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INTERPOLATING FUNCTIONAL POLYNOMIAL FOR THE APPROXIMATE SOLUTION OF THE BOUNDARY VALUE PROBLEM

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Резюме. У роботі, застосовуючи функціональний поліном Ньютона побудований на континуальній множині вузлів, будується інтерполяційний функціональний поліном n-го порядку для наближення до розв'язку крайової задачі другого порядку.

Abstract. Interpolating functional polynomial of order for the approximation to the solution of the boundary value problem of the second order is constructed and justified in this paper. This is done using Newton functional polynomial constructed on a continual set of knots.

1. Introduction

Many authors investigated the generalization of the classical theory of one variable functions interpolation to the case of nonlinear functionals and operators (see for example [1, 2, 3, 4, 5, 6, 7, 8]). In particular, in [9] it is suggested to seek for Newton-type interpolants in the class of functional polynomials of the following form

$$P_n(x(\cdot)) = K_0 + \\ + \sum_{s=1}^{n} \int_0^1 \int_{z_1}^1 \cdots \int_{z_{s-1}}^1 K_s(\overrightarrow{z}^s) \prod_{i=1}^{s} [x(z_i) - x_{i-1}(z_i)]\, dz_s \dots dz_1, \tag{1}$$

where $x_i\left(z\right) \in Q[0,1]$, $i = 0, 1, \dots$ are arbitrary, fixed elements from the space $Q[0,1]$. Which is a space of piecewise continuous on the interval $[0,1]$ functions with a finite number of discontinuity points of the first kind. For determination of the kernels K_0, $K_s\left(\overrightarrow{z}^s\right)$, $s = \overline{1,n}$ a following continual set of knots

$$x^n\left(z, \vec{\xi}^n\right) = x_0\left(z\right) + \sum_{i=1}^{n} H\left(z - \xi_i\right)\left[x_i\left(z\right) - x_{i-1}\left(z\right)\right], \quad z \in [0,1], \tag{2}$$

$$\vec{\xi}^n = (\xi_1, \xi_2, \dots, \xi_n) \in \overline{\Omega}_n = \\ = \left\{\vec{z}^n = (z_1, z_2, \dots, z_n) : 0 \le z_1 \le z_2 \le \dots \le z_n \le 1\right\},$$

was introduced and continual interpolation conditions of the form

Key words. Newton's functional polynomial, continual set of nodes, boundary value problem, interpolating functional polynomial.

$$P_n^I\left(x^n\left(\cdot,\vec{\xi^n}\right)\right)=F\left(x^n\left(\cdot,\vec{\xi^n}\right)\right)\ ,\quad \forall\vec{\xi^n}\in\overline{\Omega}_n,$$

were set, where $H(z)$ is a Heaviside function.

In the above-mentioned work, it was shown that the necessary conditions for polynomial (1) to be interpolating on the continual knots (2) are the determination of its kernels according to the following formulas

$$K_0=F\left(x_0\left(\cdot\right)\right),$$

$$K_s\left(\vec{z^s}\right)=(-1)^s\prod_{i=1}^{s}\left[x_i\left(z_i\right)-x_{i-1}\left(z_i\right)\right]^{-1}\frac{\partial^s}{\partial z_1\ldots\partial z_s}F\left(x^s\left(\cdot,\vec{z^s}\right)\right)\ ,$$
$$s=\overline{1,n}\,.$$

To ensure sufficient condition for polynomial $P_n\left(x\left(\cdot\right)\right)$ to be interpolating on continual knots (2) the following substitution rules satisfaction

$$\begin{gathered}\frac{\partial^p}{\partial z_1\partial z_2\ldots\partial z_p}\left[F\left(x^{p+1}\left(\cdot,\vec{z}^{p+1}\right)\right)\Big|_{z_{p+1}=z_p}\right]=\\=\left[\frac{\partial^p}{\partial z_1\partial z_2\ldots\partial z_p}F\left(x^{p+1}\left(\cdot,\vec{z}^{p+1}\right)\right)\right]\Bigg|_{z_{p+1}=z_p}\frac{x_{p+1}\left(z_p\right)-x_{p-1}\left(z_p\right)}{x_p\left(z_p\right)-x_{p-1}\left(z_p\right)}\ ,\\ p=\overline{1,n-1}\end{gathered}\tag{3}$$

were required.

The purpose of this paper is to develop and study the interpolating functional polynomial for approximation of the solution of the second order boundary value problem.

