Журнал обчислювальної та прикладної математики | 2013, № 3 (113) с. 107–118 | Journal of Computational & Applied Mathematics

UDC 519.624.2

A POSTERIORI ERROR ESTIMATIONS FOR FINITE ELEMENT APPROXIMATIONS ON QUADRILATERAL MESHES

Heorgiy Shynkarenko, Olexandr Vovk

Резюме. Основною метою цієї праці є побудова простих апостеріорних оцінювачів похибок частинами білінійних апроксимацій методу скінченних елементів, здатних надійно та ефективно обчислювати двосторонні границі похибок наближення розв'язків еліптичних крайових задач. За допущення, що схема методу скінченних елементів спроможна обчислити точні значення розв'язку у вузлах сітки, запропоновано поелементно визначені оцінювачі похибок Діріхле та Неймана, які послідовно обчислюються як наближені розв'язки задачі про лишок апроксимації методу скінченних елементів. Перший з них знаходить нижню границю похибки апроксимації методу скінченних елементів, а другий – верхню границю. Ми доповнюємо характеризацію цих оцінювачів детальними результатами числових експериментів з слабко нелінійною та сингулярно збуреними задачами з примежевими і внутрішніми шарами.

Abstract. The main goal of this paper is to construct the simple a posteriori error estimators for piecewise bilinear approximations of finite element method which are able to reliably and efficiently calculate the two-sided confidence interval for the approximation error of the elliptic boundary value problems. Under assumption that finite element method scheme can calculate the exact values of a solution at mesh nodes, we propose the element-wise error estimators of Dirihlet and Neuman, which are calculated in succession as the approximated solutions of the residual problem of finite element method approximations. The first of them evaluate the lower bound of the finite element approximation error and second evaluate the upper bound. We supplement the characteristics of this estimators by the detailed results of the numerical experiments with semi-linear and singularly perturbed problems with boundary and internal layers.

1. Introduction

A posteriori error estimations of finite element method (FEM) approximations is the important component of a modern science calculations. The Babuška's and Rheinboldt's original conception of a posteriori error estimation (1978) in the last decades generates a large family of various a posteriori error estimators (AEEs), which are able to qualitatively describe the errors of obtained approximations by FEM and create the foundation for local triangulation refinement and\or local refinement of approximations rates such that to find

Key words. Semi-linear diffusion-advection-reaction equation, variational problem, finite element method, Newton's method, generalized minimum residual method, element-wise a posteriori error estimator, efficiency index, convergence rate.

approximative solutions with guaranteed accuracy and minimal computational cost, see [2], [3], and also [4].

Following the previous work [8] we build element-wise Dirihlet ε_h^{Dir} and Neuman ε_h^{Neu} a posteriori error estimators for piece-wise bilinear finite element approximations on quadrilateral meshes. These estimators are able to qualitatively calculate the lower and upper bounds of exact error in terms of the following inequality

$$\varepsilon_h^{Dir} \leq \| u - u_h \| \leq \varepsilon_h^{Neu}. \tag{1}$$

This paper is structured in the following manner. In Section 2 we formulate the variational problem for elliptic diffusion-advection-reaction equation with semi-linearity and describe its features. The numerical scheme with quadrilateral finite elements is presented in Section 3. The next (Section 4) is devoted to the problem of the error estimation of FEM approximations. In Sections 5 and 6 we present element-wise solutions of this problem as the polynomial Dirihlet and Neuman indicator functions. The rest of the paper is devoted to the analysis of numerical experiments with the boundary value problems which require some efforts for solving because they are semi-linear or singularly perturbed. A comparison of characteristics of the estimators and they analogues, which are calculated for exact values of errors confirms the possibility of the calculation of two-sided error estimates (1) and expected convergence rates of FEM approximations.

2. Problem statement

To construct the cheap a posteriori error estimators for two-sided error estimates of finite element approximations we consider a singular perturbed and\or semi-linear boundary value problems with second order elliptic equation

$$\begin{cases} -\nabla.(\mu\nabla u) + \beta.\nabla u + \sigma u = f[u] \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_D, \\ -(\mu\nabla u).\nu = \bar{q} \text{ on } \Gamma_N = \partial\Omega\backslash\Gamma_D. \end{cases} \tag{2}$$

This semi-linear boundary value problem has the following variational formulation

$$\begin{cases} \textit{find } u \in V = \{v \in H^1(\Omega) : \ v = 0 \textit{ on } \Gamma_D\} \text{ such that} \\ a(u,v) = \langle N[u], v\rangle \quad \forall v \in V, \end{cases} \tag{3}$$

where

$$\begin{cases} a(u,v) := \int_\Omega [(\mu\nabla u).\nabla v + v(\beta.\nabla u + \sigma u)]dx, \\ \langle N[w], v\rangle := \int_\Omega f[w]vdx - \int_{\Gamma_N} \bar{q}vd\gamma. \end{cases}$$

Below we assume that the domain $\Omega \subset \mathbb{R}^2$ is a bounded polygon and other problem data are sufficiently regular functions to guarantee existence and uniqueness of the solution $u = u(x,y)$ that satisfies (3). We note here that the problem (3) becomes singularly perturbed in the case $\|\beta\|_{L^\infty(\Omega)} \to +\infty$ or/and $\|\sigma\|_{L^\infty(\Omega)} \to +\infty$, for the details we refer to [7].

