and two-dimensional heat equations are considered.

In the present paper we study the effect of the mixed boundary conditions when the Neumann boundary condition is given on the left side of the unit square and the Dirichlet one is on its three other sides.

We consider the problem

Key words. Heat equation, mixed boundary condition, grid method, weighted error estimate.

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + f(x,t), \quad (x,t) \in Q_T = Q \times (0,T),\\
&\frac{\partial u(x,t)}{\partial x_1} = 0, \quad (x,t) \in \Gamma_{-1} \times (0,T),\\
&u(x,t) = 0, \quad (x,t) \in (\Gamma \backslash \Gamma_{-1}) \times (0,T),\\
&u(x,0) = \varphi(x), \quad x \in Q,
\end{aligned} \tag{1}
$$

where $x = (x_1, x_2)$, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, $Q = \{x = (x_1, x_2) : \ 0 < x_\alpha < 1, \ \alpha = 1, 2\}$ is a unit square, $\Gamma = \partial Q$ is the boundary of Q, $\Gamma_{-1} = \{x = (x_1, x_2) : x_1 = 0, \ 0 < x_2 < 1\}$ is the left side of Q.

Denoting the grid sets

$$
\omega_\alpha = \{x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, \ i_\alpha = \overline{1, \ N_\alpha - 1}, \ h_\alpha = 1/N_\alpha, \ N_\alpha \geqslant 2,
$$

$$
N_\alpha \text{ is an integer number}\},
$$

$$
\omega_\alpha^- = \omega_\alpha \cup \{0\}, \quad \omega_\alpha^+ = \omega_\alpha \cup \{1\}, \quad \bar{\omega}_\alpha = \omega_\alpha \cup \{0\} \cup \{1\},
$$

$$
\omega = \omega_1 \times \omega_2, \quad \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2, \quad \gamma = \bar{\omega} \backslash \omega,
$$

$$
\gamma_{-\alpha} = \{x_\alpha = 0, \ x_{3-\alpha} \in \omega_{3-\alpha}\}, \quad \gamma_{+\alpha} = \{x_\alpha = 1, \ x_{3-\alpha} \in \omega_{3-\alpha}\}, \quad \alpha = 1, 2,
$$

$$
\omega_\tau = \{t_j = j\tau, \ \ j = \overline{1, \ M - 1}, \ \ \tau = T/M, \ \ M \geqslant 2, \ M \text{ is an integer number}\},
$$

$$
\omega_{Q_T} = (\omega \cup \gamma_{-1}) \times \omega_\tau.
$$

and making use of the operators

$$
(T_2 v)(x_1, x_2) = \frac{1}{h_2^2} \int_{x_2 - h_2}^{x_2 + h_2} \big(h_2 - |x_2 - \xi_2|\big) v(x_1, \xi_2)\, d\xi_2\,, \quad x \in \omega \cup \gamma_{-1},
$$

$$
(T_1 v)(x_1, x_2) = \begin{cases} \dfrac{1}{h_1^2} \displaystyle\int_{x_1 - h_1}^{x_1 + h_1} (h_1 - |x_1 - \xi_1|)\, v(\xi_1, x_2) d\xi_1\,, & x \in \omega, \\ \dfrac{2}{h_1^2} \displaystyle\int_0^{h_1} (h_1 - \xi_1)\, v(\xi_1, x_2) d\xi_1\,, & x \in \gamma_{-1}, \end{cases}
$$

we approximate problem (1) with the following finite-difference scheme

$$
\begin{aligned}
&y_{\bar{t}}(x,t) + (Ay)(x,t) = (T_1 T_2 f)(x,t), \quad (x,t) \in \omega_{Q_T},\\
&y(x,t) = 0, \quad (x,t) \in (\gamma \backslash \gamma_{-1}) \times \omega_\tau,\\
&y(x,0) = \varphi(x), \quad x \in \omega,
\end{aligned} \tag{2}
$$

where $A = A_1 + A_2$,

$$
A_1 y = \begin{cases} y_{\bar{x}_1 x_1}, & x \in \omega, \\ 2h_1^{-1} y_{x_1}, & x \in \gamma_{-1}, \end{cases}
$$

$A_2 y = -y_{\bar{x}_2 x_2}, \ \ x \in \omega \cup \gamma_{-1}$.

For the error $z(x,t)=y(x,t)-u(x,t)$ we then have the problem

$$\begin{aligned} z_{\bar t}(x,t)+(Az)(x,t)=\psi(x,t),\quad & (x,t)\in\omega_{Q_T},\\ z(x,t)=0,\quad & (x,t)\in(\gamma\backslash\gamma_{-1})\times\omega_\tau,\\ z(x,0)=0,\quad & x\in\omega, \end{aligned} \tag{3}$$

where $\psi=T_1T_2f-u_{\bar t}-Au=A_1\eta_1+A_2\eta_2+\eta_3$ is the approximation error, $\eta_\alpha(x,t)=u(x,t)-(T_{3-\alpha}u)(x,t)$, $\alpha=1,2$, $\eta_3(x,t)=\dfrac{d(Tu)}{dt}(x,t)-u_{\bar t}(x,t)$, $(x,t)\in(\omega\cup\gamma_{-1})\times\omega_\tau$.

2. The Green's function estimate

Denoting by H_h the space of grid functions defined on $\bar\omega$ and vanishing on $\gamma\backslash\gamma_{-1}$ we introduce the inner product and the associate norm:

$$(y,v)=\sum_{x\in\omega}h_1h_2y(x)v(x)+\frac{h_1}{2}\sum_{x\in\gamma_{-1}}h_2y(x)v(x),$$

$$\|v\|=\sqrt{(v,v)}=\Big(\sum_{x\in\omega}h_1h_2v^2(x)+\frac{h_1}{2}\sum_{x\in\gamma_{-1}}h_2v(x)\Big)^{1/2}.$$