2. Statement of the problem

One must apply the Newton type functional polynomial of the form (1), (2) and construct the approximation to the solution of the following boundary value problem.

$$U''\left(x;q\left(\cdot\right)\right)-q\left(x\right)U\left(x;q\left(\cdot\right)\right)=-f\left(x\right),\qquad x\in\left(0,1\right),\tag{4}$$

$$U\left(0;q\left(\cdot\right)\right)=0,\qquad U\left(1;q\left(\cdot\right)\right)=0.\tag{5}$$

3. Solution of the problem

When the function $f(x)$ is fixed, one can consider solution of the problem (4), (5) as non-linear operator with respect to $q(x)$. We introduce the following continual interpolating knots

$$q^n\left(x,\overrightarrow{\xi}^n\right)=\sum_{i=1}^{n}\frac{1}{n}H\left(x-\xi_i\right),\tag{6}$$

where

$$0\le\xi_1\le\xi_2\le\ldots\le\xi_n\le1\tag{7}$$

and the frame of these knots are

$$q_i(x) = \frac{i}{n}, \quad i = \overline{0,n}.$$

Let us write the following $n-$ degree interpolating functional polynomial of Newton type

$$U_n(x; q(\cdot)) = \sum_{i=0}^{n} \int_0^1 \int_{z_1}^1 \dots \int_{z_{i-1}}^1 K_i(x; q(\cdot)) \prod_{p=1}^{i} n\left(q(z_p) - \frac{p}{n}\right) d\overrightarrow{z}_p, \tag{8}$$

where

$$K_i(x; q(\cdot)) = (-1)^i \frac{\partial^i}{\partial z_1 \dots \partial z_i} U\left(x; q^i\left(x; \overrightarrow{z}^i\right)\right), \tag{9}$$

$$i = \overline{1,n}, \qquad K_0(x; q(\cdot)) = U(x; 0).$$

According to Theorem 2.1 from [9] the necessary and sufficient condition for polynomial (8), (9) to be interpolating for solution of the boundary problem (4), (5) on a continual set of interpolating knots (6), (7), i.e. the following conditions were met

$$U\left(x; q^n\left(\cdot, \overrightarrow{\xi}^n\right)\right) = U_n\left(x; q^n\left(\cdot, \overrightarrow{\xi}^n\right)\right), \quad \forall\, \overrightarrow{\xi}^n \in \Omega_n, \tag{10}$$

is the following substitution rules to be applicable

$$\left[\frac{\partial}{\partial \xi_{i-1}} U\left(x; q^i\left(\cdot, \overrightarrow{\xi}^i\right)\right)\right]_{\xi_i = \xi_{i-1}} = \frac{1}{2} \frac{\partial}{\partial \xi_{i-1}} U\left(x; q^i\left(\cdot; \overrightarrow{\xi}^i\Big|_{\xi_i = \xi_{i-1}}\right)\right), \quad i = \overline{2,n}. \tag{11}$$

The following statement is fulfilled.

Lemma 1. *Let the solution of boundary value problem (4), (5) be considered as non-linear operator with respect to $q(x)$. Then it satisfies the substitution rule (11).*

Proof. Consider the following boundary problem

$$U''\left(x; q^i\left(\cdot; \overrightarrow{\xi}^i\right)\right) - \sum_{p=1}^{i} \frac{1}{n} H(x - \xi_p)\, U\left(x; q^i\left(x; \overrightarrow{\xi}^i\right)\right) = -f(x), \quad x \in (0,1), \tag{12}$$

$$U\left(0; q^i\left(\cdot; \overrightarrow{\xi}^i\right)\right) = 0, \qquad U\left(1; q^i\left(\cdot; \overrightarrow{\xi}^i\right)\right) = 0. \tag{13}$$

As consequences from (12), (13) we have following two boundary value problems with the same differential operator

$$\frac{d^2}{dx^2}\left[\frac{\partial}{\partial \xi_{i-1}} U\left(x; q^i\left(\cdot; \overrightarrow{\xi}^i\right)\right)\right]_{\xi_i = \xi_{i-1}} -$$

$$-\sum_{p=1}^{i}\frac{1}{n}\,H\left(x-\xi_p\right)\big|_{\xi_i=\xi_{i-1}}\left[\frac{\partial}{\partial\xi_{i-1}}U\left(x;q^i\left(\cdot;\overrightarrow{\xi}^{\,i}\right)\right)\right]_{\xi_i=\xi_{i-1}}= \tag{14}$$

$$=\frac{1}{n}\frac{d}{d\xi_{i-1}}\,H\left(x-\xi_{i-1}\right)U\left(x;q^i\left(\cdot,\overrightarrow{\xi}^{\,i}\right)\right)\Big|_{\xi_i=\xi_{i-1}},$$

