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EXTENSION OF A CLASS OF NONLINEAR HAMMERSTAIN INTEGRAL EQUATIONS WITH SOLUTIONS REPRESENTED BY COMPLEX POLYNOMIALS

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Резюме. В роботі розглядається нелінійне інтегральне рівняння типу Гаммерштейна з довільною залежністю від модуля невідомої функції. Розв'язки рівнянь такого типу подаються через поліноми скінчених степенів, параметри яких визначаються із системи, що складається із одного інтегрального і скінченного числа трансцендентних рівнянь. Встановлено існування еквівалентних груп розв'язків нелінійних інтегральних рівнянь, що розглядаються. Одержано необхідні умови для точок галуження і системи рівнянь для їх обчислення. Наведено числові результати для конкретної задачі.

Abstract. An approach, developed before for nonlinear integral Hammerstein equations with the linear dependence on the modulus of unknown function, is generalized to the case of arbitrary differentiable dependency. The approach is based on presentation of the solutions via a complex polynomials of finite degrees. The problem is reduced to a system of integro-transcendental equations. The systems of linear homogeneous equations for the branching points and integro-transcendental equations for the parameters of the solution branches are obtained. Numerical results for a concrete problem are presented.

1. Introduction

Let us consider the nonlinear integral equation of the Hammerstein type

$$\alpha f(\xi) = B[W(|f|)e^{i\arg f}] \equiv \int_a^b K(\xi,\xi')W(|f(\xi')|)\exp(i\arg f(\xi'))d\xi' \qquad (1)$$

with the kernel

$$K(\xi,\xi',c) = \frac{s(\xi)q(\xi') - s(\xi')q(\xi)}{\tau(\xi) - \tau(\xi')} \qquad (2)$$

generated by the linear positive defined integral operator $B : L_2(a,b) \to L_2(a,b)$,

$$(Bg, g) > 0 \qquad (3)$$

for any $g \in L_2(a,b)$;

Key words. Nonlinear integral equation of Hammerstein type, finite-parametric solutions, branching of solutions, phase optimization problem.

$s(\xi)$, $q(\xi)$, $\tau(\xi)$ are real continuous functions such that the function sets $\{\tau^n(\xi)s(\xi)\}$, $\{\tau^n(\xi)q(\xi)\}$ $(n = 0, 1, \ldots)$ are linearly independent;

$W\,(|f(\xi)\,| \in L_2(a,b)$ is a given real piecewise differentiated function.

The general theory of nonlinear integral equations and numerical methods for their solving was intensively developed in recent years (see e.g. [1], [7], [9], [10] and the literature cited there). In previous papers we have considered the nonlinear integral Hammerstein equations without any dependency of the integrand on the modulus of unknown function [11] or with a linear dependency on the modulus [6]. Such types of equations arise in different applications, in particular, in the phase optimization problems of antennas or quasioptical transmitting lines with different restrictions on the solution phase. It was established that the solutions to such equations depend on the finite number of complex parameters which are inverse zeros of polynomials of appropriate degrees (generating polynomials). These parameters are calculated from a system of transcendental equations.

In this paper the approach is generalized to equations with a nonlinear dependence of the integrand on the modulus of unknown function. The results presented here were particularly annonced in [5] and [4].

2. Finite-parametric representation of the solutions

We confine ourselves to the case when the solutions to (1) have no zeros at $\xi \in (a,b)$, and assume that they can be represented in the form

$$f(\xi) = \beta\frac{|f(\xi)|P_N(\tau)}{|P_N(\tau)|}, \tag{4}$$

where β is any complex constant with $|\beta| = 1$ (without loss of generality, we further put $\beta = 1$);