It is well known (see [1], [4]) that the difference operator A is symmetric and positive definite in H_h, and therefore the inverse operator A^{-1} exists. We find

$$\begin{aligned} \|Ay\|^2=(Ay,Ay)=\sum_{x\in\omega}h_1h_2(-y_{\bar x_1x_1}-y_{\bar x_2x_2})^2+\\ +\frac{h_1}{2}\sum_{x\in\gamma_{-1}}h_2\Big(-\frac{2}{h_1}y_{x_1}-y_{\bar x_2x_2}\Big)^2=\\ =\sum_{x\in\omega}h_1h_2(y^2_{\bar x_1x_1}+y^2_{\bar x_2x_2})+\frac{h_1}{2}\sum_{x\in\gamma_{-1}}h_2\Big(\frac{4}{h_1^2}y^2_{x_1}+y^2_{\bar x_2x_2}\Big)+\\ +\sum_{x\in\omega}h_1h_22y_{\bar x_1x_1}y_{\bar x_2x_2}+\frac{h_1}{2}\sum_{x\in\gamma_{-1}}h_22\frac{2}{h_1}y_{x_1}y_{\bar x_2x_2}=\\ =\sum_{x\in\omega}h_1h_2y^2_{\bar x_1x_1}+\frac{h_1}{2}\sum_{x\in\gamma_{-1}}h_2\Big(\frac{2}{h_1}y_{x_1}\Big)^2+\sum_{x\in\omega}h_1h_2y^2_{\bar x_2x_2}+\\ +\frac{h_1}{2}\sum_{x\in\gamma_{-1}}h_2y^2_{\bar x_2x_2}+2\sum_{x\in\omega_1^-\times\omega_2^-}h_1h_2y^2_{x_1x_2}=\|A_1y\|^2+\|A_2y\|^2+2\,\|B_1^*y\|_*^2, \end{aligned} \tag{4}$$

where $B_1^*y=-y_{x_1x_2}$, $x\in\omega_1^-\times\omega_2^-$, is a difference operator acting from H_h into the space H_h^* of the grid functions which are defined on the set of the node points $\omega_1^-\times\omega_2^-$; $(y,v)_*=\sum\limits_{x\in\omega_1^-\times\omega_2^-}h_1h_2y(x)v(x)$ is the inner product and $\|y\|_*=\sqrt{(y,y)_*}=\sum\limits_{x\in\omega_1^-\times\omega_2^-}h_1h_2y^2(x)$ is the corresponding norm in H_h^*. Then we have

$$(B_1^* y, w)_* = -\sum_{x\in\omega_1^-\times\omega_2^-} h_1 h_2 y_{x_1 x_2} w =$$

$$= -\sum_{x\in\omega} h_1 h_2 y w_{\bar{x}_1\bar{x}_2} - \frac{h_1}{2}\sum_{x\in\gamma_{-1}} h_2 y \frac{2}{h_1} w_{\bar{x}_2} = (y, B_1 w),$$

where $B_1 : H_h^* \to H_h$ is a conjugate operator for $B_1^* : H_h \to H_h^*$,

$$B_1 w = -\begin{cases} w_{\bar{x}_1\bar{x}_2}, & x \in \omega, \\ \dfrac{2}{h_1} w_{\bar{x}_2}, & x \in \gamma_{-1}. \end{cases} \tag{5}$$

Similarly to that we can also deduce the relation

$$\|Ay\|^2 = (Ay, Ay) = \sum_{x\in\omega} h_1 h_2 (-y_{\bar{x}_1 x_1} - y_{\bar{x}_2 x_2})^2 +$$

$$+\frac{h_1}{2}\sum_{x\in\gamma_{-1}} h_2 \left(-\frac{2}{h_1} y_{x_1} - y_{\bar{x}_2 x_2}\right)^2 =$$

$$= \sum_{x\in\omega} h_1 h_2 y_{\bar{x}_1 x_1}^2 + \frac{h_1}{2}\sum_{x\in\gamma_{-1}} h_2 \left(\frac{2}{h_1} y_{x_1}\right)^2 + \sum_{x\in\omega} h_1 h_2 y_{\bar{x}_2 x_2}^2 + \tag{6}$$

$$+\frac{h_1}{2}\sum_{x\in\gamma_{-1}} h_2 y_{\bar{x}_2 x_2}^2 + 2\sum_{x\in\omega_1^-\times\omega_2^+} h_1 h_2 y_{x_1\bar{x}_2}^2 = \|A_1 y\|^2 + \|A_2 y\|^2 + 2\,\|B_2^* y\|_*^2,$$

where $B_2^* y = -y_{x_1\bar{x}_2}$, $x \in \omega_1^- \times \omega_2^+$, is a difference operator acting from H_h into the space H_h^* of the grid functions which are defined on the set of the node points $\omega_1^- \times \omega_2^+$; $(y, v)_* = \sum_{x\in\omega_1^-\times\omega_2^+} h_1 h_2 y(x) v(x)$ is the inner product and $\|y\|_* = \sqrt{(y, y)_*} = \sum_{x\in\omega_1^-\times\omega_2^+} h_1 h_2 y^2(x)$ is the corresponding norm in H_h^*. We have

$$(B_2^* y, w)_* = -\sum_{x\in\omega_1^-\times\omega_2^+} h_1 h_2 y_{x_1\bar{x}_2} w =$$

$$= -\sum_{x\in\omega} h_1 h_2 y w_{\bar{x}_1 x_2} - \frac{h_1}{2}\sum_{x\in\gamma_{-1}} h_2 y \frac{2}{h_1} w_{x_2} = (y, B_2 w),$$

where $B_2 : H_h^* \to H_h$ is a conjugate operator for $B_2^* : H_h \to H_h^*$,

$$B_2 w = -\begin{cases} w_{\bar{x}_1 x_2}, & x \in \omega, \\ \dfrac{2}{h_1} w_{x_2}, & x \in \gamma_{-1}. \end{cases} \tag{7}$$

In what follows we need the assertion from [4] (see p. 54 therein).

