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A GRADIENT METHOD OF SOLVING INVERSE EIGENVALUE PROBLEM

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Резюме. Розглядається обернена задача на власні значення, яка включає класичні адитивні та мультиплікативні спектральні задачі. Показано спосіб зведення оберненої задачі до багатопараметричної задачі на власні значення. Запропоновано чисельний метод відшукання наближеного розв'язку спектральної задачі шляхом розв'язання еквівалентної їй варіаційної задачі. Проведено числові експерименти для ілюстрації роботи методу.

Abstract. It is investigated the inverse eigenvalue problem that includes classic additive and multiplicative spectral problems. It is presented the method of transformation of the inverse eigenvalue problem to the direct multi-parameter one. It is proposed the numerical method of calculating the approximate solution of the spectral problem by solving the equivalent variation problem. There are several numerical experiments presented in order to illustrate the behavior of the method.

1. Introduction

The problem of reconstruction of the matrix of some given structure based on the given spectral data is well known as the inverse eigenvalue problem, or in other words, the inverse spectral problem.

Such problems arise in a wide area of analysis investigations and mathematical physics, namely in the systems of control and identifications, the structural analysis, the modeling of mechanical systems and so on.

The major common point of all these applications is the fact that the physical parameters of some system should be restored based on the given dynamical parameters of the same system. If we describe the physical parameters mathematically and present them in a form of a matrix, we get an inverse eigenvalue problem.

As it was mentioned above, the needed matrix should have some given structure. Such structural constraints are not unsubstantial – they add sense to the spectral problem. Beside that, they define the different types of inverse spectral problem: additive, multiplicative, multi-parameter, structural etc.

There are two main questions regarding the eigenvalue problem: theoretical one, concerning the existence of the solution, and practical one, about the numerical method of finding this solution. There is provided a lot of literature concerning the conditions of solubility and uniqueness of the solution for different types of inverse spectral problem. A variety of methods of calculating the approximate solution of the mentioned problem is also listed in different

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sources. See, for example, [1] - [11] and the cited literature). In this article we will discuss another numerical algorithm of solving the inverse eigenvalue problem, assuming that the solution exists.

2. Inverse Eigenvalue Problem

Let's consider the following inverse spectral problem.

Problem GIEP (General Inverse Eignevalue Problem).

Provided it is given the complex matrices of dimensions $n \times n$: A_0, A_1, ..., $A_m \in C^{n\times n}$ and the collection of numbers $\lambda = \{\lambda_1, ..., \lambda_m\} \in C^m$.

Find such parameters $p = \{p_1, ..., p_m\} \in C^m$ that the eigenvalues of the matrix

$$A(p) = A_0 + p_1 A_1 + ... + p_m A_m \tag{1}$$

coincide with the given set of numbers $\lambda = \{\lambda_1, ..., \lambda_m\} \in C^m$.

This problem involves classic partial cases of *additive* and *multiplicative* inverse spectral problems:

Problem AIEP (Additive Inverse Eigenvalue Problem).

Let A be a given matrix and $\lambda = \{\lambda_1, ..., \lambda_m\} \in C^m$ be a given set of numbers.

Find the diagonal matrix $D = \operatorname{diag}(p_1, \ldots, p_m)$, $p_1, p_2, \ldots, p_m \in C^m$, such that the matrix $A + D$ has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Problem MIEP (Multiplicative Inverse Eigenvalue Problem).

Let A be a given matrix and $\lambda = \{\lambda_1, ..., \lambda_m\} \in C^m$ be a given set of numbers.

Find the diagonal matrix $D = \operatorname{diag}(p_1, \ldots, p_m)$, $p_1, p_2, \ldots, p_m \in C^m$, such that the matrix AD has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$.

The question of solvability of such kind of problems, namely AIEP, is widely explored in the literature (see, for example, [2], [4], [11]). Beside the theoretical results there is a lot of numerical methods constructed for solving the additive inverse eigenvalue problem (see, for example, [1], [3], [5], [7] - [9]).

In this survey we propose another method of finding the approximate solution of the problem (2.1) in the real Euclidian space. This method is based on a gradient procedure.

3. Preliminary

Consider the multi-parameter spectral problem in the Euclidian space E^n:

$$T(\lambda)\, x \equiv Ax - \lambda_1 B_1 x - ... - \lambda_m B_m x = 0 \tag{2}$$

where $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$ are spectral parameters, $x = (x_1, ..., x_n) \in E^n$, $A, B_1, ..., B_m$ are some linear operators that act in the real Euclidian space E^n.