3. Finite element approximations

In order to obtain approximations of the solution u of the variational problem (3) we use the family of quasiuniform meshes $\{\mathcal{T}_h\}$, which are composed of quadrilateral elements Q, $\mathcal{T}_h = \{Q\}$, $h_Q = diam\, Q$, $h = \max h_Q$. Now for each $m \in \mathbb{N}$ we can construct the finite element space

$$V_h^1 := \Big\{ v \in V \cap \mathbb{C}(\Omega) : v = \sum\nolimits_{i,j=0,1} a_{ij} x^i y^j \quad \forall a_{ij} \in \mathbb{R}, \\ \forall (x,y) \in Q, \ \forall Q \in \mathcal{T}_h \Big\}$$

with usual basis functions

$$\varphi_1, ..., \varphi_M \in V_h^1, \quad supp\, \varphi_i := \Omega_i = \{\cup\, Q : \ A_i \in \bar{Q}\}, \quad \varphi_j(A_i) = \delta_{ij}, \tag{4}$$

where M is a number of nodes $A_i = (x_i, y_i)$ of the mesh $\mathcal{T}_h$, which does not lie on a boundary patch Γ_D.

Then, using Galerkin discretization procedure, we reduce (3) to the following problem

$$\begin{cases} \textit{find } u_h \in V_h^1 \textit{ such that} \\ a(u_h, v) = \langle N[u_h], v \rangle \quad \forall v \in V_h^1 \end{cases} \tag{5}$$

or in the algebraic form:

$$\begin{cases} \textit{find } u_h = \displaystyle\sum_{i=1}^{M} q_i \varphi_i \textit{ such that} \\ \textit{the coefficients } \{q_i\}_{i=1}^M \in \mathbb{R}^M \textit{satisfy} \\ \sum\nolimits_{j=1}^{M} a(\varphi_j, \varphi_i) q_j = \langle N[u_h], \varphi_i \rangle, \quad i = 1, ..., M. \end{cases} \tag{6}$$

In order to unify computing process of the coefficients $q_i \in \mathbb{R}$, $i = 1, ..., M$, we use the so called 'master element' $Q_0 = \{(\alpha, \beta) \in \mathbb{R}^2 : \ |\alpha|, |\beta| \le 1\}$ with the mapping $\Phi : Q_0 \to Q$ as follows

$$\begin{cases} x(\alpha, \beta) = \displaystyle\sum_{i,j=\pm 1} x_{\frac{1}{2}(5+2j-ij)} (1 + i\alpha)(1 + j\beta), \\ y(\alpha, \beta) = \displaystyle\sum_{i,j=\pm 1} y_{\frac{1}{2}(5+2j-ij)} (1 + i\alpha)(1 + j\beta), \end{cases}$$

where $A_m = (x_m, y_m)$, $m = 1, ..., 4$ are the vertices of the quadrilateral Q. The integrals from (6) that defined in the variational problem (3) are calculated numerically by using Gauss quadratures on master element Q_0.

To solve the problem (6) we rewrite it in the following matrix form:

$$\textit{find vector } q \in \mathbb{R}^M \textit{such that } Sq = F[q], \tag{7}$$

where the matrix $S = \{S_{km}\}_{k,m=1}^{M}$ and the vector $F[q] = \{F_k[q]\}_{k=1}^{M}$ are obtained in the following rules

$$\begin{cases} S_{km} := \displaystyle\int_{\Omega_{mk}} [(\mu \nabla \phi_m).\nabla \phi_k + (\beta.\nabla \phi_m + \sigma \phi_m)\phi_k] dx, \quad \Omega_{mk} := \Omega_m \cap \Omega_k, \\ F_k[w] := \displaystyle\int_{\Omega_k} f\left[\sum_{i=1}^{N} w_i \phi_i\right] \phi_k dx + \int_{\Gamma_q \cap \partial\Omega_k} g\phi_k d\gamma, \quad k, m = 1, ..., M. \end{cases}$$

