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APPROXIMATING A CONTOUR BY LINEAR SEGMENTS AND CIRCULAR ARCS WITH GIVEN CURVATURE BOUNDS[1]

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Abstract. A method of approximation of a contour by a set of linear segments and circular arcs is proposed. Starting from the vertex with minimal scalar product of the vectors issuing from a vertex to neighbouring ones, we consecutively construct the next element which is either a linear segment or a circular arc. In each such step we guarantee that constraints on distance to the initial contour and constraints on circular arc radii are satisfied.

1. Introduction

In a number of problems of placement, cutting and packing, the information on detailes (objects) enters after digitising their contour (frontier) in a form of a vertex sequence. The contour is a polygon or it includes circular arcs as well as linear segments. Such information is often a result of operating of appropriate equipment: digitizers, etc. and contains quite great, in most cases redundant, amount of elements (segments).

A complete class of Φ-functions [1, 2] has been constructed for a series of basic objects whose frontier consists of linear segments and concave or convex (with respect to the interior of the object) circular arcs.

Composed objects can be represented by a set of basic objects of four types [1, 2].

The Φ-functions allow us to use effective solution methods of solving optimization problems of placement, cutting and packing.

At the same time, effectiveness of a series of solution methods depends essentially on complexity of objects. Knowing admissible error of contour definition we may signigicantly reduce the number of elemantary curves defining the contour, and therefore optimization problem runtime.

Optimization problem solution methods are often such that their solution quality depends on quantity of local extrema explored.

Hence, reduction of contour element number for many problem statements provides summary improvement of solution as well.

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2. Review of Existing Methods

The simplest contours are linear segments and circular arcs and the need of approximating scattered points by a circle or a circular arc arises in physics, biology and medicine, archeology, industry, etc [5].

Due to its importance the problem of approximation of a contour by a set of linear segments and/or circular arcs was investigated in numerous works.

Paper [4] addresses the problem of identifying perceptually significant segments on general planar curvilinear contours. The problem posed there is to identify all segments of a contour that can be interpreted as a "natural" or "perceptually salient" section to approximate using a circular arc model. Unfortunately, as the author writes, what it means to be perceptually salient is not specified by any formal definition. Therefore, in his paper he attempts to articulate subjective criteria for what constitutes a perceptually natural contour segment by examining a number of prototypical situations that occur on curvilinear contours and appeals to the reader's own judgements. An algorithm attempting to meet these criteria is presented in [4]. The segments delivered have the following properties: (1) each segment is well-approximated by a circular arc, (2) each pair of segments describe different sections of the contour, and (3) the curve either terminates or changes in orientation and/or curvature beyond each end of every segment. The result is a description of the contour at multiple scales in terms of circular arcs that may overlap one another.

In [5] the problem of fitting standard shapes or curves to incomplete data (which represent only a small part of the curve) is considered. Even if the curve is quite simple, such as an ellipse or a circle, it is hard to reconstruct it from noisy data sampled along a short arc.

The least squares fit (LSF) of circular arcs is studied to incomplete scattered data. The authors reveal the cause of unstable behavior of conventional algorithms and propose an algorithm that accurately fits circles to data sampled along arbitrarily short arcs.

In [6] proposed is an algorithm of constructing an approximate continuous representation of a digital contour by circular arcs with guarantees on the Hausdorff error between the digital shape and its reconstruction.

Article [7] introduces an approach for the segmentation and approximation of 2D contours. The algorithm deals with the pixel representation of the contour.

In our work we investigate the problem in a statement arising when preparing data for cutting and packing optimization algorithms. A representation by minimal number of linear segments and circular arcs of curvature bounded from above and below is convenient for further computations.

3. Problem Statement

Let us have a polygonal object frontier (contour) $K_0 = \operatorname{fr} S_0$ defined by a cyclic sequence of vertices $\mathbf{v}_i \in \mathbb{R}^2$, $i \in \mathbb{I}_n = \{1, 2, \ldots, n\}$.

It is required to construct its approximation by a cyclic sequence of segments and circular arcs of minimal quantity so that the new frontier K' lie within $G = (S_0 \oplus B_{\varepsilon_+}) \cap (\complement S_0 \oplus B_{\varepsilon_-})$ where

$\complement S_0 = \mathbb{R}^2 \backslash S_0$ is the complement of S_0,
$\oplus$ is the Minkowski sum operation [3],
ε_+ and ε_- are the required outer and inner accuracy limits respectively,
B_ε is the disc of radius ε.