Lemma 1. *Let: 1) A be a self-conjugate operator acting in the Hilbert space H, 2) B be a linear operator acting from H^* into H ($H^* \supseteq H$), 3) A^{-1} exist, 4) $\|B^* v\|_* \leqslant \gamma\|Av\|$ $\forall v \in H$, where $B^* : H \to H^*$ is a conjugate operator* *for* $B : H^* \to H$, $(y,v)_*$ *is the inner product, and* $\|v\|_* = \sqrt{(v,v)_*}$ *is a corresponding norm in* H^*. *Then*

$$\|A^{-1}Bv\| \leqslant \gamma\|v\|_* \quad \forall v \in H^*.$$

Applying Lemma 1 to the operators A, B_1, B_2 we obtain the estimate ($k = 1,2$)

$$\|A^{-1}B_k v\| \leqslant \frac{1}{\sqrt{2}}\|v\|_* \quad \forall v \in H^*. \tag{8}$$

By means of Green's function $G(x,\xi) = G(x_1,x_2;\xi_1,\xi_2)$ of the difference boundary value problem

$$\begin{gathered}
-G_{\bar{\xi}_1\xi_1}(x,\xi) - G_{\bar{\xi}_2\xi_2}(x,\xi) = \frac{\delta(x_1,\xi_1)\,\delta(x_2,\xi_2)}{h_1h_2}, \quad \xi \in \omega,\\
-\frac{2}{h_1}G_{\xi_1}(x,\xi) = \frac{2}{h_1}\frac{\delta(x_1,\xi_1)\,\delta(x_2,\xi_2)}{h_2}, \quad \xi \in \gamma_{-1},\\
G(x,\xi) = 0, \quad \xi \in \gamma\backslash\gamma_{-1},
\end{gathered} \tag{9}$$

where $\delta(r,s)$ is the Kronecker symbol, we present the solution of problem (3) as follows

$$z(x,t) = \big(G(x,\cdot),\ \psi(\cdot,t) - z_{\bar{t}}(\cdot,t)\big), \quad (x,t) \in \omega_{Q_T} = \big(\omega \cup \gamma_{-1}\big) \times \omega_\tau. \tag{10}$$

In the statement which goes next we obtain the Green's function estimate.

Lemma 2. *The following inequality holds true*

$$\|G(x,\cdot)\| \leqslant \frac{1}{\sqrt{2}}\rho(x),$$

where $\rho(x) = \min\big\{\sqrt{(1-x_1)(1-x_2)}\,,\ \sqrt{(1-x_1)x_2}\ \big\}$.

Proof. We write down problem (8) in a different way:

$$\begin{gathered}
-G_{\bar{\xi}_1\xi_1}(x,\xi) - G_{\bar{\xi}_2\xi_2}(x,\xi) = \big(H(\xi_1-x_1)H(\xi_2-x_2)\big)_{\bar{\xi}_1\bar{\xi}_2}, \quad \xi \in \omega,\\
-\frac{2}{h_1}G_{\xi_1}(x,\xi) = \frac{2}{h_1}\big(H(\xi_1-x_1)H(\xi_2-x_2)\big)_{\bar{\xi}_2}, \quad \xi \in \gamma_{-1},\\
G(x,\xi) = 0, \quad \xi \in \gamma\backslash\gamma_{-1},
\end{gathered}$$

where $H(x) = \begin{cases} 1, & s \geqslant 0,\\ 0, & s < 0, \end{cases}$ is the Heaviside step function. That can be reduced to the operator equation

$$A_\xi G(x,\xi) = -B_{1\xi}\big(H(\xi_1-s_1)H(\xi_2-x_2)\big).$$

Then we have

$$\begin{aligned}\|G(x,\cdot)\| &= \left\|-A_\xi{}^{-1}B_{1\xi}\big(H(\cdot-x_1)H(\cdot-x_2)\big)\right\| \leqslant \\ &\leqslant \frac{1}{\sqrt{2}}\|H(\cdot-x_1)H(\cdot-x_2)\|_* = \\ &= \frac{1}{\sqrt{2}}\left(\sum_{\xi\in\omega_1^-\times\omega_2^-} h_1h_2H^2(\xi_1-x_1)H^2(\xi_2-x_2)\right)^{1/2} = \\ &= \frac{1}{\sqrt{2}}\left(\sum_{\xi_1=0}^{1-h_1} h_1H^2(\xi_1-x_1)\right)^{1/2}\left(\sum_{\xi_2=0}^{1-h_2} h_2H^2(\xi_2-x_2)\right)^{1/2} = \qquad (11)\\ &= \frac{1}{\sqrt{2}}\left(\sum_{\xi_1=x_1}^{1-h_1} h_1\right)^{1/2}\left(\sum_{\xi_2=x_2}^{1-h_2} h_2\right)^{1/2} = \frac{1}{\sqrt{2}}\sqrt{(1-x_1)(1-x_2)}\ .\end{aligned}$$

Similarly we rewrite problem (9) as follows

$$\begin{aligned}-G_{\bar\xi_1\xi_1}(x,\xi)-G_{\bar\xi_2\xi_2}(x,\xi) &= -\big(H(\xi_1-x_1)H(x_2-\xi_2)\big)_{\bar\xi_1\xi_2}, \quad \xi\in\omega,\\ -\frac{2}{h_1}G_{\xi_1}(x,\xi) &= -\frac{2}{h_1}\big(H(\xi_1-x_1)H(x_2-\xi_2)\big)_{\xi_2}, \quad \xi\in\gamma_{-1},\\ G(x,\xi) &= 0, \quad \xi\in\gamma\backslash\gamma_{-1}.\end{aligned}$$

That can be reduced to the operator equation

$$A_\xi G(x,\xi) = B_{2\xi}\big(H(\xi_1-x_1)H(x_2-\xi_2)\big),$$

from where we get the relation

$$\begin{aligned}\|G(x,\cdot)\| &= \left\|A_\xi{}^{-1}B_{2\xi}\big(H(\cdot-x_1)H(x_2-\cdot)\big)\right\| \leqslant \frac{1}{\sqrt{2}}\|H(\cdot-x_1)H(x_2-\cdot)\|_* = \\ &= \frac{1}{\sqrt{2}}\left(\sum_{\xi\in\omega_1^-\times\omega_2^+} h_1h_2H^2(\xi_1-x_1)H^2(x_2-\xi_2)\right)^{1/2} = \\ &= \frac{1}{\sqrt{2}}\left(\sum_{\xi_1=0}^{1-h_1} h_1H^2(\xi_1-x_1)\right)^{1/2}\left(\sum_{\xi_2=h_2}^{1} h_2H^2(x_2-\xi_2)\right)^{1/2} = \qquad (12)\\ &= \frac{1}{\sqrt{2}}\left(\sum_{\xi_1=x_1}^{1-h_1} h_1\right)^{1/2}\left(\sum_{\xi_2=h_2}^{x_2} h_2\right)^{1/2} = \frac{1}{\sqrt{2}}\sqrt{(1-x_1)x_2}\ .\end{aligned}$$