$\tau = \tau(\xi)$, $\tau' = \tau(\xi')$;

$$P_N(\tau) = \prod_{k=1}^{N}(1-\eta_{Nk}\tau) \tag{5}$$

is a polynomial of a finite degree N with complex pairwise non-conjugated zeros η_{Nk}^{-1}:

$$\eta_{Nk} - \bar{\eta}_{Nm} \neq 0, \quad k,m = 1,2,...,N. \tag{6}$$

We call $P_N(\tau)$ as the *generating polynomial.*

It follows from (4) that

$$\exp(i\arg f(\xi)) = \beta\frac{P_N(\tau)}{|P_N(\tau)|}. \tag{7}$$

Introduce the symmetrical polynomial of two real variables

$$R_{N-1}(\tau,\tau') = \frac{2i[P_N(\tau')\bar{P}_N(\tau)]}{(\tau-\tau')} = \sum_{n,m=1}^{N} d_{nm}\tau^{n-1}(\tau')^{m-1} \tag{8}$$

and denote the matrix of its coefficients by $D = \{d_{nm}\}$. The determinant of D equals

$$\det D = (-1)^{[N/2]} \prod_{k,m=1}^{N} (\bar{\eta}_{Nm} - \eta_{Nk}), \tag{9}$$

where the square brackets mean the integer part of the value. This fact follows from the condition 4^0 of the Bezudiant from [8]. Its immediate proof is given in [11]. Due to condition (6), $\det D \neq 0$.

The conditions for the function $f(\xi)$ of the form (4) to be a solution to equation (1) are stated by the following theorem.

Theorem 1. *Let a function $f(\xi)$ of the form (4) have no zeros at $\xi \in [a, b]$. In order that it is a solution to equation (1), it is necessary and sufficient that the parameters η_{Nk} satisfy the following system of the transcendental equations:*

$$|f(\xi)| = \int_a^b K(\xi, \xi')W(|f(\xi')|)\frac{\mathrm{Re}[P_N(\tau')\bar{P}_N(\tau)]}{|P_N(\tau')||P_N(\tau)|}d\xi', \tag{10a}$$

$$\Phi_{Nn}(|f(\xi)|, \eta_{N1}, \eta_{N2,\dots} \eta_{NN}) = 0, \quad n = 1, 2, ..., N, \tag{10b}$$

$$\Psi_{Nn}(|f(\xi)|, \eta_{N1}, \eta_{N2,\dots} \eta_{NN}) = 0, \quad n = 1, 2, ..., N, \tag{10c}$$

where

$$\Phi_{Nn} = \int_a^b \tau^{n-1}s(\xi)\frac{W(|f(\xi)|)}{|P_N(\tau)|}d\xi, \tag{11a}$$

$$\Psi_{Nn} = \int_a^b \tau^{n-1}q(\xi)\frac{W(|f(\xi)|)}{|P_N(\tau)|}d\xi. \tag{11b}$$

Proof. Necessity. Let function (4) be a solution to equation (1). Substituting (4) into (1) and multiplying the both sides of this equality by $\bar{P}_N(\tau)$, we have

$$\alpha\frac{|f(\xi)|\,|P_N(\tau)|^2}{|P_N(\tau)|} = \bar{P}_N(\tau)\int_a^b K(\xi, \xi')W(|f(\xi)|)\frac{P_N(\tau')}{|P_N(\tau')|}d\xi'. \tag{12}$$

After dividing both its sides by $|P_N(\tau)|$ this equation becomes of the form (10a). On the other hand, after taking the imaginary part from the same result, we have

$$\int_a^b \frac{[s(\xi)q(\xi') - s(\xi')q(\xi)]R_{N-1}(\tau, \tau')}{|P_N(\tau')|}W(|f(\xi)|) \equiv 0. \tag{13}$$

Then, substituting (8) into (13) with interchanging the variables ξ and ξ', we have

$$\sum_{n,m=1}^{N} d_{nm}\left[q(\xi')\int_a^b \frac{\tau^{n-1}s(\xi)W(|f(\xi)|)}{|P_N(\tau)|}d\xi - \right.$$
$$\left. - s(\xi')\int_a^b \frac{\tau^{n-1}q(\xi)W(|f(\xi)|)}{|P_N(\tau)|}d\xi\right](\tau')^{m-1} \equiv 0. \tag{14}$$