It is supposed that G has no self-intersections of its contour curves consisting of two components.

We also have a restriction on minimal and maximal radii of circles, separately for convex and concave arcs:
$\hat{r}_- \leq \hat{r} \leq \hat{r}_+$,
$\breve{r}_- \leq \breve{r} \leq \breve{r}_+$.

4. Solution Algorithm

The following algorithm to solve the problem is proposed.

1. Renumber the vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of the object P cyclically. We write i_{++} and i_{--} instead of $\begin{cases} i+1, & \text{if } x < n; \\ 1, & \text{if } x = n \end{cases}$ and $\begin{cases} i-1, & \text{if } x > 1; \\ n, & \text{if } x = 1 \end{cases}$ respectively.
2. Start with $i = 1$.
3. Let $U_\varepsilon(\mathbf{v}_i)$ be a small enough neighbourhood of $\mathbf{v}_i$ of radius ε, $\varepsilon \ll \varepsilon_+$, $\varepsilon \ll \varepsilon_-$. Find the so called effective neighbours $\mathbf{v}_{i^-}$, $\mathbf{v}_{i^+}$ of the current vertex $\mathbf{v}_i$. For that find the first index k among $k \in \mathbb{I}_n = \{1, 2, \ldots, n\}$ such that $\mathbf{v}_{i_{(--)^k}} \notin U_\varepsilon(\mathbf{v}_i)$ and denote it as $\mathbf{v}_{i^-}$. Notation $i_{(--)^k}$ means here that operation $--$ for i, i. e. $i \to i_{--}$, is performed k times. Similarly find $\mathbf{v}_{i^+}$.
4. Renumber the vertices $\mathbf{v}_i$ of the object P so that only effective neighbours of $\mathbf{v}_i$ be included.
5. Form normalised vectors to the effective neighbours of a vertex $\mathbf{v}_i$ as follows:
   $\mathbf{a}_i^\sigma = (\mathbf{v}_{i^\sigma} - \mathbf{v}_i) / \|(\mathbf{v}_{i^\sigma} - \mathbf{v}_i)\|$, $\sigma \in \{-, +\}$.
6. Choose $\mathbf{v}_i$ with minimal scalar product $(\mathbf{a}_i^-, \mathbf{a}_i^+)$ (the sharpest corner, no matter whether convex or concave).
7. Renumber the vertices starting from the found sharpest corner.
8. For each vertex $\mathbf{v}_i$ define $\mathbf{a}_i^* = (\mathbf{a}_{i^+} - \mathbf{a}_{i^-}) / \|(\mathbf{a}_{i^+} - \mathbf{a}_{i^-})\|$, then define outward and inward normals $\mathbf{n}_i^+$, $\mathbf{n}_i^-$ so that $(\mathbf{a}_i^*, \mathbf{n}_i^+)$ and $(\mathbf{n}_i^-, \mathbf{a}_i^*)$ would form a right orthgonal basis.
9. For each vertex $\mathbf{v}_i$ construct $\mathbf{v}_i^+ = \mathbf{v}_i + \varepsilon_+ \mathbf{n}_i^+$ and $\mathbf{v}_i^- = \mathbf{v}_i + \varepsilon_- \mathbf{n}_i^-$
10. Starting from the new first vertex $\mathbf{v}_i = \mathbf{v}_1$ assume $\mathbf{v'}_i = \mathbf{v}_i$ and try to construct a section of the frontier consecutively, constructing linear segment and circular arcs issuing from $\mathbf{v}_i'$ to $\mathbf{v}_{i+k}$, $\mathbf{v}_{i+k}^+$ and $\mathbf{v}_{i+k}^-$, $k = 2, 3, \ldots$.

    In so doing, check a possibility to construct a linear segment as well as a concave or convex circular arc without violating approximation restrictions (outer and inner ones) on distances between the original and constructed frontiers.

