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IMPLICIT ITERATION METHOD OF SOLVING LINEAR EQUATIONS WITH APPROXIMATING RIGHT-HAND MEMBER AND APPROXIMATELY SPECIFIED OPERATOR

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Резюме. У гільбертовому просторі досліджується неявний метод ітерацій розв'язування лінійних рівнянь з ненегативним самоспряженим і несамоспряженим обмеженим оператором. Доведено збіжність методу у випадку апріорного вибору числа ітерацій у вихідній нормі гільбертового простору, в припущенні, що похибки є не тільки в правій частині рівняння, а й в операторі. Отримано оцінки похибки і апріорний момент зупинки.

Abstract. The article deals with the study of the implicit method of solving linear equations with nonnegative self-adjoint and nonself-adjoint limited operator in Hilbert space. It aims at proving the method convergence in case of a priori choice of the number of iterations in the basic norm of Hilbert space on the assumption of existing errors not only in the equation right-hand member but in the operator as well. Error estimation and a priori stop moment are obtained.

1. Problem statement

Let H and F be Hilbert spaces and $A \in \pounds(H, F)$, i. e. A is a linear continuous operator functioning from H to F. It is assumed that zero belongs to operator spectrum A, but it is not its characteristic constant. The following equation is solved

$$Ax = y. \tag{1}$$

The problem of searching for element $x \in H$ by element $y \in F$ is incorrect, for arbitrary small disturbances in the right-hand member y may result in arbitrary disturbances in solution.

Let us suppose that the accurate development $x^* \in H$ of equation (1) exists and is the unique one. We shall search for it with the help of iteration process

$$(E + \alpha^2 A^{2k})x_{n+1} = (E - \alpha A^k)^2 x_n + 2\alpha A^{k-1} y, x_0 = 0, k \in N, \tag{2}$$

where E is an identity operator while α is an iteration parameter.

We consider that operator A and the right-hand member of equation (1) are specified approximately, i.e. approximation $y_\delta, \|y - y_\delta\| \le \delta$ is known instead of y, and operator $A_\eta, \|A - A_\eta\| \le \eta$ is known instead of operator A. Suppose $0 \in Sp(A_\eta), Sp(A_\eta) \subseteq [0, M]$. Then method (2) will look

$$(E + \alpha^2 A_\eta^{2k})x_{n+1} = (E - \alpha A_\eta^k)^2 x_n + 2\alpha A_\eta^{k-1} y_\delta, x_0 = 0, k \in N. \tag{3}$$

Key words. Regularization, iteration method, incorrect problem, Hilbert space, self-conjugated and non self-conjugated approximately operator.

The case of approximate right-member of equation y_δ and faithful operator A for the method under consideration (3) has been studied in monograph [1]. It deals with a priori and a posteriori choice of a regularization parameter and the case of non-unique solution of problem (1), as well as with proving the method convergence in Hilbert space energy norm.

Let us prove the method convergence (3) in case of a priori choice of a regularization parameter in solving the equation $A_\eta x = y_\delta$ with the approximate operator A_η and the approximate right-hand member y_δ and obtain a priori estimated errors.

2. The case of self-adjoint nonnegative operators

Let H equal F, $A = A^* \geq 0$, $A_\eta = A_\eta^* \geq 0$, $Sp(A_\eta) \subseteq [0, M]$, $0 < \eta \leq \eta_0$. The iteration method (3) will be presented in the following way:

$$x_\eta = g_n(A_\eta) y_\delta, \tag{4}$$

where $g_n(\lambda) = \lambda^{-1}\left[1 - \dfrac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n}\right]$. There have been obtained in [1-2] the conditions for functions $g_n(\lambda)$ with $\alpha > 0$:

$$\sup_{0\leq\lambda\leq M} |g_n(\lambda)| \leq \gamma n^{1/k}, \gamma = 2k\alpha^{1/k}, n > 0, \tag{5}$$

$$\sup_{0\leq\lambda\leq M} \lambda^s |1 - \lambda g_n(\lambda)| \leq \gamma_s n^{-s/k}, (n > 0), 0 < s < \infty, \gamma_s = \left(\frac{s}{2k\alpha e}\right)^{s/k}, \tag{6}$$

(here s is the degree of source representability of exact solution $x^* = A^s z, s > 0, \|z\| \leq \rho$),

$$\sup_{0\leq\lambda\leq M} |1 - \lambda g_n(\lambda)| \leq \gamma_0, \gamma_0 = 1, n > 0, \tag{7}$$

$$\sup_{0\leq\lambda\leq M} \lambda |1 - \lambda g_n(\lambda)| \to 0, n \to \infty. \tag{8}$$

