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MATRIX CONTINUED FRACTIONS FOR SOLVING THE POLYNOMIAL MATRIX EQUATIONS

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Резюме. Розглянуто алгоритм розв'язування поліноміальних матричних рівнянь. Запропонована рекурентна формула розвинення розв'язку в ланцюговий матричний дріб. Доведено збіжність методу. Наведено результати чисельних експериментів, що підтверджують справедливість теоретичних викладок..

Abstract. The article deals with the algorithm for solving the polynomial matrix equations. Recurrent formula for decomposition solution by the matrix continued fractions is proposed. The convergence of the method is proved and results of the numerical experiments that confirm the validity of the calculations are provided.

1. Introduction

The most simple matrix equations were being solved in the second half of the nineteenth century [1]. In default of a common approach polynomial matrix equations were resolved for a specific partial case.

A new approach for solving equations of the form

$$A_nX^n + A_{n-1}X^{n-1} + \ldots + A_1X + A_0 = 0, \tag{1}$$

is proposed in this paper. Here the coefficients $A_i \in \mathbb{R}^{p\times p}$ $\left(i=\overline{1,m}\right)$ and unknowns $X \in \mathbb{R}^{p\times p}$ are set on the ring of no commutative matrices.

For example we can consider quadratic equation

$$XAX + X + B = 0, \tag{2}$$

where A and B are nonzero square matrices of order n with constant coefficients and X is unknown square matrix of order n.

The equation can be written in the form

$$(XA + E)\,X = -B.$$

Or, assuming the existence of the inverse matrix, in form $(XA+E)^{-1}$,

$$X = -\,(XA+E)^{-1}\,B.$$

For convenience here this notation will be used:

$$-\,(XA+E)^{-1}\,B = -\frac{B}{E+XA}.$$

Key words. Polynomial matrix equations; The matrix continued fractions; The convergence of the method.

Then, using the insertion method to solve equation (2), the following expansion of X into a continued fraction is written:

$$X = -\cfrac{B}{E - \cfrac{BA}{E - \cfrac{BA}{E - \cfrac{BA}{\ddots}}}} \tag{3}$$

Using the similar transformations to solve the matrix equation

$$AX + XB + XFX + C = 0 \tag{4}$$

we obtain formal expansion of X into the following continued fraction

$$X = -F^{-1}B + \cfrac{AF^{-1}B - C}{A - F^{-1}BF + \cfrac{AF^{-1}BF - CF}{-F^{-1}BF + \cfrac{AF^{-1}B - C}{\ddots \atop A - F^{-1}BF + \cfrac{AF^{-1}BF - CF}{\ddots}}}} \tag{5}$$

Or using the Prinhcheym's notation for continued fractions

$$X = -F^{-1}B + \frac{AF^{-1}B - C|}{|A - F^{-1}BF} + \frac{AF^{-1}BF - CF|}{|-F^{-1}BF} + \ldots + \frac{AF^{-1}B - C|}{|A - F^{-1}BF} + \ldots$$

It is known [1] that the problem of optimal control for discrete stationary control system is reduced to a discrete Riccati equation

$$A^T XA - X - A^T XB \left(R + B^T XB\right)^{-1} B^T XA + Q = 0. \tag{6}$$

Here matrices A with dimension $n \times n$ and B with dimension $n \times m$ describes the state of the system

$$x(k+1) = Ax(k) + Bu(k).$$

And symmetric matrices Q and R defines quality criteria

$$J = \sum_{k=0}^{\infty} \left[x^T(k)\, Qx(k) + u^T(k)\, Ru(k)\right].$$

Herewith R is positive defined and Q is positive semi defined.