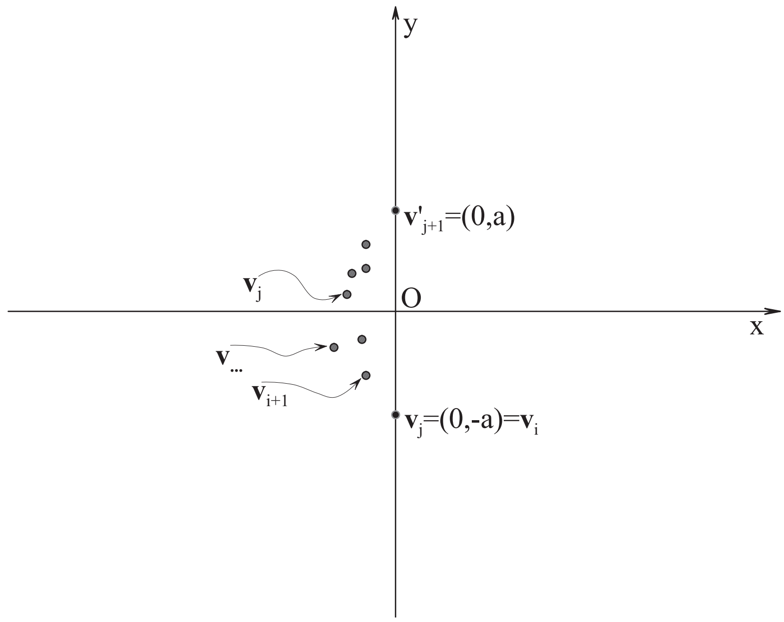


FIG. 1. Approximating the contour by an arc

If there is no admissible choice for k, but there is one for $k-1$, assume $\mathbf{v'}_{i+1} = \mathbf{v}^{+}_{i+k-1}$, and choose as the curve connecting $\mathbf{v'}$ and $\mathbf{v'}_{i+1}$ such curve that privides minimal deviation from the original frontier.

5. Admissibility of an Arc

Let us consider now the question of detecting admissibility of an arc (from $\mathbf{v'}_i$ to $\mathbf{v'}_{i+1}$ as an approximating contour element).

1. By translation and rotation, transform object P so that $\mathbf{v'}_i$, $\mathbf{v'}_{i+1}$ move into $(0,a)$, $(0,-a)$ respectively (Fig. 1).

Each intermediate vertex $\mathbf{v}_j$ defines (with account of tolerances $\varepsilon_+, \varepsilon_-$) possible bounds of the radius of the circular arcs $r_j^{\min}$, $r_j^{\max}$ respectively (Fig. 2).

The values $r_j^{\max}$, $r_j^{\min}$ can be obtained from the next equations, in which $(x_c, 0)$ are the center coordinates of the required arc.

$$\begin{cases} a^2 + x_c^2 = r_j^{\max} \\ (x_c - x_j)^2 + (0 - y_j)^2 = (r_j^{\max} + \varepsilon_-)^2, \end{cases} \tag{1}$$

$$\begin{cases} a^2 + x_c^2 = r_j^{\min} \\ (x_c - x_j)^2 + (0 - y_j)^2 = (r_j^{\min} - \varepsilon_+)^2, \end{cases} \tag{2}$$