$$\left[\frac{\partial}{\partial\xi_{i-1}}U\left(x;q^i\left(\cdot;\overrightarrow{\xi}^{\,i}\right)\right)\right]_{\xi_i=\xi_{i-1}}\Bigg|_{x=0,1}=0, \tag{15}$$

$$\frac{d^2}{dx^2}\left[\frac{\partial}{\partial\xi_{i-1}}U\left(x;q^i\left(\cdot;\overrightarrow{\xi}^{\,i}\right)\right)\right]_{\xi_i=\xi_{i-1}}-$$

$$-\sum_{p=1}^{i}\frac{1}{n}\,H\left(x-\xi_p\right)\big|_{\xi_i=\xi_{i-1}}\frac{\partial}{\partial\xi_{i-1}}U\left(x;q^i\left(\cdot;\overrightarrow{\xi}^{\,i}\Big|_{\xi_i=\xi_{i-1}}\right)\right)= \tag{16}$$

$$=\frac{2}{n}\frac{d}{d\xi_{i-1}}H\left(x-\xi_{i-1}\right)U\left(x;q^i\left(\cdot,\overrightarrow{\xi}^{\,i}\Big|_{\xi_i=\xi_{i-1}}\right)\right),$$

$$\frac{\partial}{\partial\xi_{i-1}}U\left(x;q^i\left(\cdot;\overrightarrow{\xi}^{\,i}\Big|_{\xi_i=\xi_{i-1}}\right)\right)\Bigg|_{x=0,1}=0. \tag{17}$$

Note that right hand sides of their differential equations differ only by numerical multiplier. Comparison of boundary value problems (14), (15) and (16), (17) proves the lemma.

To construct the interpolant (8), (9) one must find the solution of the problems (12), (13) at $i=\overline{0,n}$. Then we have

$$U\left(x;0\right)=K_0\left(x;q\left(\cdot\right)\right)=\int_0^1 G_0\left(x,\xi\right)f\left(\xi\right)d\xi,$$

$$U\left(x;q^i\left(\cdot;\overrightarrow{\xi}^{\,i}\right)\right)=\int_0^1 G_i\left(x,\xi\right)f\left(\xi\right)d\xi,\qquad i=\overline{1,n},$$

where $G_i\left(x,\xi\right)$, $i=\overline{0,n}$ are Green's functions of the corresponding boundary value problems

$$G_0\left(x,\xi\right)=\begin{cases} x\left(1-\xi\right), & 0\le x\le\xi,\\ \xi\left(1-x\right), & \xi\le x\le 1,\end{cases}$$

.

$$G_i\left(x,\xi\right)=\frac{1}{V_{1,i}\left(1\right)}\begin{cases} V_{1,i}\left(x\right)V_{2,i}\left(\xi\right), & 0\le x\le\xi,\\ V_{2,i}\left(x\right)V_{1,i}\left(\xi\right), & \xi\le x\le 1.\end{cases}$$

Here $V_{1,i}\left(x\right)$, $V_{2,i}\left(x\right)$ are solutions of the following Cauchy problems:

$$\frac{d^2V_{\alpha i}\left(x\right)}{dx^2}-\sum_{p=1}^{i}\frac{1}{n}H\left(x-\xi_p\right)V_{\alpha i}\left(x\right)=0,\qquad x\in\left(0,1\right),\qquad \alpha=1,2;$$

$$V_{1i}\left(0\right)=0;\qquad \frac{dV_{1i}\left(0\right)}{dx}=1;\qquad V_{2i}\left(1\right)=0;\qquad \frac{dV_{2i}\left(1\right)}{dx}=-1.$$

It is quite simple to find functions $V_{1,i}(x)$, $V_{2,i}(x)$ in explicit form because the differential equations which they satisfy have a piecewise constant coefficient. In particular at $i = 1$ we obtain

$$V_{11}(x) = \begin{cases} x, & 0 \le x \le \xi_1, \\ \sqrt{n}\sinh\dfrac{1}{\sqrt{n}}(x-\xi_1) + x\cosh\dfrac{1}{\sqrt{n}}(x-\xi_1), & \xi_1 \le x \le 1, \end{cases}$$

$$V_{21}(x) = \begin{cases} \sqrt{n}\sinh\dfrac{1}{\sqrt{n}}(1-x), & \xi_1 \le x \le 1, \\ -\cosh\dfrac{1}{\sqrt{n}}(1-\xi_1)(x-\xi_1) + \sqrt{n}\sinh\dfrac{(1-\xi_1)}{\sqrt{n}}, & 0 \le x \le \xi_1. \end{cases}$$

4. Conclusions

Thus, Newton type interpolating functional polynomial of $n-$degree of form (8), (9) was obtained. This polynomial will be the approximation to the solution of the boundary value problem (4), (5).

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