The following is valid:

Lemma 1. *Let $A = A^* \geq 0$, $A_\eta = A_\eta^* \geq 0$, $\|A_\eta - A\| \leq \eta$, $Sp(A_\eta) \subseteq [0, M]$, $(0 < \eta \leq \eta_0)$, $\alpha > 0$ and conditions (7), (8) be satisfied. Then $\|G_{n\eta} v\| \to 0$ at $n \to \infty$, $\eta \to 0$ $\forall v \in N(A)^\perp = \overline{R(A)}$, where $N(A) = \{x \in H | Ax = 0\}$ and $G_{n\eta} = E - A_\eta g_n(A_\eta)$.*

Proof. We have

$$\begin{aligned}
\|G_{n\eta} v\| &= \|(E - A_\eta g_n(A_\eta))v\| = \\
&= \left\| \int_0^M (1 - \lambda g_n(\lambda)) dE_\lambda v \right\| = \left\| \int_0^M \frac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n} dE_\lambda v \right\| \leq \\
&\leq \left\| \int_0^\varepsilon \frac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n} dE_\lambda v \right\| + \left\| \int_\varepsilon^M \frac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n} dE_\lambda v \right\|.
\end{aligned}$$

$$\left\| \int_{\varepsilon}^{M} \frac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n} dE_\lambda v \right\| \leq q^n(\varepsilon) \left\| \int_{\varepsilon}^{M} dE_\lambda v \right\| \to 0, n \to \infty,$$

as for $\lambda \in [\varepsilon, M]$

$$\frac{(1-\alpha\lambda^k)^2}{(1+\alpha^2\lambda^{2k})^n} \leq q(\varepsilon) < 1.$$

$$\left\| \int_{0}^{\varepsilon} \frac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n} dE_\lambda v \right\| \leq \left\| \int_{0}^{\varepsilon} dE_\lambda v \right\| = \|E_\varepsilon v\| \to 0, \quad \varepsilon \to 0$$

owing to integrated spectrum properties [3-4]. Consequently, $\|G_{n\eta}v\| \to 0$ at $n \to \infty$, $\eta \to 0$. Lemma 1 is proved. □

The convergence condition for method (3) is given by

Theorem 1. *Let* $A = A^* \geq 0$, $A_\eta = A_\eta^* \geq 0$, $\|A_\eta - A\| \leq \eta$, $Sp(A_\eta) \subseteq [0, M]$, $(0 < \eta \leq \eta_0)$, $\alpha > 0$, $y \in R(A)$, $\|y - y_\delta\| \leq \delta$ *and conditions* (5), (7), (8) *be satisfied. Let us choose parameter* $n = n(\delta, \eta)$ *in approximation* (3) *so that* $(\delta + \eta)n^{1/k}(\delta, \eta) \to 0$ *at* $n(\delta, \eta) \to \infty$, $\delta \to 0$, $\eta \to 0$. *Then* $x_{n(\delta,\eta)} \to x^*$ *at* $\delta \to 0$, $\eta \to 0$.

Proof. According to (4) we have $x_n = g_n(A_\eta)y_\delta$. Then

$$x_n - x^* = g_n(A_\eta)y_\delta - x^* = -G_{n\eta}x^* + G_{n\eta}x^* + g_n(A_\eta)y_\delta - x^* =$$

$$= -G_{n\eta}x^* + (E - A_\eta g_n(A_\eta))x^* + g_n(A_\eta)y_\delta - x^* = -G_{n\eta}x^* + g_n(A_\eta)(y_\delta - A_\eta x^*).$$

Condition (5) being as follows $\|g_n(A_\eta)\| \leq \sup\limits_{0\leq\lambda\leq M} |g_n(\lambda)| \leq \gamma n^{1/k}$, then

$$\|y_\delta - A_\eta x^*\| \leq \|y_\delta - y\| + \|y - A_\eta x^*\| =$$

$$= \|y_\delta - y\| + \|Ax^* - A_\eta x^*\| \leq \delta + \|A - A_\eta\|\|x^*\| \leq \delta + \eta\|x^*\|.$$

Consequently,

$$\|x_{n(\delta,\eta)} - x^*\| \leq \|G_{n\eta}x^*\| + \|g_n(A_\eta)(y_\delta - A_\eta x^*)\| \leq \|G_{n\eta}x^*\| + \gamma n^{1/k}(\delta + \eta\|x^*\|).$$