The multi-parameter eigenvalue problem, linear towards the spectral parameters, consists in finding a vector of spectral parameters $\lambda = \{\lambda_1, \ ... \ , \lambda_m\} \in E^m$ such that there exists a non-trivial solution $x \in E^n \backslash \{0\}$ of the equation (3.1).

Let's put the variation problem of minimization of the following functional in correspondence to the spectral problem (3.1):

$$F(u) = \frac{1}{2} \left\| T(\lambda)\, x \right\|_H^2, \ \forall u = \{x, \lambda\} \in H = (E^n \backslash \{0\}) \oplus E^m \tag{3}$$

The problem of minimization of the functional (3.2) consists in finding such set of parameters $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$ and corresponding vector $x \in E^n \backslash \{0\}$ that the functional $F(u)$ reaches its minimum value:

$$F(u) \to \min_u, \quad u \in U \subset H, \tag{4}$$

where U is the set with points $u = \{x, \lambda\}$ that satisfy the equation (3.1), H is an Euclidian space with the scalar product and the norm defined in a standard way:

$$(u, v)_H = (u_1, u_2)_{E^n} + (v_1, v_2)_{E^m}, \|u\|_H = \sqrt{\|u_1\|^2_{E^n} + \|v_1\|^2_{E^m}},$$

$$u = \{u_1, v_1\}, v = \{u_2, v_2\}, u_1, u_2 \in E^n, v_1, v_2 \in E^m.$$

In the article [8] it is shown that the spectral problem (3.1) and the variation problem (3.3) are equivalent. This means that each eigen pair $\{x, \lambda\}$ of the problem (3.1) is the point of minimum $u = \{x, \lambda\}$ of the functional (3.2), and vice-versa.

This result lets us construct the gradient procedure of numerical solving of the problem (3.3) and thus, of the problem (3.1), in the following form:

$$u_{k+1} = u_k - \gamma(u_k) \nabla F(u_k) \ , \quad k = 0, \ 1, \ 2, \ ... \tag{5}$$

The relation (3.4) describes the whole class of methods that differ only by the choice of the step value $\gamma(u_k)$.

In this article we will calculate the value $\gamma_k = \gamma(u_k)$ at each step of the process by using the formula:

$$\gamma_k = \frac{F(u_k)}{||\nabla F(u_k)||^2_H} \tag{6}$$

From this point here, in order to make the formulas more easy to read, we will omit the index H in the denotation of the scalar product and the norm.

So, the iteration process can be written as following:

$$u_{k+1} = u_k - \frac{F(u_k)}{||\nabla F(u_k)||^2} \nabla F(u_k) \ , \tag{7}$$

where the gradient of the functional has the structure

$$\nabla F(u) = \left\{ (T^* T x, e_1), ..., (T^* T x, e_n), \left(Tx, \frac{\partial T}{\partial \lambda_1} x \right), ..., \left(Tx, \frac{\partial T}{\partial \lambda_m} x \right) \right\} \tag{8}$$

Here $T \equiv T(\lambda)$, and $e_i \in E^n$ is the vector, the i-th co-ordinate of which is equal to 1 and all the others co-ordinates are 0.

If the starting approximation is chosen in some sense close enough to the eigenvector and the vector of eigenvalues, then the iteration process (3.6) converges to the stationary point of the functional (3.2) $u^* = \{x^*, \lambda^*\}$. In this point the minimum of the functional is reached. Note, this means that the

process converges to the eigenvector x^* and the vector of eigenvalues λ^* of the problem (3.1).

Thus, for the iteration process, described above, the following theorem is true:

Theorem 1. [8] *Let the gradient of the functional (3.6) satisfies the Lipchitz condition*

$$\|\nabla F(u) - \nabla F(z)\| \leq L \|u - z\|, \quad \forall u, z \in U, \;\; L > 0 \tag{9}$$

where U is a closed convex set that contains the solution u^*. If for some starting approximation $u_0 = (x_0, \lambda^{(0)}) \in U$ the following condition is true

$$0 < \gamma_0 \equiv \gamma(u_0) \leq 1/2L, \tag{10}$$

then the iteration process (3.6) converges to the point of minimum of the functional (3.2) $u^* = \{x^*, \lambda^*\}$ and, thus, to the eigenvector x^* and the vector of eigenvalues λ^* of the problem (3.1). Which means that the relations below are true:

$$\lim_{k\to\infty} \rho(u_k, U_*) = \lim_{k\to\infty} \rho(u_k, u^*) = 0 \tag{11}$$

$$\lim_{k\to\infty} F(u_k) = F(u^*) = 0 \tag{12}$$

4. Algorithm of solving an inverse spectral problem

Consider an inverse eigenvalue problem of type (2.1) with the real matrices $A_0, A_1, ..., A_m \in E^{n\times n}$, and where the pairs $\{\lambda_k, x^k\}_{k=1}^m$ are the eigen pairs of the matrix $A(p)$. Here $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$, $x^k \in H = E^n \backslash \{0\}$, $k = 1, 2, ..., m$, E is the real Euclidian space.