Inequalities (11) and (12) lead to the assertion of the lemma. □

3. The weighted error estimate

From (10), we deduce the relation

$$\begin{aligned}|z(x,t)| &= \left|\big(G(x,\cdot),\ \psi(\cdot,t)-z_{\bar t}(\cdot,t)\big)\right| \leqslant \|G(x,\cdot)\|\cdot\|\psi(\cdot,t)-z_{\bar t}(\cdot,t)\| \leqslant \\ &\leqslant \|G(x,\cdot)\|\left(\|\psi(\cdot,t)\|+\|z_{\bar t}(\cdot,t)\|\right) \leqslant \frac{1}{\sqrt{2}}\rho(x)\big(\|\psi(\cdot,t)\|+\|z_{\bar t}(\cdot,t)\|\big),\end{aligned}$$

which subsequently gives the inequality

$$\frac{|z(x,t)|^2}{\rho^2(x)} \leqslant \frac{1}{2}\big(\|\psi(\cdot,t)\| + \|z_{\bar t}(\cdot,t)\|\big)^2 \leqslant \|\psi(\cdot,t)\|^2 + \|z_{\bar t}(\cdot,t)\|^2.$$

From the above, we obtain the inequation

$$\sum_{\eta=\tau}^{t}\tau\frac{|z(x,\eta)|^2}{\rho^2(x)} \leqslant \sum_{\eta=\tau}^{t}\tau\big(\|\psi(\cdot,\eta)\|^2 + \|z_{\bar t}(\cdot,\eta)\|^2\big) \leqslant 2\sum_{\eta=\tau}^{t}\tau\|\psi(\cdot,\eta)\|^2, \qquad (13)$$

which comes out from the following

$$\|z_{\bar t}(\cdot,t)\|^2 + 2\big(z_{\bar t}(\cdot,t),\,(Az)(\cdot,t)\big) + \|(Az)(\cdot,t)\|^2 = \|\psi(\cdot,t)\|^2,$$

$$2\sum_{\eta=\tau}^{t}\tau\big(z_{\bar\eta}(\cdot,\eta),\,(Az)(\cdot,\eta)\big) =$$

$$= 2\sum_{\eta=\tau}^{t}\tau\Bigg(\sum_{\xi\in\omega} h_1h_2 z_{\bar\eta}(\xi,\eta)\Big(-z_{\bar\xi_1\xi_1}(\xi,\eta) - z_{\bar\xi_2\xi_2}(\xi,\eta)\Big) +$$

$$+\frac{h_1}{2}\sum_{\xi\in\gamma_{-1}} h_2 z_{\bar\eta}(\xi,\eta)\Big(-\frac{2}{h_1}z_{\xi_1}(\xi,\eta) - z_{\bar\xi_2\xi_2}(\xi,\eta)\Big)\Bigg) =$$

$$= \tau\sum_{\xi\in\omega_1^+\times\omega_2} h_1h_2\sum_{\eta=\tau}^{t}\tau z^2_{\bar\xi_1\bar\eta}(\xi,\eta) + \sum_{\xi\in\omega_1^+\times\omega_2} h_1h_2 z^2_{\bar\xi_1}(\xi,t) +$$

$$+\tau\sum_{\xi\in\omega_1\times\omega_2^+} h_1h_2\sum_{\eta=\tau}^{t}\tau z^2_{\bar\xi_2\bar\eta}(\xi,\eta) =$$

$$+\sum_{\xi\in\omega_1\times\omega_2^+} h_1h_2 z^2_{\bar\xi_2}(\xi,t) + \frac{h_1}{2}\tau\sum_{\substack{\xi_2\in\omega_2^+\\(\xi_1=0)}} h_2\sum_{\eta=\tau}^{t}\tau z^2_{\bar\xi_2\eta}(\xi,\eta) +$$

$$+\frac{h_1}{2}\sum_{\substack{\xi_2\in\omega_2^+\\(\xi_1=0)}} h_2 z^2_{\bar\xi_2}(\xi,t) > 0\,.$$

For the approximation error $\psi = T_1T_2f - u_{\bar t} - Au = A_1\eta_1 + A_2\eta_2 + \eta_3$, we have

$$\|\psi(\cdot,t)\|^2 = \|(A_1\eta_1)(\cdot,t) + (A_2\eta_1)(\cdot,t) + \eta_3(\cdot,t)\|^2 \leqslant$$
$$\leqslant 3\left(\|(A_1\eta_1)(\cdot,t)\|^2 + \|(A_2\eta_1)(\cdot,t)\|^2 + \|\eta_3(\cdot,t)\|^2\right),$$

from where (see also (13)) the following inequality comes out

$$\sum_{\eta=\tau}^{t}\tau\frac{|z(x,\eta)|^2}{\rho^2(x)} \leqslant 6\sum_{\eta=\tau}^{t}\tau\left(\|(A_1\eta_1)(\cdot,t)||^2 + \|(A_2\eta_1)(\cdot,t)\|^2 + \|\eta_3(\cdot,t)\|^2\right). \quad (14)$$

Now we investigate the summand $\|(A_1\eta_1)(\cdot,t)\|^2$. For the node $x\in\omega\cup\gamma_{-1}$ we have (see [3])

$$\eta_1(x,t)=(T_2u)(x,t)-u(x,t)=$$
$$=-\frac{1}{\tau h_2^3}\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}}d\xi_2\int\limits_{\eta}^{t}d\eta_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi|\big)d\xi\int\limits_{\xi}^{x_2}\frac{\partial^2u(x_1,\xi_1,\eta_1)}{\partial\eta_1\partial\xi_1}d\xi_1-$$
$$-\frac{1}{\tau h_2^3}\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}}d\xi_2\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi|\big)d\xi\int\limits_{\xi}^{x_2}d\xi_1\int\limits_{\xi_2}^{\xi_1}\frac{\partial^2u(x_1,\xi_3,\eta)}{\partial\xi_3^2}d\xi_3.$$