As appears from Lemma 1, $\|G_{n\eta}x^*\| \to 0$ at $n \to \infty$, $\eta \to 0$, and according to the condition of Theorem 1, $n^{1/k}(\delta + \eta) \to 0$ at $\delta \to 0$, $\eta \to 0$. Thus, $\|x_{n(\delta,\eta)} - x^*\| \to 0$, $\delta \to 0$, $\eta \to 0$. Theorem 1 is proved. □

Theorem 2. *Let* $A = A^* \geq 0$, $A_\eta = A_\eta^* \geq 0$, $\|A_\eta - A\| \leq \eta$, $Sp(A_\eta) \subseteq [0, M]$, $(0 < \eta \leq \eta_0)$, $\alpha > 0$, $y \in R(A)$, $\|y_\delta - y\| \leq \delta$ *and conditions* (5), (6) *be satisfied. If the exact solution is source representable, i.e.* $x^* = A^s z$, $s > 0$, $\|z\| \leq \rho$, *then error estimation is equitable*

$$\|x_{n(\delta,\eta)} - x^*\| \leq \gamma_0 c_s \eta^{\min(1,s)}\rho + \gamma_s n^{-s/k}\rho + \gamma n^{1/k}(\delta + \eta\|x^*\|), 0 < s < \infty.$$

Proof. Using the source representability of the exact solution we have

$$\begin{aligned} \|G_{n\eta}x^*\| = \|G_{n\eta}A^s z\| \leq \|G_{n\eta}(A^s - A_\eta^s)z\| + \|G_{n\eta}A_\eta^s z\| \leq \\ \leq \gamma_0 c_s \eta^{\min(1,s)}\rho + \gamma_s n^{-s/k}\rho, \end{aligned} \tag{9}$$

as according to Lemma 1.1 [5, p. 91] $\|A_\eta^s - A^s\| \le c_s \eta^{\min(1,s)}$, $c_s = const$, ($c_s \le 2$ for $0 < s \le 1$). Then

$$\|x_{n(\delta,\eta)} - x^*\| \le \gamma_0 c_s \eta^{\min(1,s)}\rho + \gamma_s n^{-s/k}\rho + \gamma n^{1/k}(\delta + \eta\|x^*\|), 0 < s < \infty. \quad (10)$$

Theorem 2 is proved. □

If the right side of estimation (10) is minimized by n, we get the meaning of a priori stop moment:

$$n_{opt} = \left[\frac{s\gamma_s\rho}{\gamma\left(\delta + \|x^*\|\eta\right)}\right]^{k/(s+1)} = d_s \rho^{k/(s+1)} \left[\delta + \eta\|x^*\|\right]^{-k/(s+1)},$$

where $d_s = \left(\dfrac{s\gamma_s}{\gamma}\right)^{k/(s+1)} = \left(\dfrac{s}{2k}\right)^{(s+k)/(s+1)} \alpha^{-1} e^{-s/(s+1)}$. Consequently,

$$n_{opt} = \left(\frac{s}{2k}\right)^{(s+k)/(s+1)} \alpha^{-1} e^{-s/(s+1)} \rho^{k/(s+1)} \left[\delta + \eta\|x^*\|\right]^{-k/(s+1)}.$$

Let us substitute n_{opt} in estimation (10) to get

$$\|x_{n(\delta,\eta)} - x^*\|_{opt} \le \gamma_0 c_s \eta^{\min(1,s)}\rho + \gamma_s\rho\left(d_s\rho^{k/(s+1)}\right)^{-s/k}\left(\delta + \eta\|x^*\|\right)^{s/(s+1)} +$$

$$+\gamma\left(\delta + \eta\|x^*\|\right) d_s^{1/k}\rho^{1/(s+1)}\left(\delta + \eta\|x^*\|\right)^{-1/(s+1)} =$$

$$= \gamma_0 c_s \eta^{\min(1,s)}\rho + \left(\delta + \eta\|x^*\|\right)^{s/(s+1)}\left(d_s^{-s/k}\gamma_s\rho^{1/(s+1)} + \gamma d_s^{1/k}\rho^{1/(s+1)}\right) =$$

$$= \gamma_0 c_s \eta^{\min(1,s)}\rho + \rho^{1/(s+1)} c_s' \left(\delta + \eta\|x^*\|\right)^{s/(s+1)},$$