Since the functions $\{\tau^n s\}$, $\{\tau^n q\}$, $n = 0, \ldots, N-1$, are linearly independent, (14) gives

$$\sum_{n=1}^{N} d_{nm}\int_a^b \frac{\tau^{n-1}s\,(\xi)\,W(|f(\xi)|)}{|P_N(\tau)|}d\xi = 0, \quad n = 1, 2, ..., N, \tag{15a}$$

$$\sum_{n=1}^{N} d_{nm}\int_a^b \frac{\tau^{n-1}q\,(\xi)\,W(|f(\xi)|)}{|P_N(\tau)|}d\xi = 0, \quad n = 1, 2, ..., N. \tag{15b}$$

Equalities (15) can be considered as two independent systems of linear algebraic equations with respect to the unknown integrals. The determinant of their common matrix D does not equal zero owing to conditions (6), so that the systems have only zero solutions, that is, the transcendental equations (10) are satisfied.

*Sufficiency***.** Let (10) hold at a certain integer N and complex η_{Nk}, $k = 1, 2, ..., N$, satisfying conditions (6). Then, of course, equalities (15) are satisfied, too, and, hence, the identities (14) and (13) hold as well. With the aid of (8), we obtain from (13)

$$\mathrm{Im}\left[\bar{P}_N(\tau)\int_a^b K(\xi,\xi')\frac{W(|f(\xi')|)}{|P_N(\tau')|}P_N(\tau')d\xi'\right] = 0 \tag{16}$$

or, after adding the real function $\alpha|f(\xi)|\,|P_N(\tau)|$ under the imaginary sign,

$$\mathrm{Im}\left[\alpha|f(\xi)|\,|P_N(\tau)| + \bar{P}_N(\tau)\int_a^b K(\xi,\xi')\frac{W(|f(\xi')|)}{|P_N(\tau')|}P_N(\tau')d\xi'\right] = 0. \tag{17}$$

Dividing the both sides of (17) by the real positive function $|P_N(\tau)|$, we obtain

$$\mathrm{Im}\left[\alpha|f(\xi)| + \frac{\bar{P}_N(\tau)}{|P_N(\tau)|}\int_a^b K(\xi,\xi')\frac{W(|f(\xi')|)}{|P_N(\tau')|}P_N(\tau')d\xi'\right] = 0. \tag{18}$$

On the other hand, integral equation (10a) can be written in the form

$$\mathrm{Re}\left[\alpha\,|f(\xi)| + \frac{\bar{P}_N(\tau)}{|P_N(\tau)|}\int_a^b K(\xi,\xi')\frac{W(|f(\xi')|)}{|P_N(\tau')|}P_N(\tau')d\xi'\right] = 0. \tag{19}$$

Equalities (18) and (19) together imply that the expression in their square brackets equals zero, that is, function (4) solves integral equation (1).

End of proof.

Theorem 2. *If the function $f(\xi)$ of the form (4) with $\beta = 1$ solves equation (1), then the functions*

$$f_n(\xi) = \frac{|f(\xi)|P_N(\tau)}{|P_N(\tau)|}\frac{1-\bar{\eta}_{Nn}\tau}{1-\eta_{Nn}\tau}, \quad n = 1, 2, ..., N,$$

solve this equation, too.

Proof. The proof of this theorem is analogous to the proof of the Theorem 2.2 in [6] with substitution $W(|f(\xi)|) = F(\xi) - |f(\xi)|$.

In the simplest case, the theorem is complitely adjusted with the obvious property that if the function $f(\xi)$ solves equation (1), then $\bar{f}(\xi)$ solves this equation, too.

Corollary 1. *The solutions to integral equation (10a) and the system of transcendental equations (10b,10c) make up the equivalent groups inside which the function $|f(\xi)|$ remains the same and the polynomials $P_N(\tau)$ differ only by substitution of any number $s < N$ of the parameters η_k by the complex conjugated ones:*

$$P_N^{(s)}(\tau) = \prod_{m=1}^{s}(1-\eta_{n_m}\tau)\prod_{m=s+1}^{N}(1-\bar{\eta}_{n_m}\tau),$$

where $n_{m_1} \neq n_{m_2}$ if $m_1 \neq m_2$. Such polynomials generate the solutions to (1) with the same $|f(\xi)|$.