By using the definition of an eigen value and a corresponding eigen vector, we can write as following:

$$A(c)\, x^k = \lambda_k x^k, \quad x^k \in H, \quad k = 1, ..., m$$

Thus, we get the system of m equations to find the parameters $p_1, ..., p_m$:

$$\begin{cases} ((A_0 - \lambda_1 I) + p_1 A_1 + ... + p_m A_m)\, x^1 = 0, \\ \qquad\qquad ... \\ ((A_0 - \lambda_m I) + p_1 A_1 + ... + p_m A_m)\, x^m = 0, \end{cases} \tag{13}$$

Let's transform this system so that it has the structure (3.1). For this reason consider the matrix operators $\mathbf{A},\ \mathbf{B}_i\ :\ \mathbf{H} \to \mathbf{H}$, $\mathbf{H} = \bigoplus_{k=1}^{m} E^{n\times n}$, $i = 1, ..., m$:

$$\mathbf{A} = \begin{pmatrix} (A_0 - \lambda_1 I) & & 0 \\ & \ddots & \\ 0 & & (A_0 - \lambda_m I) \end{pmatrix} \tag{14}$$

$$\mathbf{B}_i = \begin{pmatrix} -A_i & & 0 \\ & \ddots & \\ 0 & & -A_i \end{pmatrix} \tag{15}$$

In case $\mathbf{x} = (x^1,\ x^2,\ \ldots\ ,\ x^m)^T \in \mathbf{H}$, we get

$$\mathbf{Ax} = ((A_0 - \lambda_1 I)x^1,\ (A_0 - \lambda_2 I)x^2,\ \ldots\ ,\ (A_0 - \lambda_m I)x^m),$$

$$\mathbf{B}_i\mathbf{x} = (-A_i x^1,\ -A_i x^2,\ \ldots\ ,\ -A_i x^m).$$

Now we can proceed from the problem (4.1) to the problem of type (3.1) in the space $\mathbf{H}$.

$$T(p) \equiv \mathbf{Ax} - p_1\mathbf{B}_1\mathbf{x} - \ldots - p_m\mathbf{B}_m\mathbf{x} = 0 \tag{16}$$

So, we configured the problem of finding such set of parameters $p_1, \ldots, p_m$, that the equation (4.4) has a non-trivial solution $\mathbf{x} \in \mathbf{H}\backslash\{\mathbf{0}\}$.

Now let's put a variation problem in correspondence to the problem (4.4):

$$F(\mathbf{u}) \to \min_{\mathbf{u}}, \quad \mathbf{u} \in \mathbf{U} \subset \mathbf{H},$$

where

$$F(\mathbf{u}) = \frac{1}{2}\left\|T(p)\,\mathbf{x}\right\|^2_{\tilde{\mathbf{H}}},\ \forall \mathbf{u} = \{\mathbf{x}, p\} \in \tilde{\mathbf{H}} = \mathbf{H} \oplus E^m \tag{17}$$

The task is to find the set of parameters $p = \{p_1, \ldots, p_m\} \in E^m$ and the corresponding vector $\mathbf{x} \in \mathbf{H}\backslash\{\mathbf{0}\}$, for which the functional $F(\mathbf{u})$ reaches its minimum value. For this variation problem we will apply the iteration process (3.6).

So, the algorithm consists of the following steps:

Step 1. Select the starting approximation.

Step 2. Build the matrices A, B_i, $i = 1, \ldots, m$ by using the formulas (4.2), (4.3).

Step 3. **for** $k = 0, 1, 2, \ldots$ until the exactness is reached **do**:

Step 4. Calculate $T(p_k)$ by using the formula (4.4).

Step 5. Calculate $F(u_k)$ by using the formula (4.5).

Step 6. Calculate $\nabla F(u_k)$ by using the formula (3.7).

Step 7. Calculate the next approximation u_{k+1} of the solution by using the formula (3.6).

end *for* k

Step 8. Extract the p_{k+1} components of the vector $u_{k+1} = \{x_{k+1}, p_{k+1}\}$ which is the needed approximate solution.