It turns out that the matrix continued fractions can be used for solving the discrete Riccati equation (6). After regrouping its members obtain

$$A^T X \left(A - E - B \left(R + B^T XB\right)^{-1} B^T XA\right) + Q = 0,$$

or

$$A^T X \left(A - E - B \left(R + B^T XB\right)^{-1} B^T XBB^{-1}A\right) + Q = 0.$$

From this we obtain

$$A^T X \left[A - E - B \left(R + B^T XB\right)^{-1} \left(R + B^T XB - R\right) B^{-1}A\right] + Q = 0$$

and

$$A^T X \left[A - E - BB^{-1}A + B\left(R + B^T XB\right)^{-1} RB^{-1}A\right] + Q = 0.$$

So,

$$X = -\left(A^{-1}\right)^T Q \left[A - E - BB^{-1}A + B\left(R + B^T XB\right)^{-1} RB^{-1}A\right]^{-1}.$$

Thus, the following recurrent formula can be written for the Riccati equation:

$$X = -\frac{\left(A^{-1}\right)^T Q\Big|}{\Big|E + BB^{-1}A - A - B\dfrac{RB^{-1}A}{R + B^T XB}}. \qquad (7)$$

Using composition (7) for equation (6) with numerical or symbolic elements, the following expansion of X into a continued fraction can be written:

$$X = -\frac{\left(A^{-1}\right)^T Q\Big|}{|E + BB^{-1}A - A} - B\frac{RB^{-1}A|}{|R} - B^T\frac{\left(A^{-1}\right)^T QB\Big|}{|E + BB^{-1}A - A} - \\ - \ldots - B\frac{RB^{-1}A|}{|R} - B^T\frac{\left(A^{-1}\right)^T QB\Big|}{|E + BB^{-1}A - A} - \ldots \qquad (8)$$

It is easy to see, comparing the expansions in continued fractions for equations (2), (4) and (6), that all of them are derived from a certain kind of schemes that does not fit into the framework of a single method. Moreover, algorithms for expansions of solutions in continued fractions are not known for algebraic numeric equations with two higher orders too.

2. The computational scheme of the method

The algorithm of expansions into the periodic branched continued fraction

$$x = p_0 + \sum_{i=1}^{n-1} \frac{p_i|}{|-q_i} + \sum_{i=1}^{n-1} \frac{p_i|}{|-q_i} + \ldots + \sum_{i=1}^{n-1} \frac{p_i|}{|-q_i} + \ldots \qquad (9)$$

for polynomial numerical equations

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad (10)$$

was proposed in [2]. Unknown coefficients p_i and q_i of the fraction (10) are defined as solutions of systems of linear algebraic equations. However, this scheme cannot be trivially moved in case of solving matrix polynomial equations because non commutative multiplication of matrices. But a similar algorithm can be constructed.

Theorem 1. *A solution to equation (1) of the n th order can be represented in the form of an infinite periodic continued fraction with $(n-1)$ branches.*

Proof. Suppose that matrices $(X - Q_k)^{-1}$ $\left(k = \overline{1, n-1}\right)$ are invertibles and consider the equality

$$X = P_0 + \sum_{k=1}^{n-1} (X - Q_k)^{-1} P_k, \qquad (11)$$

were $P_k \in \mathbb{R}^{p\times p}\,(k=0,1,\ldots,n-1)$ and $Q_k \in \mathbb{R}^{p\times p}\,(k=1,2,\ldots,n-1)$ are square matrices with unknown elements. To define them, the method of undetermined coefficients can be used. We will look for such items $p_{k,i,j}$ $\big(i = 1,2,\ldots,p;\ j=1,2,\ldots,p\big)$ and $q_{k,i,j}$ $\big(i=1,2,\ldots,p;\ j=1,2,\ldots,p\big)$ of matrices P_k and Q_k accordingly, that equations (1) and (11) will be equivalent.

Put additional,

$$Q_k = q_k \cdot E, \tag{12}$$

where E identity matrices and their dimensions are equal p. Easy to see that in this case

$$(X-Q_k)\,(X-Q_1)\times\ldots\times(X-Q_{k-1})\,(X-Q_{k+1})\times\ldots\times(X-Q_{n-1}) = \prod_{k=1}^{n-1}(X-Q_k)\,.$$