The last one can be solved by Newton's method which is written as the following iterative process with the initial guess $q^0 \in \mathbb{R}^M$ and the relaxation parameter τ

$$\begin{cases} \textit{given vector } q^n \in \mathbb{R}^M, \tau = const > 0; \\ \textit{find vector } r \in \mathbb{R}^M \textit{such that} \\ \{S - \tau F_q[q^n]\} r = F[q^n] - Sq^n, \\ q^{n+1} = q^n + \tau r, \quad n = 0, 1, ..., \end{cases} \tag{8}$$

where $F_q[q] := \left\{\frac{\partial F_k[q]}{\partial q_m}\right\}_{k,m=1}^{M}$ is the Jacobi matrix with components

$$\frac{\partial F_k[q]}{\partial q_m} = \int_{\Omega_{mk}} \frac{\partial f}{\partial u}\left[\sum_{i=1}^{M} q_i \phi_i\right] \phi_m \phi_k dx, \quad k, m = 1, ..., M, \quad q \in \mathbb{R}^M.$$

At each iteration of the Newton's method we solve the system of linearized equations (8) by the iterative solver, namely the generalized minimal residual method (GMRES) [14]. A preconditioner for this linear system is constructed using incomplete LU factorization.

4. Residual element-based estimator

We define the error $e_h = u - u_h$, which is the solution of the following nonlinear error problem [1, 4, 5]:

$$\begin{cases} \textit{find } e_h \in E_h \subset E, \ V = E \oplus V_h \textit{ such that} \\ a(e_h, v) = \langle N[u_h + e_h], v\rangle - a(u_h, v) \quad \forall v \in E_h. \end{cases}$$

Applying Taylor's formula $f[e_h + u_h] = f[u_h] + f_u[u_h]e_h + O(e_h^2)$, we obtain the linear problem

$$\begin{cases} \textit{find error estimator } e_h \in E_h \textit{ such that} \\ \qquad b(u_h; e_h, v) = \langle \rho[u_h], v\rangle \quad \forall v \in E_h, \end{cases} \tag{9}$$

where

$$\begin{cases} b(w; z, v) : = a(z, v) - \displaystyle\int_\Omega f_u[w] z v dx, \\ \langle \rho[w], v\rangle : = \langle N[w], v\rangle - a(w, v) \quad \forall w, z, v \in V. \end{cases}$$

In order to obtain the two-sided confidence interval for the approximation error we introduce both Dirihlet ϵ_h^{Dir} and Neuman ϵ_h^{Neu} element-based residual error indicator functions that get lower and upper error bounds correspondingly. They are the approximate solutions of the problem (9) for two different finite dimensional subspaces $E_h^{Dir} \subseteq E_h$ and $E_h^{Dir} \subseteq E_h$ and are obtained as the solutions of the following local problems:

$$\begin{cases} \textit{find } \epsilon_Q^{Dir} \in E_h^{Dir}(Q) := \{v \in H^1(Q):\ v = 0 \textit{ on } \partial\bar{Q}\} \textit{ such that} \\ \qquad b(u_h; \epsilon_Q^{Dir}, v) = \langle \rho[u_h], v\rangle \quad \forall v \in E_h^{Dir}(Q), \quad \forall Q \in \mathcal{T}_h, \end{cases} \tag{10}$$

and

$$\begin{cases} \textit{find } \epsilon_Q^{Neu} \in E_h^{Neu}(Q) := \{v \in H^1(Q):\ v(A_i) = 0\, \forall A_i \in \bar{Q}\} \textit{ such that} \\ \qquad b(u_h; \epsilon_Q^{Neu}, v) = \langle \rho[u_h], v\rangle \quad \forall v \in E_h^{Neu}(Q), \quad \forall Q \in \mathcal{T}_h, \end{cases} \tag{11}$$

corispondingly. The solutions of the problems (10) and (11) are unique and exist on each finite element $Q \in \mathcal{T}_h$. Also, we can define the single element indicator $\eta_Q := \|\epsilon_Q\|_{1,Q} \quad \forall Q \in \mathcal{T}_h$ and the global estimator $\|\epsilon_h\|^2_{1,\Omega} := \sum_{Q\in\mathcal{T}_h} \eta_Q^2 \quad \forall \mathcal{T}_h$ for both of them. This a posteriori error estimators come from the original concept of a posteriori error estimation, which was proposed in [2, 3], and is similar to the residual estimators based on a local Dirichlet boundary value problem, see [4]. The novelty is in the behaviour of interpolation on the edges of elements: the constructed Dirihlet error estimator ϵ_Q^{Dir} (10) vanishes at all boundary of finite element Q and Neuman estimator ϵ_Q^{Neu} (11) vanishes only at the nodes of $Q \in \mathcal{T}_h$. The similar idea was proposed in [8, 9] for triangular meshes.