Bearing in mind the relation $\left(T_1\dfrac{\partial^2u}{\partial x_1^2}\right)(x,t)=u_{\bar{x}_1x_1}(x,t)$, $x\in\omega$, one can get

$$\eta_{1\bar{x}_1x_1}(x,t)=-\frac{1}{h_1^2\tau h_2^3}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_4|\big)d\xi_4\int\limits_{t-\tau}^{t}d\eta\times$$
$$\times\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}}d\xi_2\int\limits_{\eta}^{t}d\eta_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi|\big)d\xi\int\limits_{\xi}^{x_2}\frac{\partial^4u(\xi_4,\xi_1,\eta_1)}{\partial\xi_4^2\partial\eta_1\partial\xi_1}d\xi_1-$$

$$-\frac{1}{h_1^2\tau h_2^3}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_4|\big)d\xi_4\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}}d\xi_2\times$$
$$\times\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi|\big)d\xi\int\limits_{\xi}^{x_2}d\xi_1\int\limits_{\xi_2}^{\xi_1}\frac{\partial^2u(x_1,\xi_3,\eta)}{\partial\xi_3^2}d\xi_3\,.$$

Then

$$\big|\eta_{1\bar{x}_1x_1}(x,t)\big|\leqslant 4\sqrt{\frac{\tau h_2}{h_1}}\left(\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_4\int\limits_{t-\tau}^{t}d\eta_1\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(\xi_4,\xi_1,\eta_1)}{\partial\xi_4^2\partial\eta_1\partial\xi_1}\right)^2d\xi_1\right)^{1/2}+$$
$$+8\sqrt{\frac{h_2^3}{h_1\tau}}\left(\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_4\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(\xi_4,\xi_3,\eta)}{\partial\xi_4^2\partial\xi_3^2}\right)^2d\xi_3\right)^{1/2},\quad x\in\omega.$$

Applying here the inequality $(a+b)^2\leqslant 2(a^2+b^2)$ we receive

$$\left|\eta_{1\bar{x}_1 x_1}(x,t)\right|^2 \leqslant 2\left(16\frac{\tau h_2}{h_1}\int\limits_{x_1-h_1}^{x_1+h_1} d\xi_4 \int\limits_{t-\tau}^{t} d\eta_1 \int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4 u(\xi_4,\xi_1,\eta_1)}{\partial\xi_4^2\partial\eta_1\partial\xi_1}\right)^2 d\xi_1\right.$$
$$\left. +\ 64\frac{h_2^3}{h_1\tau}\int\limits_{x_1-h_1}^{x_1+h_1} d\xi_4 \int\limits_{t-\tau}^{t} d\eta \int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4 u(\xi_4,\xi_3,\eta)}{\partial\xi_4^2\partial\xi_3^2}\right)^2 d\xi_3\right), \quad x\in\omega\,. \tag{15}$$

Next we make use of the relation $\left(T_1\dfrac{\partial^2 u}{\partial x_1^2}\right)(x,t) = \dfrac{2}{h_1}u_{x_1}(x,t)$, $x\in\gamma_{-1}$, and draw the formula

$$\frac{2}{h_1}\eta_{1x_1}(x,t) = -\frac{2}{h_1^2\tau h_2^3}\int\limits_0^{h_1}(h_1-\xi_4)d\xi_4\int\limits_{t-\tau}^{t} d\eta\times$$
$$\times\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}} d\xi_2\int\limits_{\eta}^{t} d\eta_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi|\big)d\xi\int\limits_{\xi}^{x_2}\frac{\partial^4 u(\xi_4,\xi_1,\eta_1)}{\partial\xi_4^2\partial\eta_1\partial\xi_1}d\xi_1 -$$
$$-\ \frac{2}{h_1^2\tau h_2^3}\int\limits_0^{h_1}(h_1-\xi_4)d\xi_4\int\limits_{t-\tau}^{t} d\eta\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}} d\xi_2\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi|\big)d\xi\times$$
$$\times\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi|\big)d\xi\int\limits_{\xi}^{x_2} d\xi_1\int\limits_{\xi_2}^{\xi_1}\frac{\partial^2 u(x_1,\xi_3,\eta)}{\partial\xi_3^2}d\xi_3, \quad x\in\gamma_{-1},$$

which gives the inequality

$$\left|\frac{2}{h_1}\eta_{1x_1}(x,t)\right| \leqslant \frac{4\sqrt{2\tau h_2}}{\sqrt{h_1}}\left(\int\limits_0^{h_1} d\xi_4\int\limits_{t-\tau}^{t} d\eta_1\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4 u(\xi_4,\xi_1,\eta_1)}{\partial\xi_4^2\partial\eta_1\partial\xi_1}\right)^2 d\xi_1\right)^{1/2} +$$
$$+\ \frac{8\sqrt{2h_2^3}}{\sqrt{h_1\tau}}\left(\int\limits_0^{h_1} d\xi_4\int\limits_{t-\tau}^{t} d\eta\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4 u(\xi_4,\xi_3,\eta)}{\partial\xi_4^2\partial\xi_3^2}\right)^2 d\xi_3\right)^{1/2}, \quad x\in\gamma_{-1},$$

from where we obtain

$$\left|\frac{2}{h_1}\eta_{1x_1}(x,t)\right|^2 \leqslant 2\left(32\frac{\tau h_2}{h_1}\int\limits_0^{h_1} d\xi_4\int\limits_{t-\tau}^{t} d\eta_1\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4 u(\xi_4,\xi_1,\eta_1)}{\partial\xi_4^2\partial\eta_1\partial\xi_1}\right)^2 d\xi_1+\right.$$
$$\left. +\ 128\frac{h_2^3}{h_1\tau}\int\limits_0^{h_1} d\xi_4\int\limits_{t-\tau}^{t} d\eta\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4 u(\xi_4,\xi_3,\eta)}{\partial\xi_4^2\partial\xi_3^2}\right)^2 d\xi_3\right), \quad x\in\gamma_{-1}. \tag{16}$$

Thus we come to the estimate (see (15) and (16))

$$\sum_{\eta=\tau}^{t}\tau\|(A_1\eta_1)(\cdot,\eta)\|^2\leqslant$$

$$\leqslant 128\Biggl(\tau^2h_2^2\int\limits_0^t d\eta\iint\limits_D\left(\frac{\partial^4u(\xi_1,\xi_2,\eta)}{\partial\xi_1^2\partial\xi_2\partial\eta}\right)^2d\xi_1d\xi_2 \tag{17}$$

$$+\ 4h_2^4\int\limits_0^t d\eta\iint\limits_D\left(\frac{\partial^4u(\xi_1,\xi_2,\eta)}{\partial\xi_1^2\partial\xi_2^2}\right)^2d\xi_1d\xi_2\Biggr).$$