Step 9. End.

5. Numerical experiments

Let's demonstrate how the algorithm works on two examples below.

Example 1. [7]. Consider the following inverse eigenvalue problem:

$$A_0 = \begin{pmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

$$A(p) = A_0 + p_1 A_1 + p_2 A_2 + p_3 A_3 + p_4 A_4.$$

Where $n = m = 4$, the eigen values are given: $\lambda = \{0, 2, 2, 4\}$. The exact solution of the problem is also known $p^* = \{1, 1, 1, 1\}$.

Let's choose the starting approximation of the parametrs $p_i, i = 1, 2, 3, 4$.

a) $p^{(0)} = \{1.1, 0.9, 1.1, 0.9\}$, as it is proposed in [7], which is quite close to the exact solution;

b) $p^{(0)} = \{0.1, 0.2, 0.3, 0.4\}$, which strongly differs from the exact solution.

Note, that in [7] the starting approximation is given only for the parameters. However, to apply the method proposed in this article we also need to select the approximation of the eigen vectors $x^{(0)}$. In order to choose the correct values $x^{(0)}$, we did the following calculations: for the given parameters $p^{(0)} = \left\{p_1^{(0)}, ..., p_m^{(0)}\right\}$ we built the matrix $A\left(p^{(0)}\right) = A_0 + p_1^{(0)} A_1 + ... + p_m^{(0)} A_m$ and found the eigen values and the corresponding eigen vectors of this matrix by using the software application Matlab. Then we accepted these eigen vectors as the starting approximation $x^{(0)}$ for the method (3.6).

There were used two different stop conditions for the iteration process:

1. The value of the functional becomes zero, which means that $F\left(u^{(k+1)}\right) < \varepsilon, \quad \varepsilon = 10^{-9}$, where $u^{(k+1)} = \left\{x^{(k+1)}, p^{(k+1)}\right\}$ is the k-th approximation of the solution of the problem, $k = 0, 1, ...$

2. The norm of deviation between the values of the parameters on two iterations $p^{(k)}$ and $p^{(k+1)}$ becomes sufficiently small: $\left\|p^{(k)} - p^{(k+1)}\right\| < \varepsilon$, $\varepsilon = 10^{-9}$, $k = 0, 1, ...$

The results received in the cases a) and b) of starting approximations are presented in the Table 1 and the Table 2 respectively. Note, that each table contains two approximate solutions that correspond to two stop conditions of the iteration process.

Tabl. 1. Approximate solutions of Example 1, case a

	p^*	$p^{(0)}$	$p^{(m+1)}$, Stop cond 1	$p^{(m+1)}$, Stop cond 2
	1	1.1	1.0000403787	1.0000000123
	1	0.9	1.0000199351	1.0000000061
	1	1.1	1.0000266736	1.0000000081
	1	0.9	0.9999747545	0.9999999923
F	9.37e-31	0.02	8.4065983153e-11	9.2005057095e-18
$\|p - p^*\|$	0	0.2	5.8109138139e-5	1.7748612738e-8

In the tables it is also given the value of the functional in the point of starting approximation, $F^0 = F\left(u^{(0)}\right)$, the point of approximate solution, $F = F(u)$,

TABL. 2. Approximate solutions of Example 1, case b

	p^*	$p^{(0)}$	$p^{(m+1)}$, Stop cond 1	$p^{(m+1)}$, Stop cond 2
	1	0.1	0.9999582549	0.9999999786
	1	0.2	0.9999782404	0.9999999888
	1	0.3	0.9999714938	0.9999999854
	1	0.4	1.0000265629	1.0000000135
F	9.37e-31	1.89	7.1596842832e-11	1.9824859739e-17
$\|p-p^*\|$	0	1.52	6.1109094783e-5	3.1269595091e-8

and the point of exact solution, $F^* = F(u^*)$. In this way it can be seen that the value of the functional decreases, as it was expected.

Example 2. Consider the given inverse spectral problem, where the matrices A_i are the Toeplitz matrices. Note, that similarly to the previous example here $n = m$.

$$A(p) = A_0 + p_1 A_1 + ... + p_n A_n$$

$$A_0 = O, A_1 = I,$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, ..., A_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Let's solve this problem for $n = 5$. In this case the exact values of the parameters are $p^* = \{-1.8,\ 1.9,\ 2.5, 0.08,\ 1.2\}$.

The chosen starting approximations of the parameters $p_i, i = 1, ..., 5$ are the following: a) $p^{(0)} = \{-1,\ 1,\ 1,\ -1,\ 1\}$; b) $p^{(0)} = \{0,\ 1,\ 1,\ 0,\ 1\}$.