Corollary 2. *If there is a solution to equation (1) with two parameters $\eta_1 = -\eta_2$ in the polynomial P_N, which give an even polynomial argument addend, then a solution exists in the same equivalent group, which has an odd argument. In particular, if all parameters of the polynomial P_N can be devided into such symmetrical pairs, what means that the polynomial argument is an even function, then another solution exists in the same equivalent group, which have an odd argument.*

This corollary is justyfied by the following logical considerations. The argument of the factor $p_1(\tau) = (1-\eta_1\tau)(1-\eta_2\tau) = 1-\eta_1^2\tau^2$ is obviously the even function of τ. Substituting η_2 with $\overline{\eta}_2$ according to above theorem gives the factor $p_2(\tau) = (1-\eta_1\tau)(1+\overline{\eta}_1\tau) = 1-|\eta_1|^2\tau^2-(\eta_1-\overline{\eta}_1)\tau$. Its argument is :

$$\arg p_2 = \arctan\frac{2\mathrm{Im}\eta_1\tau}{1-|\eta_1|^2\tau^2}.$$

If N is even integer and $P_N(\tau) = \prod\limits_{n=1}^{N/2}(1-\eta_n^2\tau^2)$, then

$$\widetilde{P}_N(\tau) = \prod_{n=1}^{N/2}(1-|\eta_n|^2\tau^2-(\eta_n-\overline{\eta}_n)\tau)$$

and its argument is

$$\arg \widetilde{P}_N(\tau) = \sum_{n=1}^{N/2} \arctan \frac{2\mathrm{Im}\eta_n \tau}{1 - |\eta_n|^2 \tau^2},$$

which is the odd function of τ.

3. Branching of solutions

For $N = 0$ (real positive solutions) the transcendental equations (10b, 10c) disappear and the only integral equation (10a) remains, which coincides with (1), in which $f(\xi)$ must be substituted by $|f(\xi)|$. This equation has the non-trivial solution but not for all values c and N.

The number of solutions to (1) may change at some values $c = c_j$. Such values are called the branching points. The branching points of solutions to equation (1) are found from the condition that the system of the homogeneous integral equations

$$\begin{aligned}\lambda_n w_n |f| &= B\left[W(|f|)\frac{\mathrm{Im}\left(\bar{P}_N(\tau')P_N(\tau)\right)}{|P_N(\tau')|\,|P_N(\tau)|}v_n + \right. && \text{(20a)}\\ &\left. + W'(|f|)|f|\frac{\mathrm{Re}\left(\bar{P}_N(\tau')P_N(\tau)\right)}{|P_N(\tau')|\,|P_N(\tau)|}w_n\right],\\ \lambda_n v_n |f| &= B\left[W(|f|)\frac{\mathrm{Re}\left(\bar{P}_N(\tau)P_N(\tau')\right)}{|P_N(\tau')|\,|P_N(\tau)|}v_n + \right. && \text{(20b)}\\ &\left. + W'(|f|)|f|\frac{\mathrm{Im}\left(\bar{P}_N(\tau)P_N(\tau')\right)}{|P_N(\tau')|\,|P_N(\tau)|}w_n\right]\end{aligned}$$

has multiple eigenvalues $\lambda_n = 1$. Here $\{w_n, v_n\}$ are vector-functions; $W' = dW/d(|f|)$. It is easy to check that $\lambda_1 = 1$, $\{v_1 \equiv 1,\ w_1 \equiv 0\}$ is always the eigenpair of (20). These equations are obtained by application of usual pertrubations technique to equation (1) (see e.g. [6]).