5. Computable estimator for piecewise bilinear approximations

Now we consider the finite element approximation $u_h \in V_h^1$, which is written in local coordinates (α, β) of the quadrilateral $Q \in \mathcal{T}_h$ as follows

$$u_h|_Q = u_Q(\alpha,\beta) = \sum\nolimits_{i,j=\pm1} u_h(A_{\frac{1}{2}(5+2j-ij)})(1+i\alpha)(1+j\beta).$$

To compute the solutions of of the problems (10) and (11), in a general case we define the indicator function $\bar{\epsilon}_h$ on each finite element Q in the local manner

$$\bar{\epsilon}_h|_Q = \bar{\epsilon}_Q(\alpha,\beta) := \lambda_Q \phi_Q(\alpha,\beta) \in E_h(Q) \quad \forall(\alpha,\beta) \in Q_0,\ \lambda_Q \in \mathbb{R}, \tag{12}$$

where $\phi_Q(\alpha, \beta)$ is the quadratic function on master element Q_0. Then, from local problem (10) or (11) we can obtain the coefficients λ_Q on each finite element of the following general kind

$$\lambda_Q = \frac{\langle \rho[u_h], \phi_Q\rangle}{b(u_h; \phi_Q, \phi_Q)} \quad \forall Q \in \mathcal{T}_h,$$

and define the element error indicator $\bar{\eta}_Q$ and the global error estimation $\|\bar{\epsilon}_h\|_{1,\Omega}$

$$\bar{\eta}_Q = \|\bar{\epsilon}_Q\|_{1,Q} = |\lambda_Q|\, \|\phi_Q\|_{1,Q} \quad \forall Q \in \mathcal{T}_h, \quad \|\bar{\epsilon}_h\|^2_{1,\Omega} = \sum\nolimits_{Q\in\mathcal{T}_h} \bar{\eta}_Q^2 \quad \forall \mathcal{T}_h.$$

From the general view of error estimator (12) we construct the following Dirihlet estimator

$$\begin{cases} \bar{\epsilon}_Q^{Dir}(\alpha,\beta) = \lambda_Q^{Dir}\phi_Q^{Dir}(\alpha,\beta) = \lambda_Q^{Dir}(1-\alpha^2)(1-\beta^2) \in E_h^{Dir} \\ \forall(\alpha,\beta) \in Q_0,\ \Phi : Q_0 \to Q,\ \lambda_Q^{Dir} \in \mathbb{R},\ \forall Q \in \mathcal{T}_h, \end{cases} \tag{13}$$

and Neuman estimator

$$\begin{cases} \bar{\epsilon}_Q^{Neu}(\alpha,\beta) = \lambda_Q^{Neu}\phi_Q^{Neu}(\alpha,\beta) = \lambda_Q^{Neu}[1-\frac{1}{2}(\alpha^2+\beta^2)] \in E_h^{Neu} \\ \forall(\alpha,\beta)\in Q_0,\ \Phi: Q_0 \to Q,\ \lambda_Q^{Neu}\in\mathbb{R},\ \forall Q\in\mathcal{T}_h, \end{cases} \tag{14}$$

see also Fig.1.

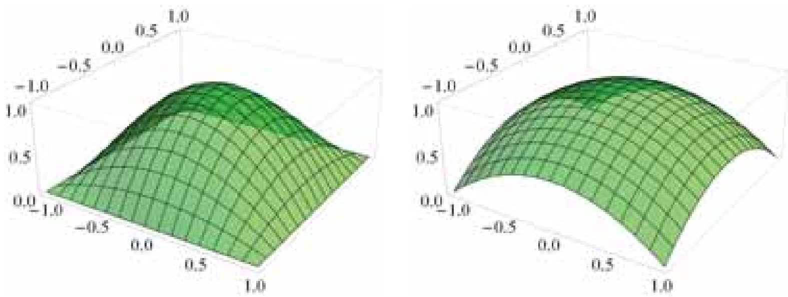

FIG. 1. Indicator functions: $\phi_Q^{Dir}(\alpha,\beta) = (1-\alpha^2)(1-\beta^2)$ (left), $\phi_Q^{Neu}(\alpha,\beta) = 1-0.5(\alpha^2+\beta^2)$ (right)

6. Convergence analysis of numerical results

To investigate abilities and features of the constructed Dirihlet ϵ_h^{Dir} (13) and Neuman ϵ_h^{Neu} (14) AEEs, we solve the model problems with known exact solutions. We present results of the numerical experiments for bilinear finite elements approximations on uniform quadrilateral meshes.

Example 1. Problem with Helmholtz equation

$$\begin{cases} -\Delta u - 10^4 u = f \text{ in } \Omega = (0,1)^2, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

that has the exact solution $u(x) = \sin(3\pi x)\sin(3\pi y)$.

At first we solve this problem using bilinear approximation on 10×10 quadrilateral mesh to illustrate the exact solution, it's approximation, error magnitude and distribution, see Fig. 2.