In order to select the starting approximations of the eigen vectors $x^{(0)}$ we did the same calculations as it was explained in the Example 1. The received results are presented in the Table 3 and the Table 4 for two cases of starting approximations.

Similarly to the Example 1, there were used two conditions to stop the iteration process. By analyzing the received values of the functional in the points of starting approximation, approximate solution and exact solution it can be seen that they go down to the minimum (zero) value, as expected.

Let us also note, that the analyzed examples had been solved by using two variants of the Newton method presented in the articles [1] and [3]. This experiment showed that the iteration processes of the Newton methods [1] and [3] do not converge to the exact solution if the selected starting approximation strongly differs from the exact values. The method presented in this survey, on the contrary, does converge to the exact solution in case of the same starting approximations.

Tabl. 3. Approximate solutions of Example 2, case a

	p^*	$p^{(0)}$	$p^{(m+1)}$, Stop cond 1	$p^{(m+1)}$, Stop cond 2
	-1.8	-1	-1.8000000001	-1.8000000001
	1.9	1	1.9000000048	1.9000000184
	2.5	1	2.4999999976	2.4999999865
	0.08	-1	0.0800000037	0.0799999959
	1.2	1	1.1999999837	1.1999999674
F	1.94e-29	12.62	3.7755871274e-10	1.1228316897e-9
$\|p - p^*\|$	0	2.22	1.7159808703e-8	3.9948275161e-8

Tabl. 4. Approximate solutions of Example 2, case b

	p^*	$p^{(0)}$	$p^{(m+1)}$, Stop cond 1	$p^{(m+1)}$, Stop cond 2
	-1.8	0	-1.8000048843	-1.8000048843
	1.9	1	1.9000646818	1.9000646818
	2.5	1	2.4999533405	2.4999533404
	0.08	0	0.0799945432	0.0799945431
	1.2	1	1.1998789745	1.1998789744
F	1.94e-29	17.56	3.5156195443e-6	3.5156195443e-6
$\|p - p^*\|$	0	2.52	1.4512640324e-4	1.4512640324e-4

Bibliography

1. Biegler-Konig F. W. A Newton iteration process for inverse eigenvalue problems / F. W. Biegler-Konig // Numer. Math. – 1981. – Vol. 37, No 3. – P. 349–354.
2. Biegler-Konig F. W. Sufficient conditions for the solubility of inverse eigenvalue problems / F. W. Biegler-Konig // Linear Algebra Appl. – 1981. – Vol. 40. – P. 89–100.
3. Podlevskyi B. M. Newton's method for the solution of inverse spectral problems / B. M. Podlevskyi, O. S. Yaroshko // J. Mathematical Sciences. – 2013. – Vol. 174, No 3. – P. 156-165.
4. Friedland S. Inverse eigenvalue problems / S. Friedland // Linear Algebra Appl. – 1977. – Vol. 17, No 1. – P. 15–51.
5. Friedland S. The formulation and analysis of numerical methods for inverse eigenvalue problems / S. Friedland, J. Nocedal, M. L. Overton // SIAM J. Numer. Anal. – 1987. – Vol. 24, No 3. – P. 634–667.
6. Graham M. L. Inverse Problems in Vibration / M. L. Graham, M. T. Gladwell. – New York: Kluwer Academic Publishers. – 2005.
7. Dai H. A Numerical Method for Solving Inverse Eigenvalue Problems / H. Dai // Mathematical Modelling and Numerical Analysis M2AN. – 1999. – Vol. 33, No 5. – P. 1003-1017.
8. Khlobystov V. V. A gradient descent method for solving of one class of nonlinear multiparameter eigenvalue problem / V. V. Khlobystov, B. M. Podlevskyi, O. S. Yaroshko // Trudy Instituta Matematiki Akad. Nayk Belarusi. – 2014 (in Russian).
9. Kublanovskaya V. N. On one approach to solving an inverse eigenvalue problem / V. N. Kublanovskaya // Zapiski Nauch. Semin. Leningr. Otdel. Mat. Inst. Akad. Nauk SSSR. – 1970. – Vol. 18. – P. 138–149 (in Russian).

10. LiL. Sufficient conditions for the solvability of algebraic inverse eigenvalue problems / L. Li // Linear Algebra Appl. – 1995. – Vol. 221. – P. 117–129.
11. Chu M. T. Inverse Eigenvalue Problems / Moody T. Chu // SIAM Rev. – 1998. – Vol. 40. – P. 1-39.

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