There is an obvious way to obtain the transcendental equation system for calculation of the branching points and the polynomial parameters in them. As a rule, the branching of solutions to equation (1) is caused by changing the degree N of the polynomial P_N by one. At the branching points the parameters η_{Nk} of the initial polynomial P_N and parameters $\eta_{N+1,k}$ of the branched polynomial P_{N+1} are connected by the equalities

$$\begin{gathered}\frac{P_N(\tau)}{|P_N(\tau)|} = \frac{P_{N+1}(\tau)}{|P_{N+1}(\tau)|}, \quad \eta_{Nk} = \eta_{N+1,k}, \qquad k = 1, 2, \ldots, N,\\ \mathrm{Im}\eta_{N+1,N+1} = 0.\end{gathered} \tag{21}$$

At the branching points two new unknown c_0 and $\mathrm{Re}\eta_{N+1,N+1}$ occur. Besides (10), system

$$|f(\xi)| = \int_a^b K(\xi,\xi')W(|f(\xi')|)\frac{\mathrm{Re}[P_{N+1}(\tau')\bar{P}_{N+1}(\tau)]}{|P_{N+1}(\tau')||P_{N+1}(\tau)|}d\xi', \tag{22a}$$

$$\int_a^b \tau^{n-1}s(\xi)\frac{W(|f(\xi)|)}{|P_{N+1}(\tau)|}d\xi = 0, \quad n = 1,2,...,N+1, \tag{22b}$$

$$\int_a^b \tau^{n-1}s(\xi)\frac{W(|f(\xi)|)}{|P_{N+1}(\tau)|}d\xi = 0, \quad n = 1,2,...,N+1 \tag{22c}$$

should hold. Since the new parameter $\eta_{N+1,N+1}$ is real, the integral equation (22a) coincides with (10a) and the kth equation of system (22b) (22c) $k = 1,2,\ldots,N$, is a linear combination of the corresponding equation of system (10b), (10c) and $(k+1)$th equation of (22b) (22c). Hence, at the branching point, besides system (10) only two additional equations

$$\int_a^b \tau^N s(\xi)\frac{F(\xi)-\beta|f(\xi)|}{|P_N(\tau)|(1-\eta_{N+1,N+1}\tau)}d\xi = 0, \tag{23a}$$

$$\int_a^b \tau^N q(\xi)\frac{F(\xi)-\beta|f(\xi)|}{|P_N(\tau)|(1-\eta_{N+1,N+1}\tau)}d\xi = 0 \tag{23b}$$

should hold. On the whole, we have one real integral equation and $2N+2$ transcendental ones for determining the real function $|f(\xi)|$, N complex parameters η_{Nk}, $k = 1,2,\ldots,N$ and real $\eta_{N+1,N+1}$ and c_j.

At the branching points where the polynomial degree changes by two, the equalities

$$\eta_{Nk} = \eta_{N+2,k}, k = 1,\ldots,N \tag{24}$$

are valid. Besides (10), the four additional equations

$$\begin{aligned} &\int_a^b \frac{\tau^{k-1}s(\xi)W(|f(\xi)|)}{|P_N(\tau)|(1-\eta_{N+2,N+1}\tau)(1-\eta_{N+2,N+2}\tau)}d\xi=0,\ n = N+1, N+2; \\ &\int_a^b \frac{\tau^{k-1}q(\xi)W(|f(\xi)|)}{|P_N(\tau)|(1-\eta_{N+2,N+1}\tau)(1-\eta_{N+2,N+2}\tau)}d\xi=0,\ n = N+1, N+2, \end{aligned} \tag{25}$$

should be fulfilled with $\eta_{N+2,N+1}$, $\eta_{N+2,N+2}$ satisfying the conditions

$$\eta_{N+2,N+1} = \bar{\eta}_{N+2,N+2} \tag{26}$$

or

$$\mathrm{Im}\eta_{N+2,N+1} = \mathrm{Im}\eta_{N+2,N+2} = 0. \tag{27}$$

Hence, we have $2N+5$ equation for $2N+4$ real unknown: N complex η_{Nk}, $n = 1,2,\ldots,N$, one real c_j, one complex $\eta_{N+2,N+1}$ or two real $\eta_{N+2,N+1}$,

$\eta_{N+2,N+2}$, and $|f(\xi)|$. As it was mentioned in the preceding subsection, the existence of solutions to such a system is low-probable in general case. However, they may exist in the case when

$$W(|f(\xi)|) = W(|f(-\xi)|). \tag{28}$$

Then the solutions are possible, which generate the polynomials with the even modulus

$$|P_N(\tau)| = |P_N(-\tau)|, \qquad |P_{N+2}(\tau)| = |P_{N+2}(-\tau)|. \tag{29}$$