where

$$c_s' = d_s^{-s/k}\gamma_s + \gamma d_s^{1/k} = \left(s^{1/(s+1)} + s^{-s/(s+1)}\right)\gamma^{s/(s+1)}\gamma_s^{1/(s+1)} =$$

$$= \left(\frac{s}{2k}\right)^{s(1-k)/(k(s+1))}(1+s)e^{-s/(k(s+1))}.$$

Hence

$$\|x_{n(\delta,\eta)} - x^*\|_{opt} \le c_s\eta^{\min(1,s)}\rho +$$

$$+\left(\frac{s}{2k}\right)^{s(1-k)/(k(s+1))}(1+s)e^{-s/(k(s+1))}\rho^{1/(s+1)}\left(\delta + \eta\|x^*\|\right)^{s/(s+1)}.$$

Note. *Optimal error estimation does not depend on α, whereas n_{opt} depends on α. Since there are no contingencies concerning α upwards ($\alpha > 0$), it is possible to choose α so as to make $n_{opt} = 1$. For that it is enough to take*

$$\alpha_{opt} = \left(\frac{s}{2k}\right)^{(s+k)/(s+1)} e^{-s/(s+1)}\rho^{k/(s+1)}\left[\delta + \eta\|x^*\|\right]^{-k/(s+1)}.$$

3. The case of nonself-adjoint operators

In case of nonself-adjoint problem iteration method (3) will be presented as

$$\begin{aligned}\left[E + \alpha^2 (A_\eta^* A_\eta)^{2k}\right] x_{n+1} = \left[E - \alpha (A_\eta^* A_\eta)^k\right]^2 x_n +\\ +2\alpha (A_\eta^* A_\eta)^{k-1} A_\eta^* y_\delta, \quad x_0 = 0,\ k \in N.\end{aligned} \quad (11)$$

It can be written as follows:

$$x_n = g_n(A_\eta^* A_\eta) A_\eta^* y_\delta. \tag{12}$$

It follows from Lemma 1 that

Lemma 2. *Let $A, A_\eta \in \pounds(H,F), \|A_\eta - A\| \le \eta, \|A_\eta\|^2 \le M, \alpha > 0$ and conditions (7), (8) be satisfied. Then*

$$\|K_{n\eta} v\| \to 0 \; at \; n \to \infty, \eta \to 0, \forall v \in N(A)^\perp = \overline{R(A^*)}, \tag{13}$$

$$\|\tilde{K}_{n\eta} z\| \to 0 \; at \; n \to \infty, \eta \to 0, \forall z \in N(A^*)^\perp = \overline{R(A)}, \tag{14}$$

where $K_{n\eta} = E - A_\eta^* A_\eta g_n(A_\eta^* A_\eta)$, $\tilde{K}_{n\eta} = E - A_\eta A_\eta^* g_n(A_\eta A_\eta^*)$.

Lemma 2 is used for proving the following theorem.

Theorem 3. *Let $A, A_\eta \in \pounds(H,F)$, $\|A - A_\eta\| \le \eta$, $\|A_\eta\|^2 \le M$, $(0 < \eta \le \eta_0)$, $\alpha > 0$, $y \in R(A)$, $\|y_\delta - y\| \le \delta$ and conditions (5), (7), (8) be satisfied. Parameter $n = n(\delta, \eta)$ is chosen so as to get*

$$(\delta + \eta)^2 n^{1/k}(\delta, \eta) \to 0 \; at \; n(\delta, \eta) \to \infty, \delta \to 0, \eta \to 0. \tag{15}$$

Then $x_{n(\delta,\eta)} \to x^*$ at $\delta \to 0, \eta \to 0$.

Proof. For approximation error $x_{n(\delta,\eta)}$ we have

$$x_{n(\delta,\eta)} - x^* = -K_{n\eta} x^* + g_n(A_\eta^* A_\eta) A_\eta^* (y_\delta - A_\eta x^*). \tag{16}$$

We see $\|g_n(A_\eta^* A_\eta) A_\eta^*\| = \|g_n(A_\eta^* A_\eta)(A_\eta^* A_\eta)^{1/2}\| \le \gamma_* n^{1/(2k)}$, where

$$\gamma_* = \sup_{n>0} \left(n^{-1/(2k)} \sup_{0 \le \lambda \le M} \lambda^{1/2} |g_n(\lambda)| \right) \le 2k^{1/2} \alpha^{1/(2k)} \;\; [1, \; p. \; 141].$$