Now we look closely at the summand $\|(A_2\eta_2)(\cdot,t)\|^2$ in (14). Acting as before (see (15)) for the node $x\in\omega$ one can establish the relation

$$\bigl|\eta_{2\bar{x}_2x_2}(x,t)\bigr|^2\leqslant 2\Biggl(16\frac{\tau h_1}{h_2}\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(\xi_1,\xi_2,\eta)}{\partial\xi_2^2\partial\eta\partial\xi_1}\right)^2d\xi_2$$

$$+\ 64\frac{h_1^3}{h_2\tau}\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(\xi_1,\xi_2,\eta)}{\partial\xi_1^2\partial\xi_2^2}\right)^2d\xi_2\Biggr),\quad x\in\omega. \tag{18}$$

For the node $x\in\gamma_{-1}$ we have

$$\eta_2(x,t)=(T_1u)(x,t)-u(x,t)=$$

$$=\ \frac{2}{h_1^2}\int\limits_0^{h_1}(h_1-\xi)\bigl(u(\xi,x_2,t)-u(0,x_2,t)\bigr)d\xi=$$

$$=\frac{2}{h_1^2}\int\limits_0^{h_1}(h_1-\xi)d\xi\int\limits_0^{\xi}\frac{\partial u(\xi_1,x_2,t)}{\partial\xi_1}d\xi_1=$$

$$=\frac{2}{\tau h_1^3}\int\limits_0^{h_1}(h_1-\xi)d\xi\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}\left(\frac{\partial u(\xi_1,x_2,t)}{\partial\xi_1}-\frac{\partial u(\xi_2,x_2,\eta)}{\partial\xi_2}\right)d\xi_2+$$

$$+\ \frac{2}{\tau h_1^3}\int\limits_0^{h_1}(h_1-\xi)d\xi\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}\frac{\partial u(\xi_2,x_2,\eta)}{\partial\xi_2}d\xi_2\ =$$

$$=\ \frac{2}{\tau h_1^3}\int\limits_0^{h_1}(h_1-\xi)d\xi\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}d\xi_2\int\limits_{\eta}^{t}\frac{\partial^2u(\xi_1,x_2,\eta_1)}{\partial\eta_1\partial\xi_1}d\eta_1+$$

$$+\frac{2}{\tau h_1^3}\int\limits_0^{h_1}(h_1-\xi)d\xi\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}d\xi_2\int\limits_{\xi_2}^{\xi_1}\frac{\partial^2u(\xi_3,x_2,\eta)}{\partial\xi_3^2}d\eta_1+$$

$$+\frac{2}{\tau h_1^3}\int\limits_0^{h_1}(h_1-\xi)d\xi\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}\frac{\partial u(\xi_2,x_2,\eta)}{\partial\xi_2}d\xi_2\,.$$

Taking into account the relation $\left(T_2\dfrac{\partial^2 u}{\partial x_2^2}\right)(x,t)=u_{\bar{x}_2x_2}(x,t)$, $x\in\omega\cup\gamma_{-1}$, one can get the representation

$$\begin{aligned}
\eta_{2\bar{x}_2x_2}(x,t) &= \\
=\frac{2}{h_2^2\tau h_1^3}&\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_4|\big)d\xi_4\int\limits_0^{h_1}(h_1-\xi)d\xi\times\\
&\times\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}d\xi_2\int\limits_{\eta}^{t}\frac{\partial^4 u(\xi_1,\xi_4,\eta_1)}{\partial\xi_4^2\partial\eta_1\partial\xi_1}d\eta_1+\\
+\frac{2}{h_2^2\tau h_1^3}&\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_4|\big)d\xi_4\int\limits_0^{h_1}(h_1-\xi)d\xi\times\\
&\times\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}d\xi_2\int\limits_{\xi_2}^{\xi_1}\frac{\partial^4 u(\xi_3,\xi_4,\eta)}{\partial\xi_4^2\partial\xi_3^2}d\eta_1+\\
+\frac{2}{h_2^2\tau h_1^3}&\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_4|\big)d\xi_4\int\limits_0^{h_1}(h_1-\xi)d\xi\times\\
&\times\int\limits_0^{\xi}d\xi_1\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}\frac{\partial^3 u(\xi_2,\xi_4,\eta)}{\partial\xi_4^2\partial\xi_2}d\xi_2,
\end{aligned}$$

where we apply the inequality $(a+b+c)^2\leqslant 3(a^2+b^2+c^2)$ and then have

$$\begin{aligned}
\big|\eta_{2\bar{x}_2x_2}(x,t)\big|^2\leqslant 3\Bigg(&2\frac{\tau h_1}{h_2}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_0^{h_1}d\xi_1\int\limits_{t-\tau}^{t}\left(\frac{\partial^4 u(\xi_1,\xi_2,\eta)}{\partial\xi_2^2\partial\eta\partial\xi_1}\right)^2d\eta+\\
&+\frac{2h_1^3}{9\tau h_2}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}\left(\frac{\partial^4 u(\xi_1,\xi_2,\eta)}{\partial\xi_1^2\partial\xi_2^2}\right)^2d\xi_1+\qquad(19)\\
&+\frac{2h_1}{9\tau h_2}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}\left(\frac{\partial^3 u(\xi_1,\xi_2,\eta)}{\partial\xi_2^2\partial\xi_1}\right)^2d\xi_1\Bigg),\quad x\in\gamma_{-1}.
\end{aligned}$$

As a result (see (18) and (19)) the following estimate is substantiated

$$\sum_{\eta=\tau}^{t}\tau\|(A_2\eta_2)(\cdot,\eta)\|^2\leqslant 128\Bigg(\tau^2h_1^2\int\limits_0^t d\eta\iint\limits_D\left(\frac{\partial^4u(\xi_1,\xi_2,\eta)}{\partial\xi_1\partial\xi_2^2\partial\eta}\right)^2d\xi_1d\xi_2+$$
$$+4h_1^4\int\limits_0^t d\eta\iint\limits_D\left(\frac{\partial^4u(\xi_1,\xi_2,\eta)}{\partial\xi_1^2\partial\xi_2^2}\right)^2d\xi_1d\xi_2\Bigg)+\tag{20}$$
$$+\frac{2h_1^2}{3}\int\limits_0^t d\eta\iint\limits_{D_h}\left(\frac{\partial^3u(\xi_1,\xi_2,\eta)}{\partial\xi_2^2\partial\xi_1}\right)^2d\xi_1d\xi_2,$$

where $D_h=\{x=(x_1,x_2):\ 0\leqslant x_1\leqslant h_1,\ 0\leqslant x_2\leqslant 1\}$.