This equality decreases the number of unknowns twice: the parameters $\eta_{N+2,k}$ become imaginary or appear by couples with opposite signs and $\eta_{N+2,n}$, $n = N+1, N+2$ are always imaginary with opposite signs:

$$\mathrm{Re}\eta_{N+2,N+1} = \mathrm{Re}\eta_{N+2,N+2} = 0, \tag{30a}$$

$$\eta_{N+2,N+1} = \bar{\eta}_{N+2,N+2}. \tag{30b}$$

On the other hand, conditions (29) decrease the number of equations twice, as well: N equations of system (10b), (10c) and two additional equations (25) become identities, because they have odd integrands in the left-hand side.

Finally, at fulfilling (28), (29) the solution branching is possible with decreasing the polynomial degree by two if the following transcendental equation system holds:

$$\int_a^b \tau^{2n-1} s(\xi) \frac{W(|f(\xi)|)}{|P_N(\tau)|} d\xi = 0, \quad n = 1, 2, \ldots [N/2], \tag{31a}$$

$$\int_a^b \tau^{2n-2} q(\xi) \frac{W(|f(\xi)|)}{|P_N(\tau)|} d\xi = 0, \quad n = 1, 2, \ldots [(N+1)/2], \tag{31b}$$

$$\int_a^b \frac{\tau^{2[(N+2)/2]-1} s(\xi) W(|f(\xi)|)}{|P_N(\tau)|(1-\eta_{N+2,N+1}\tau)(1-\eta_{N+2,N+2}\tau)} d\xi = 0, \tag{31c}$$

$$\int_a^b \frac{\tau^{2[(N+1)/2]} q(\xi) W(|f(\xi)|)}{|P_N(\tau)|(1-\eta_{N+2,N+1}\tau)(1-\eta_{N+2,N+2}\tau)} d\xi = 0, \tag{31d}$$

where η_{Nk}, $k = 1, \ldots, N$, are either imaginary or appear by couples with alternative signs, and $\eta_{N+2,k}$, $k = N+1, N+2$ are subject to conditions (30). As a result, we have $N+3$ real equations with respect to $N+3$ real unknowns.

4. Numerical results

As an example, we show the numerical results obtained for $W(|f(\xi)|) = 1/2 - |f(\xi)|^2$ and $\alpha = 0.5$. This problem arises in the case when the linear antenna should create the uniform power pattern $F^2 \equiv 1/2$. The calculations were carried out by the Newton method.

The real and imaginary parts of η_{Nk} are shown in Fig. 1. The real parts of solutions are drawn by the dashed lines, the imaginary ones – by the solid lines. The curve numbering corresponds to the indexes Nk at these parameters.

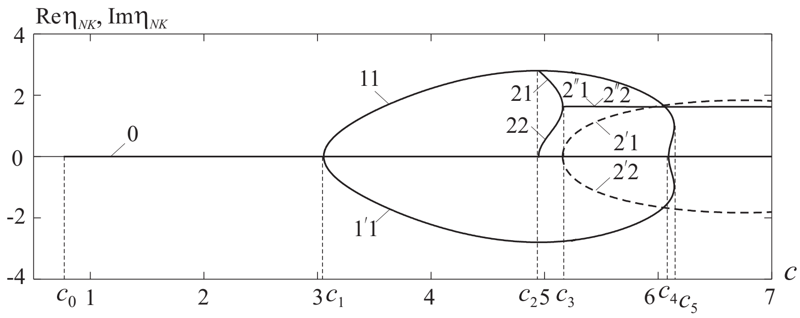

FIG. 1. Real and imag parts of parametrs η_{Nk};
$W\left(|f(\xi)|\right) = 1/2 - |f(\xi)|^2$, $\alpha = 0.5$

For $c < c_0 = 0.84$ there are no nontrivial solutions to equation (1) at this α. At $c = c_0$ the solution $f_0(\xi)$ with $N = 0$ arises (curve 0). It starts from $f_0(\xi) \equiv 0$.