We reduce fractions in (11) to a common denominator and get

$$\begin{gathered} X = \left[\prod_{k=1}^{n-1}(X-Q_k)\right]^{-1}\cdot\left[\prod_{k=1}^{n-1}(X-Q_k)\,P_0 + \prod_{k=2}^{n-1}(X-Q_k)\,P_1 + \right. \\ + (X-Q_1)\prod_{k=3}^{n-1}(X-Q_k)\,P_2+\ldots+\prod_{k=1}^{l-1}(X-Q_k)\prod_{k=l+1}^{n-1}(X-Q_k)\,P_l+\ldots+ \\ \left. +\prod_{k=1}^{n-2}(X-Q_k)\,P_{n-1}\right]. \end{gathered} \tag{13}$$

Whence we obtain the following equation:

$$\begin{gathered} \left[\prod_{k=1}^{n-1}(X-Q_k)\right]X - \left[\prod_{k=1}^{n-1}(X-Q_k)\,P_0 + \prod_{k=2}^{n-1}(X-Q_k)\,P_1 + \right. \\ + (X-Q_1)\prod_{k=3}^{n-1}(X-Q_k)\,P_2+\ldots+\prod_{k=1}^{l-1}(X-Q_k)\prod_{k=l+1}^{n-1}(X-Q_k)\,P_l+\ldots+ \\ \left. +\prod_{k=1}^{n-2}(X-Q_k)\,P_{n-1}\right]. \end{gathered}$$

For each of the products we can write:

$$\begin{aligned} &-\prod_{k=1}^{n-1}(X-Q_k) = -\Big[X^n + X^{n-1}\,(-1)^{n-1}\,Q_1Q_2\ldots Q_{n-1} + \\ &+ X^{n-2}(-1)^{n-2}(Q_1Q_2\ldots Q_{n-2}+Q_1Q_2\ldots Q_{n-3}Q_{n-1}+\ldots+Q_2Q_3\ldots Q_{n-1})+ \\ &+\ldots+X^2\,(Q_1Q_2+Q_1Q_3+\ldots+Q_{n-2}Q_{n-1}) - X\,(Q_1+Q_2+\ldots+Q_{n-1})\Big]\,; \end{aligned}$$

$$\begin{aligned} &\prod_{k=1}^{n-1}(X-Q_k)\,P_0 = X^{n-1}P_0 + X^{n-2}\,(-1)^{n-1}\,Q_1Q_2\ldots Q_{n-1}P_0 + \\ &+ X^{n-3}(-1)^{n-3}(Q_1Q_2\ldots Q_{n-2}+Q_1Q_2\ldots Q_{n-3}Q_{n-1}+\ldots+Q_2Q_3\ldots Q_{n-1})P_0 \\ &+\ldots+X(Q_1Q_2+Q_1Q_3+\ldots+Q_{n-2}Q_{n-1})P_0-(Q_1+Q_2+\ldots+Q_{n-1})P_0; \end{aligned}$$

$$\prod_{k=2}^{n-1}(X-Q_k)P_1=X^{n-2}P_1+X^{n-3}(-1)^{n-2}Q_2Q_3\dots Q_{n-2}P_1+X^{n-4}(-1)^{n-3}\cdot$$
$$\cdot(Q_2Q_3\dots Q_{n-2}+Q_2Q_3\dots Q_{n-3}Q_{n-1}+\dots+Q_3Q_4\dots Q_{n-1})P_1+\dots+$$
$$+(Q_2Q_3+Q_2Q_4+\dots Q_{n-2}Q_{n-1})P_1-(Q_2+Q_3+\dots+Q_{n-1})P_1;$$

$$\prod_{k=2}^{l-1}(X-Q_k)\prod_{k=l+1}^{n-1}(X-Q_k)P_l=X^{n-2}P_l+X^{n-3}(-1)^{n-2}\cdot$$
$$\cdot Q_1Q_2\dots Q_{l-1}Q_{l+1}\dots Q_{n-1}P_l+X^{n-4}(-1)^{n-3}(Q_1Q_2\dots Q_{l-1}Q_{l+1}\dots Q_{n-2}+$$
$$+Q_1Q_2\dots Q_{l-1}Q_{l+1}\dots Q_{n-3}Q_{n-1}+\dots+Q_2Q_3\dots Q_{l-1}Q_{l+1}\dots Q_{n-2}Q_{n-1})P_l$$
$$+\dots+X(Q_2Q_3+Q_2Q_4+\dots+Q_{l-1}Q_{l+1}+Q_{l-1}Q_{l+2}+\dots+Q_{n-2}Q_{n-1})P_l-$$
$$-(Q_1+Q_2+\dots+Q_{l-1}+Q_{l+1}+\dots+Q_{n-1})P_l;$$