Finally we examine the summand $\|\eta_3(\cdot,t)\|^2$ in (14). For the node $x\in\omega$ we get

$$\eta_3(x,t)=\frac{d(Tu)}{dt}(x,t)-u_{\bar t}(x,t)=\frac{1}{\tau h_1^2h_2^2}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_1|\big)d\xi_1\times$$
$$\times\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_{\eta}^{t}\frac{\partial^2u(\xi_1,\xi_2,\eta_1)}{\partial\eta_1^2}d\eta_1+$$
$$+\frac{1}{\tau h_1^3h_2^2}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_1|\big)d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_1}^{\xi_1}d\xi_3\times$$
$$\times\int\limits_{x_1-\frac{h_1}{2}}^{x_1+\frac{h_1}{2}}d\xi_5\int\limits_{\xi_5}^{\xi_3}\frac{\partial^3u(\xi_8,\xi_2,\eta)}{\partial\xi_8^2\partial\eta}d\xi_8+$$
$$+\frac{1}{\tau h_1^3h_2^3}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_1|\big)d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2}^{\xi_2}d\xi_4\times$$
$$\times\int\limits_{x_1-\frac{h_1}{2}}^{x_1+\frac{h_1}{2}}d\xi_6\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}}d\xi_7\int\limits_{x_6}^{x_1}\frac{\partial^3u(\xi_9,\xi_4,\eta)}{\partial\xi_9\partial\xi_4\partial\eta}d\xi_9+$$
$$+\frac{1}{\tau h_1^3h_2^3}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_1|\big)d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2}^{\xi_2}d\xi_4\times$$

$$\times \int\limits_{x_1-\frac{h_1}{2}}^{x_1+\frac{h_1}{2}} d\xi_6 \int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}} d\xi_7 \int\limits_{\xi_7}^{\xi_4} \frac{\partial^3 u(\xi_6,\xi_{10},\eta)}{\partial\xi_{10}^2\partial\eta} d\xi_{10}\,.$$

Making use of the inequality $(a+b+c+d)^2 \leqslant 4(a^2+b^2+c^2+d^2)$ we draw from here the estimate

$$\begin{aligned}
\big|\eta_3(x,t)\big|^2 \leqslant 4\Bigg(&\frac{16\tau}{9h_1h_2}\int\limits_{x_1-h_1}^{x_1+h_1} d\xi_1 \int\limits_{x_2-h_2}^{x_2+h_2} d\xi_2 \int\limits_{t-\tau}^{t} \left(\frac{\partial^2 u(\xi_1,\xi_2,\eta_1)}{\partial\eta_1^2}\right)^2 d\eta_1 + \\
&+ \frac{64h_1^3}{\tau h_2}\int\limits_{x_2-h_2}^{x_2+h_2} d\xi_2 \int\limits_{t-\tau}^{t} d\eta \int\limits_{x_1-h_1}^{x_1+h_1} \left(\frac{\partial^3 u(\xi_8,\xi_2,\eta)}{\partial\xi_8^2\partial\eta}\right)^2 d\xi_8 + \\
&+ \frac{32h_1h_2}{\tau}\int\limits_{t-\tau}^{t} d\eta \int\limits_{x_2-h_2}^{x_2+h_2} d\xi_4 \int\limits_{x_1-\frac{h_1}{2}}^{x_1+\frac{h_1}{2}} \left(\frac{\partial^3 u(\xi_9,\xi_4,\eta)}{\partial\xi_9\partial\xi_4\partial\eta}\right)^2 d\xi_9 + \qquad (21)\\
&+ \frac{128h_2^3}{\tau h_1}\int\limits_{t-\tau}^{t} d\eta \int\limits_{x_1-\frac{h_1}{2}}^{x_1+\frac{h_1}{2}} d\xi_6 \int\limits_{x_2-h_2}^{x_2+h_2} \left(\frac{\partial^3 u(\xi_6,\xi_{10},\eta)}{\partial\xi_{10}^2\partial\eta}\right)^2 d\xi_{10}\Bigg), \quad x\in\omega\,.
\end{aligned}$$

Now we consider the term $\eta_3(x,t)$ for $x \in \gamma_{-1}$. Omitting some details we can present it in the form

$$\begin{aligned}
\eta_3(x,t) &= \frac{d(Tu)}{dt}(x,t) - u_{\bar t}(x,t) = \\
&= \frac{2}{h_1^2h_2^2}\int\limits_{0}^{h_1}(h_1-\xi_1)d\xi_1 \int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)\frac{\partial u(\xi_1,\xi_2,t)}{\partial t}d\xi_2 - \\
&\quad - \frac{u(0,x_2,t)-u(0,x_2,t-\tau)}{\tau} = \\
&= \frac{2}{h_1^2h_2^2}\int\limits_{0}^{h_1}(h_1-\xi_1)d\xi_1 \int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2 \int\limits_{t-\tau}^{t}\Big[\frac{\partial u(\xi_1,\xi_2,t)}{\partial t} - \\
&\quad - \frac{\partial u(0,x_2,\eta)}{\partial \eta}\Big]d\eta = \\
&= \frac{2}{h_1^2h_2^2}\int\limits_{0}^{h_1}(h_1-\xi_1)d\xi_1 \int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2 \int\limits_{t-\tau}^{t} d\eta \int\limits_{\eta}^{t}\frac{\partial^2 u(\xi_1,\xi_2,\eta_1)}{\partial\eta_1^2} +
\end{aligned}$$

$$+\frac{2}{h_1^2h_2^2}\int\limits_0^{h_1}(h_1-\xi_1)d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{\xi_1}\frac{\partial^2u(\xi_3,\xi_2,\eta)}{\partial\xi_3\partial\eta}d\xi_3+$$

$$+\frac{2}{h_1^3h_2^3}\int\limits_0^{h_1}(h_1-\xi_1)d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)d\xi_2\times$$

$$\times\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2}^{\xi_2}d\xi_4\int\limits_0^{h_1}d\xi_5\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}}d\xi_6\int\limits_{\xi_5}^{0}\frac{\partial^3u(\xi_7,\xi_4,\eta)}{\partial\xi_7\partial\xi_4\partial\eta}d\xi_7+$$