At the point $c_1 = 3.05$ two complex conjugate solutions $f_1(\xi)$, $f_{1'}(\xi)$ with $N = 1$ and imaginary η_{11}, $\eta_{1'1}$ respectively, branch off from $f_0(\xi)$ (curves 11, $1'1$). At the point $c_2 = 4.95$, two solutions with $N = 2$ branch off from each solution with $N = 1$. All they make up an equivalent group; we analyze only one of them denoted by $f_2(\xi)$. The solutions $f_1(\xi)$, $f_{1'}(\xi)$ continue to exist. Two more characteristic points, related to them, are c_4 and c_5.

The solution $f_2(\xi)$, arising at $c = c_2$ has two imaginary parameters η_{21}, η_{22} (curves $21, 22$). At $c_3 = 5.16$ the solution $f_2(\xi)$ transforms into $f_{2'}(\xi)$, which has two complex parameters η'_{21}, η'_{22} with $\mathrm{Re}\eta'_{22} = -\mathrm{Re}\eta'_{21}$, $\mathrm{Im}\eta'_{21} = \mathrm{Im}\eta'_{22}$. Curves $2'1$, $2''1$, correspond to $\mathrm{Re}\eta'_{21}$, $\mathrm{Im}\eta'_{21}$ and curves $2'2$, $2''2$, – to $\mathrm{Re}\eta'_{22}$, $\mathrm{Im}\eta'_{22}$, respectively.

When c increases, the solutions with larger N appear, similarly as in the problem of antenna synthesis according to the amplitude pattern [3].

BIBLIOGRAPHY

1. Bauschke H. H. Phase retrieval, error reduction algorithm, and Fienup variants: A view from convex optimization / H. H. Bauschke, P. L. Combettes, D. R. Luke // J. of Optical Society of America. – 2002. – Vol. 19, No 7. – P. 1334-13345.
2. Bulatsyk O. O. Phase Optimization Problems / O. O. Bulatsyk, B. Z. Katsenelenbaum, Yu. P. Topolyuk, N. N. Voitovich. – Weinheim: WILEY-VCH, 2010. – 301 p.
3. Bulatsyk O. O. Properties of nonlinear Hammerstein integral equations connected with modified phase problem / O. O. Bulatsyk, N. N. Voitovich // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2003). – Proc. of Int. Seminar/Workshop, Lviv. – 2003. – P. 135-138.
4. Bulatsyk O. O. Complex polynomials representation of solutions to the antenna synthesis problem by the power pattern / O. O. Bulatsyk // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2013). – Proc. of Int. Seminar/Workshop, Lviv-Tbilisi. – 2013. – P. 166-170.

5. Bulatsyk O. Extension of the class of nonlinear integral equations which permit to present solutions by the complex polynomials / O. Bulatsyk // Modern Problems of Mechanics and Mathematics. Lviv, IAPMM NASU. – 2013. – Vol. 3. – P. 16-18.
6. Bulatsyk O. O. Finite-parametric solutions of a class of Hammerstein nonlinear integral equations / O. O. Bulatsyk, Yu. P. Topolyuk, N. N. Voitovich // J. Numer Appl. Math. – 2010. – Vol. 1 (100). – P. 10-28.
7. Fienup J. R. Phase retrieval algorithm: a comparison 2. / J. R. Fienup // Appl. Opt. – 1982. – Vol. 21. – P. 2758-2769.
8. Lander F. J. The Bezoitiant and inversion of Hankel and Toeplitz matrices / F. J. Lander // Matematicheskie Issledovaniya. – 1974. – Vol. 2. – P. 69-87 (in Russian).
9. Masujima M. Applied Mathematical Methods in Theoretical Physics / M. Masujima. – Weinheim: WILEY- VCH, 2005. – 377 p.
10. Precup R. Methods in Nonlinear Integral Equation / R. Precup. – Dordrecht: Kluver Acad. Publ, 2002. – 232 p.
11. Voitovich N. N. Approximation of compactly supported functions with free phase by functions with bounded spectrum / N. N. Voitovich, Y. P. Topolyuk, O. O. Reshnyak // Fields Institute Communications. – 2000. – Vol. 25. – P. 531-541.

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