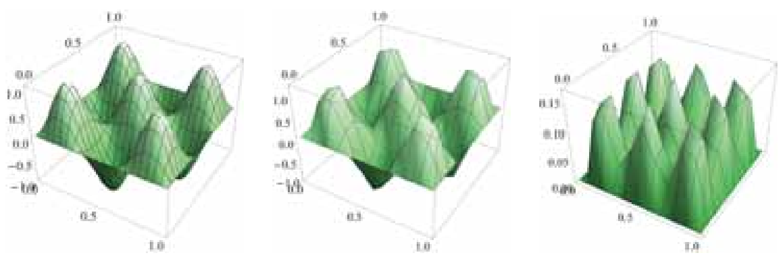

FIG. 2. Plot of the exact solution u (left), it's approximation u_h (middle) and the error $|u-u_h|$ (right) on 10×10 quadrilateral mesh for Example 1

Fig. 2 shows that even coarse 10×10 quadrilateral mesh gives the approximation with a good precision. But what about a posteriori estimation of this solving precision? To explore the bilinear approximations error and it's estimation more precisely, hereinafter we construct the convergence tables which include numerical results for uniform meshes with the variable mesh size h by rows. The columns of this tables correspond to the following characteristics: k denotes the refinement step of the solution process with estimation, Nod $\mathcal{T}_h$ is the number of nodes in the mesh, Card $\mathcal{T}_h$ is the number of finite elements in the mesh, $\varepsilon := \|u - u_h\|_{1,\Omega}\|u\|_{1,\Omega}^{-1}100\%$ is the exact relative error, $\varepsilon_h^{Dir} := \|\epsilon_h^{Dir}\|_{1,\Omega}\|u_h\|_{1,\Omega}^{-1}100\%$ is the relative error estimate by Dirihlet AEE, $\varepsilon_h^{Neu} := \|\epsilon_h^{Neu}\|_{1,\Omega}\|u_h\|_{1,\Omega}^{-1}100\%$ is the relative error estimate by Neuman AEE, $\kappa^{\mathbf{Dir}} := \|\epsilon_h^{Dir}\|_{1,\Omega}\|u - u_h\|_{1,\Omega}^{-1}$ is the efficiency index of the Dirihlet error estimator, $\kappa^{\mathbf{Neu}} := \|\epsilon_h^{Neu}\|_{1,\Omega}\|u - u_h\|_{1,\Omega}^{-1}$ is the efficiency index of the Neuman error estimator,

$$p^{Dir} := 2\frac{\ln \|\epsilon_{h,k}^{Dir}\|_{1,\Omega} - \ln \|\epsilon_{h,k+1}^{Dir}\|_{1,\Omega}}{\ln M_{k+1} - \ln M_k}, \quad p^{Neu} := 2\frac{\ln \|\epsilon_{h,k}^{Neu}\|_{1,\Omega} - \ln \|\epsilon_{h,k+1}^{Neu}\|_{1,\Omega}}{\ln M_{k+1} - \ln M_k}$$

denote the convergence rate of the Dirihlet and Neuman error estimators norms correspondingly.

TABL. 1. Convergence of bilinear approximations, it's errors and a posteriori error estimators (13), (14) for Example 1 on uniform quadrilateral meshes

k	Nod $\mathcal{T}_h$	Card $\mathcal{T}_h$	ε_h^{Dir}	ε	ε_h^{Neu}	κ^{Dir}	κ^{Neu}	p^{Dir}	p^{Neu}
1	1 681	1 600	8.025	6.796	66.045	1.19	1.73	-	-
2	6 561	6 400	3.268	3.393	9.336	0.96	2.77	1.3	3.2
3	25 921	25 600	1.568	1.696	3.638	0.93	2.16	1.1	1.4
4	103 041	102 400	0.776	0.848	1.725	0.92	2.04	1.0	1.1
5	410 881	409 600	0.387	0.424	0.851	0.91	2.01	1.0	1.0
6	1 640 961	1 638 400	0.194	0.212	0.424	0.91	2.00	1.0	1.0

Table 1 shows that the efficiency index $\kappa^{\mathbf{Dir}}$ is less then 1.0 and $\kappa^{\mathbf{Neu}}$ is greater then 1.0. It means that Dirihlet (13) and Neuman (14) estimators provide the lower and upper bounds of exact error correspondingly. The same result can be observed for the relative errors ε, ε_h^{Dir} and ε_h^{Neu}. Simultaneously, the efficiency indices are in a close neighbourhood of 1.0 and, consequently, are close to each other. So the constructed a posteriori error estimators provide a narrow interval that contain an exact error. The convergence rates p^{Dir} and p^{Neu} are equal to the expected theoretical rate 1.0. Note that, hereinafter, all conclusions from the convergence tables are true for sufficient fine meshes and, consequently, small approximation errors. In other words, they are true starting from certain table row.

Example 2. Problem with a boundary layer

$$\begin{cases} -10^{-2}\Delta u + \{2,3\}.\nabla u = f \text{ in } \Omega = (0,1)^2, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

with the solution $u(x,y) = xy^2 - y^2 g(2,x) + g(3,y)[g(2,x) - x]$, $g(\gamma,t) := exp(10^2\gamma(t-1))$.