$$+\frac{2}{h_1^3h_2^3}\int\limits_0^{h_1}(h_1-\xi_1)d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\left(h_2-|x_2-\xi_2|\right)d\xi_2\int\limits_{t-\tau}^{t}d\eta\times$$

$$\times\int\limits_{x_2}^{\xi_2}d\xi_4\int\limits_0^{h_1}d\xi_5\int\limits_{x_2-\frac{h_2}{2}}^{x_2+\frac{h_2}{2}}d\xi_6\int\limits_{\xi_6}^{\xi_4}\frac{\partial^3u(\xi_5,\xi_8,\eta)}{\partial\xi_8^2\partial\eta}d\xi_8\,,$$

which leads to the inequality

$$\begin{aligned}\big|\eta_3(x,t)\big|^2\leqslant 4\Bigg(&\frac{32\tau}{9h_1h_2}\int\limits_0^{h_1}d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_{t-\tau}^{t}\left(\frac{\partial^2u(\xi_1,\xi_2,\eta_1)}{\partial\eta_1^2}\right)^2d\eta_1+\\&+\frac{2h_1}{\tau h_2}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}\left(\frac{\partial^2u(\xi_3,\xi_2,\eta)}{\partial\xi_3\partial\eta}\right)^2d\xi_3+\\&+\frac{2h_1h_2}{\tau}\int\limits_{t-\tau}^{t}d\eta\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_4\int\limits_0^{h_1}\left(\frac{\partial^3u(\xi_7,\xi_4,\eta)}{\partial\xi_7\partial\xi_4\partial\eta}\right)^2d\xi_7+\\&+\frac{2h_2^3}{\tau h_1}\int\limits_{t-\tau}^{t}d\eta\int\limits_0^{h_1}d\xi_5\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^3u(\xi_5,\xi_8,\eta)}{\partial\xi_8^2\partial\eta}\right)^2d\xi_8\Bigg),\quad x\in\gamma_{-1}\,.\end{aligned}\tag{22}$$

Combining (21) and (22) we conclude that

$$\begin{aligned}\sum_{\eta=\tau}^{t}\tau\|\eta_3(\cdot,\eta)\|^2\leqslant\frac{256}{9}\tau^2\int\limits_0^{t}d\eta\iint\limits_D\left(\frac{\partial^2u(\xi_1,\xi_2,\eta)}{\partial\eta^2}\right)^2d\xi_1d\xi_2+\\+1024h_1^4\int\limits_0^{t}d\eta\iint\limits_D\left(\frac{\partial^3u(\xi_1,\xi_2,\eta)}{\partial\xi_1^2\partial\eta}\right)^2d\xi_1d\xi_2+\end{aligned}\tag{23}$$

$$+ 256h_1^2h_2^2\int\limits_0^t d\eta \iint\limits_D \left(\frac{\partial^3 u(\xi_1,\xi_2,\eta)}{\partial\xi_1\partial\xi_2\partial\eta}\right)^2 d\xi_1 d\xi_2 +$$

$$+ 1024h_2^4\int\limits_0^t d\eta \iint\limits_D \left(\frac{\partial^3 u(\xi_1,\xi_2,\eta)}{\partial\xi_2^2\partial\eta}\right)^2 d\xi_1 d\xi_2 +$$

$$+ 8h_1^2\int\limits_0^t d\eta \iint\limits_{D_h} \left(\frac{\partial^2 u(\xi_1,\xi_2,\eta)}{\partial\xi_1\partial\eta}\right)^2 d\xi_1 d\xi_2\,,$$

where $D_h = \{x = (x_1, x_2):\ 0 \leqslant x_1 \leqslant h_1,\ 0 \leqslant x_2 \le 1\}$.

Assembling (14), (17), (20), (23) we arrive at the final conclusion.

Theorem 1. *Let the solution $u(x_1, x_2, t)$ of problem (1) satisfy the conditions*

$$\frac{\partial^4 u}{\partial x_1^2\partial x_2\partial t},\ \frac{\partial^4 u}{\partial x_1^2\partial x_2^2},\ \frac{\partial^4 u}{\partial x_1\partial x_2^2\partial t},$$

$$\frac{\partial^3 u}{\partial x_1 x_2^2},\ \frac{\partial^3 u}{x_1^2\partial t},\ \frac{\partial^3 u}{x_2^2\partial t},\ \frac{\partial^3 u}{\partial x_1\partial x_2\partial t},\ \frac{\partial^2 u}{\partial t^2},\ \frac{\partial^2 u}{\partial x_1\partial t} \in L_2(Q_T).$$

Then for the accuracy of scheme (2) the weighted a priory estimate holds true

$$\left(\sum_{\eta=\tau}^{t}\tau\frac{|z(x,\eta)|^2}{\rho^2(x)}\right)^{1/2} \le \qquad (24)$$
$$M\left(\tau^2h_2^2 + h_2^4 + \tau^2h_1^2 + h_1^4 + \tau^2 + h_1^2h_2^2 + h_1^2\right)^{1/2},$$

where the constant M is expressed through the norms of the above listed derivatives of the solution $u(x,t)$.

Remark 4. *One can apply the results from* [4] *(see p.161 therein) to the integrals*

$$h_1^2\int\limits_0^t d\eta \iint\limits_{D_h} \left(\frac{\partial^3 u(\xi_1,\xi_2,\eta)}{\partial\xi_2^2\partial\xi_1}\right)^2 d\xi_1 d\xi_2\,,\quad h_1^2\int\limits_0^t d\eta \iint\limits_{D_h} \left(\frac{\partial^2 u(\xi_1,\xi_2,\eta)}{\partial\xi_1\partial\eta}\right)^2 d\xi_1 d\xi_2$$

in (20) and (23) respectively. If the assumptions of theorem 1 are satisfied and, in addition, $\dfrac{\partial^4 u}{\partial x_1\partial x_2^3} \in L_2(Q_T)$, then instead of (24) the following estimate can be proved

$$\left(\sum_{\eta=\tau}^{t}\tau\frac{|z(x,\eta)|^2}{\rho^2(x)}\right)^{1/2} \leqslant M\left(\tau^2h_2^2 + h_2^4 + \tau^2h_1^2 + h_1^4 + \tau^2 + h_1^2h_2^2 + h_1^3\right)^{1/2}.$$

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Taras Shevchenko National University of Kyiv,
64, Volodymyrs'ka Str., Kyiv, 01601, Ukraine

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