The following solution was obtained in the Maple 13 package.
Designate

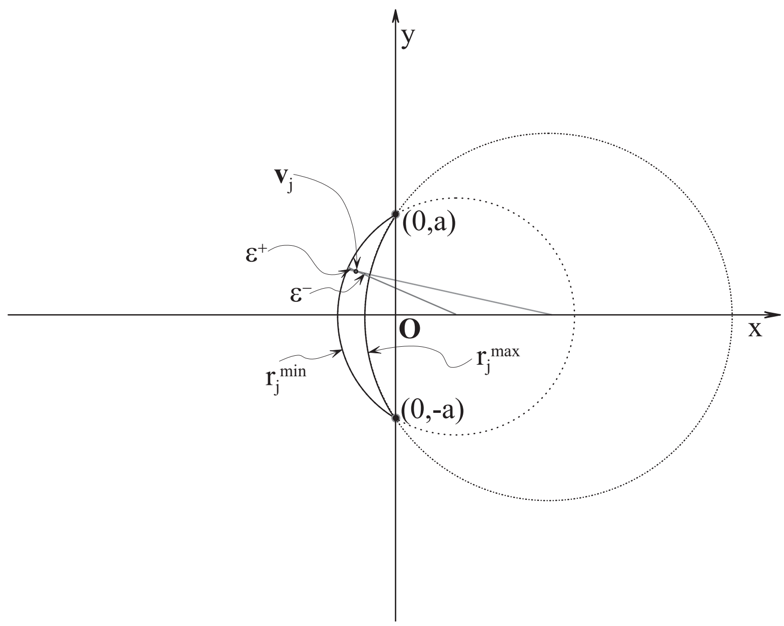


FIG. 2. Feasible arc radius limits

$$\begin{aligned} D &= 2x_j^2y_j^2\varepsilon_+^2 + 2\varepsilon_+^2x_j^2a^2 - 2\varepsilon_+^2a^2y_j^2 + x_j^4\varepsilon_+^2 \\ &\quad -2\varepsilon_-^4x_j^2 - 2\varepsilon_-^4y_j^2 + \varepsilon_+^2y_j^4 + \varepsilon_-^6 + a^4\varepsilon_+^2 - 2\varepsilon_-^4a^2. \end{aligned} \tag{3}$$

A similar expression is valid for ε_+.
Then solution of (1) is either

$$\begin{aligned} c &= \tfrac{1}{2}(-x_j^3 - x_jy_j^2 + \varepsilon_-^2x_j + a^2x_j + \sqrt{D})/(\varepsilon_-^2 - x_j^2), \\ r_j^{\max} &= -\tfrac{1}{2\varepsilon_-}(x_j\tfrac{-x_j^3 - x_jy_j^2 + \varepsilon_+^2x_j + a^2x_j + \sqrt{D}}{\varepsilon_-^2 - x_j^2} + a^2 - x_j^2 - y_j^2 + \varepsilon_-^2), \end{aligned} \tag{4}$$

or

$$\begin{aligned} x_c &= \tfrac{1}{2}(-x_j^3 - x_jy_j^2 - \varepsilon_-^2x_j + a^2x_j - \sqrt{D})/(\varepsilon_-^2 - x_j^2), \\ r_j^{\max} &= -\tfrac{1}{2\varepsilon_-}(x_j\tfrac{-x_j^3 - x_jy_j^2 + \varepsilon_+^2x_j + a^2x_j - \sqrt{D}}{\varepsilon_-^2 - x_j^2} + a^2 - x_j^2 - y_j^2 + \varepsilon_-^2), \end{aligned} \tag{5}$$

The solution of (2) is either

$$\begin{aligned} x_c &= \tfrac{1}{2}(-x_j^3 - x_jy_j^2 + \varepsilon_+^2x_j + a^2x_j + \sqrt{D})/(\varepsilon_+^2 - x_j^2), \\ r_j^{\min} &= \tfrac{1}{2\varepsilon_+}(x_j\tfrac{-x_j^3 - x_jy_j^2 + \varepsilon_+^2x_j + a^2x_j + \sqrt{D}}{\varepsilon_+^2 - x_j^2} + a^2 - x_j^2 - y_j^2 + \varepsilon_+^2), \end{aligned} \tag{6}$$

or

$$\begin{aligned} x_c &= \tfrac{1}{2}(-x_j^3 - x_jy_j^2 - \varepsilon_+^2x_j + a^2x_j - \sqrt{D})/(\varepsilon_+^2 - x_j^2), \\ r_j^{\min} &= \tfrac{1}{2\varepsilon_+}(x_j\tfrac{-x_j^3 - x_jy_j^2 + \varepsilon_+^2x_j + a^2x_j - \sqrt{D}}{\varepsilon_+^2 - x_j^2} + a^2 - x_j^2 - y_j^2 + \varepsilon_+^2). \end{aligned} \tag{7}$$

We take the second solution (5) of the system (1) and the first solution (6) of the system (2) since r has to be positive.

If the values (x_j, y_j) and ε_- (or ε_+ respectively) are still such that the obtained values $r_j^{\max} < 0$ ($r_j^{\min} < 0$ respectively), then in dependence upon situation, we assume that $r_j^{\max} = \infty$ or $r_j^{\max} = a$ ($r_j^{\min} = \infty$ or $r_j^{\min} = a$ respectively).