This problem is singularly perturbed with Peclet number $Pe = 361$. That is why we solve it on more fine mesh with 40×40 quadrilaterals.

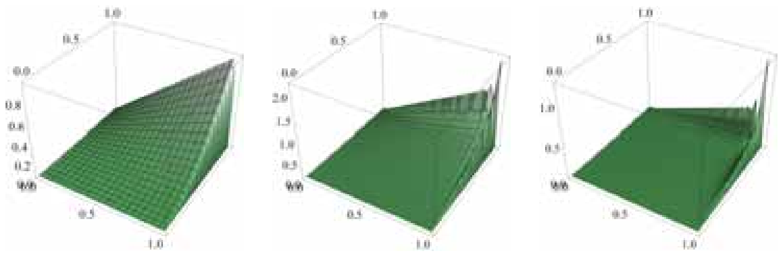

FIG. 3. Plot of the exact solution u (left), it's approximation u_h (middle) and the error $|u - u_h|$ (right) on 40×40 quadrilateral mesh for Example 2

Fig. 3 shows that the largest errors are concentrated in the boundary layer and a global error still large for such mesh density.

Now we create the following convergence table by the similar way as the previous

TABL. 2. Convergence of bilinear approximations, it's errors and a posteriori error estimators (13), (14) for Example 2 on uniform quadrilateral meshes

k	Nod $\mathcal{T}_h$	Card $\mathcal{T}_h$	ε_h^{Dir}	ε	ε_h^{Neu}	κ^{Dir}	κ^{Neu}	p^{Dir}	p^{Neu}
1	1 681	1 600	78.972	104.717	94.245	1.2	2.7	-	-
2	6 561	6 400	54.332	76.446	81.712	0.8	1.9	1.0	1.0
3	25 921	25 600	30.804	45.964	57.855	0.7	1.5	1.0	1.0
4	103 041	102 400	15.981	24.446	33.427	0.7	1.5	1.0	1.0
5	410 881	409 600	8.068	12.432	17.461	0.7	1.4	1.0	1.0
6	1 640 961	1 638 400	4.044	6.243	8.832	0.6	1.4	1.0	1.0

Table 2 confirms the conclusions (see. Table 1) about two-sided error estimates that are obtained by Dirihlet and Neuman AEEs. We also note that this problem is more difficult to solve and estimate an error than previous (Example 1.).

Example 3. Problem with two internal layers

$$\begin{cases} -\mu\Delta u - (\beta_1,\beta_2).\nabla u = 0 \text{ in } \Omega = (0,1)^2, \\ u \equiv U \text{ on } \partial\Omega, \end{cases}$$

with the solution $U = U(x,y) = G[m\beta_1(x)+v\beta_2(y)]G[m\beta_2(y)-v\beta_1(x)]$, where $\mu = 10^{-4}$, $\beta_1(x) = x - 0.6$, $\beta_2(y) = y - 0.3$, $m = \cos(\pi/6)$, $v = \sin(\pi/6)$, $G(z) = 0.5[1 - erf(z/\sqrt{2\mu})]$.

The solution of this problem include two internal layers. Peclet number for this singularly perturbed problem is approximately 8062.

Then, as before, we solve Example 1 on 40×40 mesh, plot exact solution, it's approximation, error and calculate the convergence table.

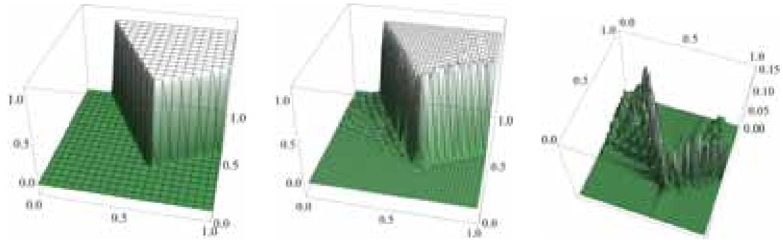

FIG. 4. Plot of the exact solution u (left), it's approximation u_h (middle) and the error $|u - u_h|$ (right) on 40×40 quadrilateral mesh for Example 3

Fig. 4 shows that the largest errors are concentrated in the internal layers. And the internal layers problem is less difficult to solve than the previous problem with boundary layer despite the fact that Peclet number in the latter is by one order of magnitude smaller.

TABL. 3. Convergence of bilinear approximations, it's errors and a posteriori error estimators (13), (14) for Example 3 on uniform quadrilateral meshes

k	Nod $\mathcal{T}_h$	Card $\mathcal{T}_h$	ε_h^{Dir}	ε	ε_h^{Neu}	κ^{Dir}	κ^{Neu}	p^{Dir}	p^{Neu}
1	1 681	1 600	87.528	51.380	94.319	3.4	5.3	-	-
2	6 561	6 400	29.316	22.044	51.412	1.4	2.7	2.5	2.2
3	25 921	25 600	8.565	10.136	17.955	0.8	1.8	1.8	1.7
4	103 041	102 400	4.106	5.057	8.896	0.8	1.8	1.1	1.0
5	410 881	409 600	2.055	2.529	4.489	0.8	1.8	1.0	1.0
6	1 640 961	1 638 400	1.028	1.264	2.251	0.8	1.8	1.0	1.0

Table 3 confirms the two-sided error estimates for FEM approximations of the internal layers problem in Example 3.