$$\prod_{k=1}^{n-2}(X-Q_k)P_{n-1}=X^{n-2}P_{n-1}+X^{n-3}(-1)^{n-2}Q_1Q_2\dots Q_{n-2}Q_{n-1}P_{n-1}+$$
$$+X^{n-4}(-1)^{n-3}(Q_1Q_2\dots Q_{n-3}+Q_1Q_2\dots Q_{n-4}Q_{n-2}+\dots+Q_2Q_3\dots Q_{n-2})P_{n-1}$$
$$+\dots+X(Q_1Q_2+Q_2Q_3+\dots+Q_{n-3}Q_{n-2})P_{n-1}-(Q_1+Q_2+\dots+Q_{n-2})P_{n-1}.$$

We now sum up the right sides of the equalities above, with simultaneously grouping the coefficients of identical powers of X. Equating coefficients of identical powers of X, we obtain the following system of equations for the determination of $P_k\,(k=0,1,2,\dots,n-1)$ and $Q_k\,(k=1,2,\dots,n-1)$:

$$\begin{aligned}
&(-1)^{n-1}Q_1Q_2\dots Q_{n-1}+P_0=A_1;\\
&(-1)^{n-2}\prod_{k=1}^{n-1}\prod_{l=1}^{k-1}Q_l\prod_{l=k+1}^{n-1}Q_l-\sum_{k=1}^{n-1}P_k+\\
&\qquad+(-1)^{n-1}Q_1Q_2\dots Q_{n-1}P_0=A_2;\\
&(-1)^{n-3}\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-2}(1-\delta_{kl})\prod_{r=1}^{k-1}Q_r\prod_{r=k+1}^{l-1}Q_r\prod_{r=l+1}^{n-2}Q_r+\\
&\qquad+\sum_{k=1}^{n-1}\prod_{r=1}^{k-1}Q_r\prod_{r=k+1}^{n-1}Q_rP_k+(-1)^{n-1}\sum_{k=1}^{n-1}\prod_{r=1}^{k-1}Q_rP_0=A_3;\\
&\dots\\
&\sum_{k=1}^{n-1}Q_k+\sum_{k=2}^{n-1}\sum_{l=k+1}^{n-1}Q_kQ_lP_1+\dots+\sum_{k=1}^{n-1}\sum_{l=k+1}^{n-1}(1-\delta_{kr})Q_kQ_lP_r+\\
&+\dots+\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-2}Q_kQ_lP_{n-1}=A_{n-1};\\
&\dots\\
&\sum_{k=1}^{n-1}Q_kP_1+\dots+\sum_{k=1}^{n-1}(1-\delta_{kr})Q_rP_r+\dots+\sum_{k=1}^{n-2}Q_kP_{n-1}+\\
&\qquad+\sum_{k=1}^{n-1}Q_kP_0=A_n,
\end{aligned}\tag{14}$$

where $\delta_{kl}=\begin{cases}1 \text{ if } k=l,\\ 0 \text{ if } k\neq l.\end{cases}$