Since $\|y_\delta - A_\eta x^*\| \le \|y_\delta - y\| + \|y - A_\eta x^*\| = \|y_\delta - y\| + \|Ax^* - A_\eta x^*\| \le \delta + \eta \|x^*\|$, it follows that $\|g_n(A_\eta^* A_\eta) A_\eta^* (y_\delta - A_\eta x^*)\| \le 2k^{1/2} \alpha^{1/(2k)} n^{1/(2k)} (\delta + \|x^*\| \eta)$. That is why

$$\|x_{n(\delta,\eta)} - x^*\| \le \|K_{n\eta} x^*\| + \|g_n(A_\eta^* A_\eta) A_\eta^* (y_\delta - A_\eta x^*)\| \le \|K_{n\eta} x^*\| +$$
$$+ 2k^{1/2} \alpha^{1/(2k)} n^{1/(2k)} (\delta + \eta \|x^*\|).$$

Let us show that $\|K_{n\eta} x^*\| \to 0$ at $n \to \infty$, $\eta \to 0$. Actually,

$$\|K_{n\eta} x^*\| = \left\|(E - A_\eta^* A_\eta g_n(A_\eta^* A_\eta)) x^*\right\| =$$

$$= \left\| \int_0^{\|A_\eta^* A_\eta\|} (1 - \lambda g_n(\lambda)) dE_\lambda x^* \right\| = \left\| \int_0^{\|A_\eta^* A_\eta\|} \frac{(1 - \alpha \lambda^k)^{2n}}{(1 + \alpha^2 \lambda^{2k})^n} dE_\lambda x^* \right\| \le$$

$$\le \left\| \int_0^{\varepsilon} \frac{(1 - \alpha \lambda^k)^{2n}}{(1 + \alpha^2 \lambda^{2k})^n} dE_\lambda x^* \right\| + \left\| \int_{\varepsilon}^{\|A_\eta^* A_\eta\|} \frac{(1 - \alpha \lambda^k)^{2n}}{(1 + \alpha^2 \lambda^{2k})^n} dE_\lambda x^* \right\|.$$

Then

$$\left\| \int\limits_{\varepsilon}^{\|A_\eta^* A_\eta\|} \frac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n} dE_\lambda x^* \right\| \le q^n(\varepsilon) \left\| \int\limits_{\varepsilon}^{\|A_\eta^* A_\eta\|} dE_\lambda x^* \right\| \to 0, \quad n\to\infty,$$

as for $\lambda \in \left[\varepsilon, \|A_\eta^* A_\eta\|\right]$, $\dfrac{(1-\alpha\lambda^k)^2}{1+\alpha^2\lambda^{2k}} \le q(\varepsilon) < 1$.

$$\left\| \int\limits_0^{\varepsilon} \frac{(1-\alpha\lambda^k)^{2n}}{(1+\alpha^2\lambda^{2k})^n} dE_\lambda x^* \right\| \le \left\| \int\limits_0^{\varepsilon} dE_\lambda x^* \right\| = \|E_\varepsilon x^*\| \to 0, \quad \varepsilon\to 0$$

owing to integrated spectrum properties [3–4].

From statement (15) $n^{1/k}(\delta+\eta)^2 \to 0$ at $n\to\infty$, $\delta\to 0$, $\eta\to 0$. Hence $2k^{1/2}\alpha^{1/(2k)}n^{1/(2k)}\left(\delta+\eta\|x^*\|\right)\to 0$, $n\to\infty$, $\delta\to 0$, $\eta\to 0$. Thus,

$$\left\|x_{n(\delta,\eta)} - x^*\right\| \to 0, \quad n\to\infty, \quad \delta\to 0, \quad \eta\to 0.$$

Theorem 3 is proved. □

The following is valid

Theorem 4. *Let $A, A_\eta \in \pounds(H,F)$, $\|A - A_\eta\| \le \eta$, $\|A_\eta\|^2 \le M$, $(0<\eta\le\eta_0)$, $\alpha>0$, $y\in R(A)$, $\|y_\delta - y\| \le \delta$. If the exact solution can be represented as $x^* = |A|^s z$, $s>0$, $\|z\|\le\rho$, $|A| = (A^*A)^{1/2}$ and conditions (5), (6) are satisfied, then estimation error is real*

$$\left\|x_{n(\delta,\eta)} - x^*\right\| \le \gamma_0 c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho +$$
$$+\gamma_{s/2} n^{-s/(2k)}\rho + 2k^{1/2}\alpha^{1/(2k)}n^{1/(2k)}\left(\delta+\|x^*\|\,\eta\right), \ 0<s<\infty.$$