For all the finite sequence of the intermediate points $\mathbf{v}_j$ we assume that

$$r^{\max} = \min_j r_j^{\max}, \tag{8}$$

$$r^{\min} = \max_j r_j^{\min}. \tag{9}$$

Find

$$[r_*^-, r_*^+] = [r^{\min}, r^{\max}] \cap [\hat{r}_-, \hat{r}_+] \tag{10}$$

in the case of a convex circular arc and

$$[r_*^-, r_*^+] = [r^{\min}, r^{\max}] \cap [\check{r}_-, \check{r}_+] \tag{11}$$

in the case of a concave one.

Denote by $\delta_j^+(r)$, $\delta_j^-(r)$ the outer and the inner deviation of the considered arc of the new contour from the vertex of the initial one respectively;

denote by $\overset{\frown}{\delta}{}_k^+(r)$, $\overset{\frown}{\delta}{}_k^-(r)$ the outer and the inner deviation of the considered arc of the new contour from the segment $[\mathbf{v}_{k-1}, \mathbf{v}_k]$ of the initial contour respectively.

In the last case for finding the values $\overset{\frown}{\delta}{}_k^+(r)$, $\overset{\frown}{\delta}{}_k^-(r)$ we construct the perpendiculars from the circle center to segments $[\mathbf{v}_{k-1}, \mathbf{v}_k]$.

Let $\delta(r) = \left\{ \frac{\max_j \delta_j^+(r)}{\varepsilon^+}, \frac{\max_j \delta_j^-(r)}{\varepsilon^-}, \frac{\max_k \overset{\frown}{\delta}{}_k^+(r)}{\varepsilon^+}, \frac{\max_k \overset{\frown}{\delta}{}_k^-(r)}{\varepsilon^-} \right\}$.

Find in the closed segment $[r_*^-, r_*^+]$ the optmial value $r = r^{\mathrm{extr}}$ so that

$$\delta(r^{\mathrm{extr}}) = \arg\min \delta(r). \tag{12}$$

Function $\delta(r)$ is a monotone function of one argument r. Its minimum $\delta(r_*)$ can be found e. g. by dichotomy.

If the obtained value $|\delta(r^{\mathrm{extr}})| \leq 1$, then the obtained solution is admissible.

6. Stable vertices

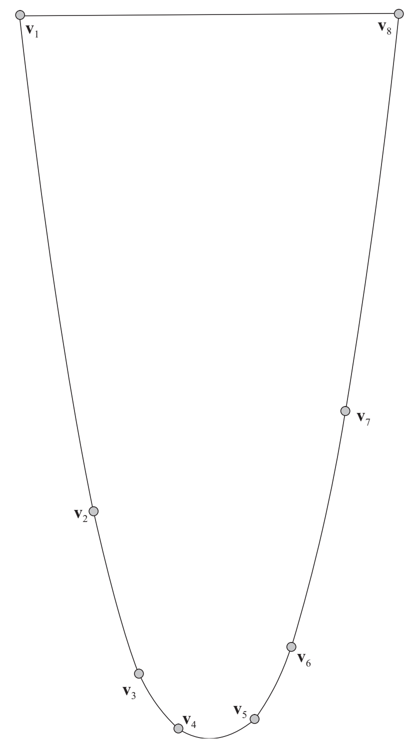
We can fix some contour vertices and retain them in the new contour. Criteria for such solution are as follows.

First, we check the condition $(\mathbf{a}_i^+, \mathbf{a}_i^-) \geq \alpha$. If it is satisfied, we attach $\mathbf{a}_i$ to the set of stable vertices.

Then, to a vertex $\mathbf{v}_i$ we can assign curvature $\kappa_i = 1/r_i$ where r_i is the radius of the circle C_i generated by vertices $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, $\mathbf{v}_{i+1}$. Denote the centre of C_i by c_i.

We can assign a sign to κ_i in dependence on whether we deal with a convex or concave arc with respect to the interior of P.

(a) The initial digitized parabola segment



(b) The approximated parabola segment

FIG. 3. Approximating a parabola segment

Value $\Delta\kappa = \|\kappa_{i-1} - \kappa_i\| + \|\kappa_i - \kappa_{i+1}\|$ characterizes curvature change in the vertex $\mathbf{v}_i$ and can be used as a criterion to consider vertex $\mathbf{v}_i$ as a stable one. We may require that $\Delta\kappa \geq \varepsilon_\kappa$ for that.