Example 4. Semi-linear problem [11]

$$\begin{cases} -\Delta u = au^3 + bu^2 \text{ in } \Omega = (0,1)^2, \\ u \equiv U \text{ on sides } x = 1, y = 1; \\ \nabla u.\nu \equiv 0 \text{ on sides } x = 0, y = 0, \end{cases}$$

with the solution $U = (\sin r^2 + 2)^{-1}$ and the coefficients $r^2 = l^2(x^2 + y^2)$, $a = -8l^2r^2\cos^2 r^2$, $b = 4l^2(\cos r^2 - r^2 \sin r^2)$, $l = 3.0$.

Finally, we demonstrate that the devised AEEs and FEM schemes are suitable to solve the semi-linear problems, see Fig. 5 and Table 4.

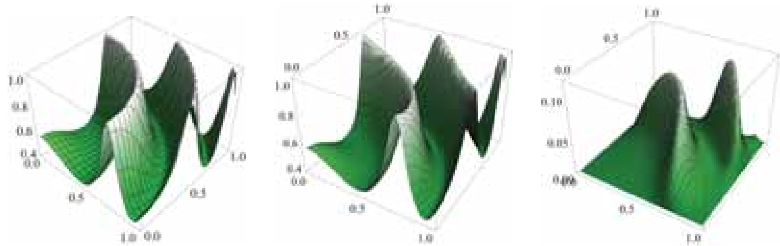

FIG. 5. Plot of the exact solution u (left), it's approximation u_h (middle) and the error $|u - u_h|$ (right) on 40×40 quadrilateral mesh for Example 4

TABL. 4. Convergence of bilinear approximations, it's errors and a posteriori error estimators (13), (14) for Example 4 on uniform quadrilateral meshes

k	Nod $\mathcal{T}_h$	Card $\mathcal{T}_h$	ε_h^{Dir}	ε	ε_h^{Neu}	κ^{Dir}	κ^{Neu}	p^{Dir}	p^{Neu}
1	1 681	1 600	10.857	18.234	23.704	0.5	1.2	-	-
2	6 561	6 400	5.577	7.869	12.210	0.7	1.5	0.9	0.9
3	25 921	25 600	2.812	3.590	6.159	0.8	1.7	1.0	1.0
4	103 041	102 400	1.409	1.739	3.087	0.8	1.8	1.0	1.0
5	410 881	409 600	0.705	0.862	1.544	0.8	1.8	1.0	1.0
6	1 640 961	1 638 400	0.353	0.430	0.772	0.8	1.8	1.0	1.0

7. CONLUSIONS

In this paper we have constructed the Dirihlet and Neuman estimators for two-sided error estimates of FEM approximations. This estimators are suitable for solving of the singularly perturbed and semi-linear diffusion-advection-reaction problems with a priori set accuracy. We use the classic Galerkin method with the piecewise linear bases of approximation spaces for uniform quadrilateral meshes. The calculation of both error indicators requires only the interior residual in the quadrilateral. The efficiency and reliability of the proposed Dirihlet and Neuman error estimators are shown by the numerical results for the boundary value problem with semi-linearity, Helmholtz equation, a boundary and interior layers.

Finally, the suggested Dirihlet and Neuman error estimators can be naturally extended to 3D case. We assume that the domain $\Omega \in \mathbb{R}^3$ is partitioned into finite hexahedral elements $\{H\}$. Then, for the 'master element'

$H_0 = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : |\alpha|, |\beta|, |\gamma| \leq 1\}$ we obtain the local Dirihlet

$$\begin{cases} e_H^{Dir}(\alpha,\beta,\gamma) = \dfrac{\langle \rho[u_h], \phi_H^{Dir}(\alpha,\beta,\gamma)\rangle}{b(u_h; \phi_H^{Dir}(\alpha,\beta,\gamma), \phi_H^{Dir}(\alpha,\beta,\gamma))} \phi_H^{Dir}(\alpha,\beta,\gamma), \\ \phi_H^{Dir}(\alpha,\beta,\gamma) = 1 - \frac{1}{2}(\alpha^2+\beta^2+\gamma^2) \qquad \forall(\alpha,\beta,\gamma) \in H_0, \end{cases}$$

and Neuman

$$\begin{cases} e_H^{Neu}(\alpha,\beta,\gamma) = \dfrac{\langle \rho[u_h], \phi_H^{Neu}(\alpha,\beta,\gamma)\rangle}{b(u_h; \phi_H^{Neu}(\alpha,\beta,\gamma), \phi_H^{Neu}(\alpha,\beta,\gamma))} \phi_H^{Neu}(\alpha,\beta,\gamma), \\ \phi_H^{Neu}(\alpha,\beta,\gamma) = (1-\alpha^2)(1-\beta^2)(1-\gamma^2) \qquad \forall(\alpha,\beta,\gamma) \in H_0, \end{cases}$$

estimators, where H is the arbitrary finite hexahedral element from the partition $\{H\}$ which is obtained from the master element H_0 using an appropriate mapping $\Psi : H_0 \to H$.