If all the chosen Q_k are pairwise different, then the latter system of n equations in n unknowns (14) will become linear relatively unknown $P_k\,(k = 0, 1, 2, \ \ldots, n-1)$ and will have a unique solution. Using composition law (11) for X, we obtain the following expansion in terms of matrix branched continued fractions. If the left matrix multiplication $(X - Q_k)^{-1} P_r$ is denoted as $\dfrac{P_r}{X - Q_k}$, the recurrent formula for X will look as such:

$$X = P_0 + \sum_{k=1}^{n-1} \frac{P_r}{X - Q_k}. \tag{15}$$

Applying now the composition (14), we obtain the expanse of matrix branched continued fraction

$$X = P_0 + \sum_{k_1=1}^{n-1} \cfrac{P_{k_1}}{P_0 - Q_{k_1} + \sum\limits_{k_2=1}^{n-1} \cfrac{P_{k_2}}{P_0 - Q_{k_2} + \sum\limits_{k_3=1}^{n-1} \cfrac{}{\ddots + \cfrac{}{\sum\limits_{k_m=1}^{n-1} \cfrac{P_{k_m}}{P_0 - Q_{k_m} + \cfrac{}{\ddots}}}}}} \tag{16}$$

which is what had to be proved. □

To calculate the solution on the computer systems the recurrent formula (15) is sufficient. But for analytical writing solution and research of its existence and convergence approaching fractions shall use the theory of branched continued fraction for expanse (16). But solving equations (1) and (2), (4) and (6) requires a detailed study of convergence and computational stability of the matrix branched continued fraction.

Some sufficient signs of convergence for matrix branched continued fractions have been proposed in [3].

But the convergence of the branched fraction does not necessarily mean the convergence to the solution of the corresponding equation (1), (3) or (5). So we will focus on this aspect in more detail and consider the branched continued fraction

$$\sum_{k_l=1}^{N} \frac{a_{k_l}|}{|b_{k_l}} + \sum_{k_2=l}^{N} \frac{a_{k_1k_2}|}{|b_{k_1k_2}} + \sum_{k_3=1}^{N} \frac{a_{k_1k_2k_3}|}{|b_{k_1k_2k_3}} + \ldots + \sum_{k_i=1}^{N} \frac{a_{k_1k_2k_3\ldots k_i}|}{|b_{k_1k_2k_3\ldots k_i}} + \ldots \tag{17}$$

Here $a_{k_1k_2k_3\ldots k_i}$ and $b_{k_1k_2k_3\ldots k_i}$ are square matrices of dimension $p \times p$. In [2] and [3] the following sufficient signs have been obtained.

Theorem 2. *If the solution of polynomial matrix equation exists and belongs to the interval $[-N, N]$, then the expansion by some iterative procedure into the matrix branched continued fraction (17) with elements that satisfy the conditions*

$$\left\| b_{k(s)}^{-1} \right\| \leq \frac{1}{\left\| a_{k(s)+N} \right\|} \ \ (k\,(s) \in [1, N]\,; s = 1, 2, 3, \ldots)$$

converges to this solution.

Theorem 3. *If the solution of polynomial matrix equation exists and belongs to the interval* $\left[-\sum\limits_{k(s)=1}^{N}\left\|a_{k(s)}\right\|, \sum\limits_{k(s)=1}^{N}\left\|a_{k(s)}\right\|\right]$, *then the expansion by some iterative procedure into the matrix branched continued fraction (17) with elements that satisfy the conditions*

$$\left\|b_{k(s)}^{-1}\right\| \leq \frac{1}{1+\sum\limits_{k(s+1)=1}^{N}\left\|a_{k(s+1)}\right\|} \quad (s=1,2,3,\ldots)$$

converges to this solution.

These signs can be used to analyze the convergence of matrix continued fractions (3), (5), (8) and (16). Also, they are simple and easy to use. The theorems 2 and 3 can be used in practice, particularly in computer algebra systems, and serve as a basis for other sufficient signs for matrix branched continued fraction.

Note also, that if signs of convergence are valid, the iterative process (16) can finish if the inequality

$$\|X_{k+1}-X_k\| \leq \epsilon$$

is valid. Here ϵ – given calculation accuracy. This follows from the fact that in conditions of the theorem 2 and the theorem 3 the absolutely convergent numerical majorizing branched fractions build for matrix branched continued fractions (16). And its approach fractions form a monotone sequence.