Conditions $\frac{\|C_{i-1}-C_i\|}{r_{i-1}+r_i} + \frac{\|C_i-C_{i+1}\|}{r_i+r_{i+1}} \geq \varepsilon_1$, and $\frac{\|C_{i-1}-C_i\|}{\max\{r_{i-1},r_i\}} + \frac{\|C_i-C_{i+1}\|}{\max\{r_{i-1},r_i\}} \geq \varepsilon_2$, where ε_1 , ε_2 are some threshold values that also may be used as criteria for vertex stability.

7. Examples

An approximation of an object formed by a part of a parabola (Fig. 3(a)) with 7 arcs and one segment is shown in Fig. 3(b).

An approximation of an object having given inner and outer approximation tolerances are shown in Fig. 4.

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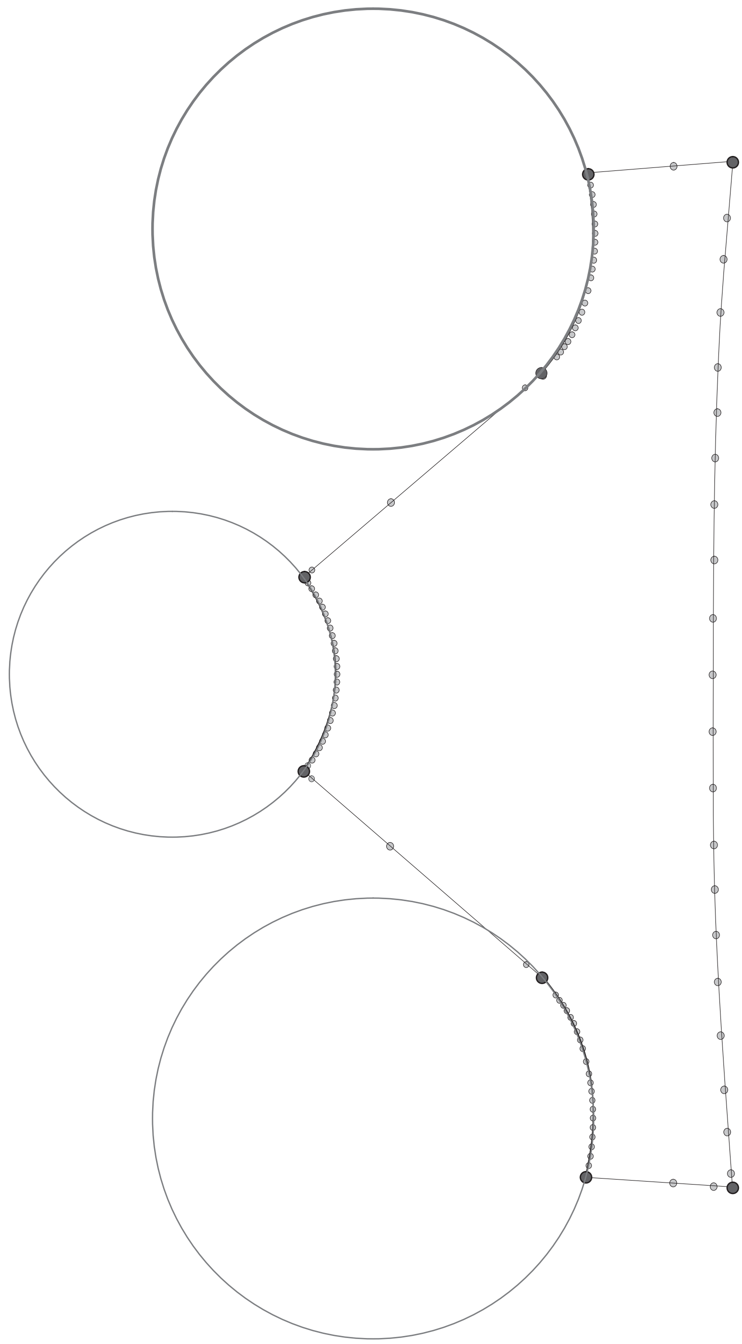

FIG. 4. Approximating an object of diameter 60.74 with parameters: outer precision 0.1, inner precision 0.1, minimal outer radius 0.1, minimal inner radius 0.1, maximal outer radius 300, maximal inner radius 300, $(\mathbf{a}_i^+, \mathbf{a}_i^-)$ threshold -0.99, curvature change threshold 0.3, circle centers' shift threshold 0.9888. The resulting object has 5 linear segments and 4 circular arcs

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