Proof. In case of sourcewise representable exact solution $x^* = |A|^s z = (A^*A)^{s/2} z$ owing to (6) we get $\sup\limits_{0\le\lambda\le M} \lambda^{s/2}|1-\lambda g_n(\lambda)| \le \gamma_{s/2} n^{-s/(2k)}$, where $\gamma_{s/2} = \left(\dfrac{s}{4k\alpha e}\right)^{s/(2k)}$. Then

$$\|K_{n\eta}|A_\eta|^s z\| = \left\||A_\eta|^s\left[E - A_\eta^*A_\eta g_n\left(A_\eta^*A_\eta\right)\right]z\right\| =$$
$$= \left\|\left(A_\eta^*A_\eta\right)^{s/2}\left[E - A_\eta^*A_\eta g_n\left(A_\eta^*A_\eta\right)\right]z\right\| \le \gamma_{s/2} n^{-s/(2k)}\rho.$$

Hence

$$\|K_{n\eta}x^*\| = \|K_{n\eta}|A|^s z\| = \|K_{n\eta}\left(|A_\eta|^s - |A|^s\right)z\| +$$
$$+\|K_{n\eta}|A_\eta|^s z\| \le \gamma_0 c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho + \gamma_{s/2}n^{-s/(2k)}\rho,$$

since according to [5, p. 92] we have $\||A_\eta|^s - |A|^s\| \le c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}$, $c_s = const$, ($c_s \le 2$ for $0<s\le 1$). Following (16)

$$\begin{aligned} \left\|x_{n(\delta,\eta)} - x^*\right\| &\le \|K_{n\eta}x^*\| + \gamma_* n^{1/(2k)}\left(\delta+\|x^*\|\,\eta\right) = \|K_{n\eta}x^*\| + \\ &+ 2k^{1/2}\alpha^{1/(2k)}n^{1/(2k)}\left(\delta+\|x^*\|\,\eta\right) \le \gamma_0 c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho + \\ &+ \gamma_{s/2}n^{-s/(2k)}\rho + 2k^{1/2}\alpha^{1/(2k)}n^{1/(2k)}\left(\delta+\|x^*\|\,\eta\right), \quad 0<s<\infty. \end{aligned} \tag{17}$$

Theorem 4 is proved. □

By minimizing the right-hand member (17) at n, the meaning of a priori stop moment is obtained:

$$n_{opt} = \left(\frac{s\gamma_{s/2}}{\gamma_*}\right)^{2k/(s+1)} \rho^{2k/(s+1)} \left(\delta + \|x^*\| \, \eta\right)^{-2k/(s+1)} =$$

$$= (4k)^{-(s+k)/(s+1)} s^{(2k+s)/(s+1)} e^{-s/(s+1)} \alpha^{-1} \rho^{2k/(s+1)} \left(\delta + \|x^*\| \, \eta\right)^{-2k/(s+1)} .$$

The substitution of n_{opt} into estimation (17) allows obtaining the optimal error estimation for the method of iterations (11)

$$\left\|x_{n(\delta,\eta)} - x^*\right\|_{opt} \le \gamma_0 c_s \left(1 + |\ln \eta|\right) \eta^{\min(1,s)} \rho +$$
$$+ c_s'' \rho^{1/(s+1)} \left(\delta + \|x^*\| \, \eta\right)^{s/(s+1)}, \quad 0 < s < \infty,$$

where

$$c_s'' = \left(s^{1/(s+1)} + s^{-s/(s+1)}\right) \gamma_*^{s/(s+1)} \gamma_{s/2}^{1/(s+1)} =$$
$$= s^{s(1-2k)/(2k(s+1))} (s+1) (4k)^{s(k-1)/(2k(s+1))} e^{-s/(2k(s+1))}.$$

To sum it up,

$$\left\|x_{n(\delta,\eta)} - x^*\right\|_{opt} \le c_s \left(1 + |\ln \eta|\right) \eta^{\min(1,s)} \rho + s^{s(1-2k)/(2k(s+1))} (s+1) \times$$
$$\times (4k)^{s(k-1)/(2k(s+1))} e^{-s/(2k(s+1))} \rho^{1/(s+1)} \left(\delta + \|x^*\| \, \eta\right)^{s/(s+1)}, 0 < s < \infty.$$

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