Bibliography

1. Ainsworth M. A Posteriori Error Estimation in Finite Element Analysis / M. Ainsworth, C. A. Brebbia, J. T. Oden.– Wiley, 2000.
2. Babuška I. A posteriori error estimates for the finite element method / I. Babuška, W. C. Rheinboldt // Int. J. Numer. Meth. Eng.– 1978.– Vol. 12.– P. 1597-1615.
3. Babuška I. Error estimates for adaptive finite element computation / I. Babuška, W. C. Rheinboldt // SIAM J. Numer. Anal.– 1978.– Vol. 15.– P. 736-754.
4. Babuška I. Finite Elements: An Introduction to the Method and Error Estimation / I. Babuška, J. R. Whiteman, T. Strouboulis.– Oxford: Oxford University Press, 2011.
5. Braess D. Finite Elements: Theory, Fast Solvers, and Applications in Elasticity Theory / D. Braess.– Cambridge: Cambridge University Press, 2007.
6. Gratsch T. A posteriori error estimation techniques in practical finite element analysis / T. Gratsch, J. Bathe // Comput. and Struct.– 2005.– Vol. 83.– P. 235-265.
7. Kozarevska J. S. Analysis of similarity criteria and sensitivity of substance migration problems solutions to coefficient perturbations / J. S. Kozarevska, H. A. Shynkarenko // Visnyk of Lviv University. Appl. Math. Comput. Science Series.– 2000.– Vol. 2.– P. 116–125 (in Ukrainian).
8. Ostapov O. Yu. A posteriori error estimator for diffusion-advection-reaction boundary value problems: piecewise linear approximations on triangles / O. Yu. Ostapov, H. A. Shynkarenko // J. Numer. Appl. Math.– 2011.– Vol. 2, № 105.– P. 111-123.
9. Ostapov O. Yu. A posteriori error estimator and h-adaptive finite element method for diffusion-advection-reaction problems / O. Yu. Ostapov, H. A. Shynkarenko, O. V. Vovk // 20th International Conference on Computer Methods in Mechanics (CMM 2013): Short Papers, August 27–31, 2013, Poznan, Poland.– Poznan: A. R. COMPRINT, 2013.– MS 10: 3-4.
10. Ostapov O. Yu. A posteriori error estimations for finite element approximations on quadrilateral meshes / O. Yu. Ostapov, H. A. Shynkarenko, O. V. Vovk // VI Int. conf. named by I. I. Lyashko "Computational and applied mathematics", September 5-6, 2013.– K.: Taras Shevchenko National University of Kyiv, 2013. – P. 31-34.
11. Ostapov O. Yu. A posteriori error estimations for serendipity finite element approximations on quadrilateral meshes / O. Yu. Ostapov, H. A. Shynkarenko, O. V. Vovk // XIX Ukrainian science conf., "Modern problems of applied mathematics and computer science", Ivan Franko National University of Lviv, October 3-4, 2013.– Lviv: Ivan Franko National University of Lviv, 2013. – P. 17-18.
12. Ostapov O. Yu. Finite element adaptive refinement techniques for diffusion-advection-reaction problems / O. Yu. Ostapov, H. A. Shynkarenko, O. V. Vovk // Manufacturing

Processes. Actual Problems 2013. M. Gajek, O. Hachkewych, A. Stanik-Besler eds. Politechnika Opolska, Opole.– 2013.– Vol. 1. Basic science applications.– P. 31-46.

13. Ostapov O. Yu. H-adaptive finite element method for nonlinear problems with mixed boundary conditions / O. Yu. Ostapov, O. V. Vovk // YSC-2013, Karpenko Physico-Mechanical Institute, October 23-25, 2013.– Lviv: P. P. Oschypok M. M, 2013. – P. 349-352.
14. Quarteroni A. Numerical Mathematics / A. Quarteroni, R. Sacco, F. Saleri.– New-York: Springer-Verlag, 2000.

Heorgiy Shynkarenko, Olexandr Vovk,
Ivan Franko National University of Lviv,
1, Universytets'ka Str., Lviv, 79000, Ukraine
Heorgiy Shynkarenko,
Opole University of Technology,
76, Proszkowska Str., Opole, 45-758, Poland

Received 2.10.2013