Estimate the complexity of the algorithm. To obtain P_k $\left(k=\overline{0,n-1}\right)$ and Q_k $\left(k=\overline{1,n}\right)$ for the system of equations (14) we need to specify the pairwise different values for all matrix elements of Q_k. Then, doing generally up to the principal term n^5p^3 operations of multiplication and n^5p^3 operations of addition, we obtain the block system of linear algebraic equations with order n to determine P_k. For its solution need to complete an additional n^3p^3 operations of multiplication and n^3p^3 operations of addition. One iteration using the recurrent formula (11) requires the implementation of $2np^3$ operations of multiplication and np^3 operations of addition.

3. Numerical experiments

To verify the practical effectiveness of this approach, a series of numerical experiments were done in Mat Lab environment. In particular matrix equation

$$X^3+A_2X^2+A_1X+A_0=0,$$

was being solved. Here matrix coefficients were equal

$$A_2=\begin{pmatrix} 2.0000 & -3.0000 & -5.0000 \\ 0.2200 & 0.2510 & 0.2500 \\ 0.2200 & -0.2340 & -0.1300 \end{pmatrix};\ A_1=\begin{pmatrix} 1.0000 & 6.0000 & -5.0000 \\ 0.2500 & 0.2200 & 0.2510 \\ 0.2340 & -0.1300 & 0.2200 \end{pmatrix};$$

$$A_0=\begin{pmatrix} 136.0000 & 139.0000 & 134.0000 \\ -272.0240 & -269.0270 & -282.0490 \\ -350.2980 & -358.7900 & -336.5740 \end{pmatrix}.$$

The recurrent formula $X = P_0 + (Q_1 + X)^{-1} P_1 + (Q_2 + X)^{-1} P_2$ was being used to calculate X.

The matrix coefficients were set as

$$Q_1 = \begin{pmatrix} 0.96 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.96 \end{pmatrix}; \; Q_1 = \begin{pmatrix} 1.92 & 0 & 0 \\ 0 & 1.92 & 0 \\ 0 & 0 & 1.92 \end{pmatrix}.$$

Then from equations (15) the following values were calculated

$$P_2 = \begin{pmatrix} 139.9739 & 121.2717 & 130.3833 \\ -285.0969 & -287.0854 & -293.3430 \\ -364.5170 & -374.3781 & -358.9099 \end{pmatrix};$$

$$P_1 = \begin{pmatrix} -141.6651 & -135.9117 & -139.7833 \\ 285.4805 & 281.1371 & 293.8120 \\ 364.9166 & 373.8342 & 351.8643 \end{pmatrix}; P_0 = \begin{pmatrix} 0.8800 & 3.0000 & 5.0000 \\ -0.2200 & 2.6290 & -0.2500 \\ -0.2200 & 0.2340 & 3.0100 \end{pmatrix};$$

For the initial approximation X_0 was chosen zero matrix and the following approximate value of the unknown matrix was received

$$X = \begin{pmatrix} 12.3600 & 147.9411 & -107.2121 \\ -28.9221 & -290.3746 & 224.4685 \\ -36.9221 & -363.6585 & 282.0369 \end{pmatrix}$$

with the following results

Number of iteration	30	40	50	60	70
Norm of difference	0.3015	$6.1725E-04$	$9.0636E-06$	$4.9470E-08$	$6.9768E-09$

Thus, this approach can be applied to solve scientific and technical problems in generalized models of V. Leontyev and so on. However, the task of building a more subtle signs of convergence for periodic matrix branched continued fractions with broader areas of convergence is still open.

Bibliography

1. Ikramov H. D. Numeric solution of matrix equations / H. D. Ikramov. – Oxford: Pergamon Press, 1984. – 192 p.
2. Nedashkovskyy M. O. Solving of non-linear polynomial equation by branching chain fractions / M. O. Nedashkovskyy // "Computing" International Scientific Journal. – ISSN 1727-6209. – 2003. – Vol. 2. – Issue 1.
3. Nedashkovskyy M. O. The convergence of branched continued fractions to solutions of polynomial matrix equations / M. O. Nedashkovskyy // VII th International Conference INTERPOR 2008. – Porous Materials. – Theory and Experiment. – Bydgoszcz. Lubostroń. – 20-22, October 2008. – Kazimerz Wielki Bydgoszcz University. – 2008. – P. 63-64.

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