Журнал обчислювальної та прикладної математики | 2015 № 2 (119) | Journal of Computational & Applied Mathematics

UDC 519.642

ABOUT MINIMAL INFORMATIONAL EFFORTS BY SOLVING EXPONENTIALLY ILL-POSED PROBLEMS

S. G. SOLODKY, E. V. SEMENOVA

Резюме. Розглядаються питання інформаційної складності для експоненціально некоректних задач. Дослідження виконані для інтегральних рівнянь Фредгольма першого роду з оператором скінченної гладкості. Запропоновані проекційні схеми дозволяють досягти оптимальний порядок точності для апостеріорного вибору параметра регуляризації за принципом рівноваги. Крім того такий підхід зберігає мінімальний обсяг інформаційних затрат.

Abstract. The issue of informational complexity for exponentially ill-posed problems is considered. The investigation is performed for Fredholm integral equations of the first kind with finite-smoothness operators. The proposed projection method allows to achieve optimal order accuracy for a posteriori selection of regularization parameter by balancing principle. Moreover such approach saves minimal volume of informational efforts.

1. Introduction

Nowadays for numerical method one of the most important issues is reduction of informational and computational efforts while saving approximation accuracy. These questions are studied in the framework of Informational Based Complexity Theory founded by J. Traub and H. Wozniakowski (see e.g. [18], [19]). The basic object of this theory is the information complexity, i.e. minimal amount of discrete information required to solve the problem with given accuracy. It was found that such amount depends on the smoothness properties of the problem. Particularly, for ill-posed problems presented by the first-kind operator equations $Ax = f$ the relation between smoothness of operator A and solution x is of primary importance. In the case of moderately ill-posed problems, when A and x are related by means of power function (i.e. A and x belong to the same smoothness scale), different efficient numeric approaches were proposed in [10], [12], [13], [14]. Owing to previous papers the exact order estimates of informational complexity for wide classes of moderately ill-posed problems (see, for example, [8]) were obtained. At the same time, much attention is paid to severely ill-posed problems where the solution has essentially worse smoothness in comparison with that of operator. Usually, in these cases A and x are related by means of logarithmic function but the corresponding equations are called exponentially ill-posed problems. For the

Key words. Severely ill-posed problems, minimal radius of Galerkin information, balancing principle.

first time severely ill-posed problems were considered by B.A. Mair [4]. Afterwards, these investigations were continued by T.Hohage [3], M.Y. Kokurin and A.B. Bakushinski [2], S.V. Pereverzev and E. Schock [17] and also in [15], [16].

It should be noted that for a long time the issue of improving effectiveness of numerical solving severely ill-posed problems (in sense of IBC theory) was not considered due to its complicatedness. The first step was done in [6], where the standard Galerkin discretization scheme was used to construct projective methods for solving different classes of problems including severely ill-posed ones. However, it was found that this approach does not provide minimal amount of computational efforts. Further investigations (see [16]) showed that amount of discrete information can be reduced in comparison with [6] for exponentially ill-posed integral equations with finite-smoothness kernels. It was done in [16] due to a modification of Galerkin scheme. In the case of a priori choice of regularization parameter it allowed not only to improve results of [6], but also provided minimal order of information efforts for mentioned Fredholm equations. The present paper is devoted to numerical solving exponentially ill-posed problems as in [16] for the case of a posteriori choice of regularization parameter. It will be shown that the absence of exact information about smoothness of solution does not influence informational complexity of problems under consideration.

2. Statement of the problem

Consider an integral equation of the first kind

$$Ax = f, \tag{1}$$

where $Ax(t) = \int_0^1 a(t,\tau)x(\tau)d\tau, \quad t \in [0,1]$, is acting continuously in $L_2 = L_2(0,1)$. Suppose that Range(A) is not closed in L_2 and $f \in$ Range(A).

Assume that instead of f we are given only $f_\delta \in L_2$ such that $\|f - f_\delta\| \le \delta$. Since, solution of problem (1) in general is not unique, we take solution of (1) with minimal norm in L_2 as element for approximation and denote it as $x^\dagger$.

Usually we call the equation (1) as severely ill-posed problem if its solution has essentially worse smoothness than that of elements from Range(A). As a rule in such case the solution $x^\dagger$ is said to satisfy the source conditions of logarithmic type and the corresponding equation (1) is called an exponentially ill-posed problem. To describe the smoothness property of solution we consider the set of smooth functions $M_p(A)$, which has the form

$$M_p(A) := \{u : \ u = \ln^{-p}(A^*A)^{-1}v, \ \|v\| \le \rho\}, \tag{2}$$

where $\rho, p > 0$ are some positive parameters and A^* is an adjoint operator to A. The exact information about smoothness, namely the value of p, is usually not available by practical experiment. So it should be assumed that the minimal-norm solution $x^\dagger$ belongs to the set

$$M(A) := \cup_{p \in (0,p_1]} M_p(A), \tag{3}$$

where $p_1 < \infty$ is an upper bound for possible values of p.

For constructing an effective numerical method for solving (1) we also need to describe smoothness properties of A. To this end let consider some orthonormal basis $\{e_i(t)\}_1^\infty$ in L_2 and denote by P_m orthogonal projection onto linear span of elements $\{e_i(t)\}_1^m$ such that

$$P_m u(t) = \sum_{i=1}^{m} (u, e_i) e_i(t).$$

Further we introduce the class of operators

$$\mathcal{H}_\gamma^r = \left\{ A : \|A\| \leq \gamma_0, \sum_{n+m=1}^{\infty} \hat{a}_{n,m}^2 (\underline{n} \cdot \underline{m})^{2r} \leq \gamma_1^2 \right\},$$

where $r > 0$, $\hat{a}_{n,m}^2 = \int_0^1 \int_0^1 e_n(t) e_m(\tau) a(t, \tau) d\tau dt$, $\gamma_0 \leq e^{-1}, \gamma = (\gamma_0, \gamma_1), \underline{n} = 1$ if $n = 0$ and $\underline{n} = n$ otherwise. As an example of operator from the class mentioned one can present integral operator A' that has the same structure as (1) with kernel $a'(t, \tau)$ that has mixed partial derivatives up to order r by each variables and for $i, j = 0, 1, \ldots, r$ it holds true that

$$\int_0^1 \int_0^1 \left[\frac{\partial^{i+j} a'(t; \tau)}{\partial t^i \partial \tau^j} \right]^2 dt d\tau < \infty.$$

It is known [7], that there is such set $\gamma = (\gamma_0; \gamma_1)$ that $A' \in \mathcal{H}_\gamma^r$. Further we assume that $A \in \mathcal{H}_\gamma^r$ for some values of γ with $\gamma_0 \leq e^{-1}$.

Every projection scheme for discretization of equation (1) with perturbed right-hand side can be associated with a set of following functionals

$$(Ae_j, e_i), \quad (i, j) \in \Omega, \tag{4}$$

$$(f_\delta, e_k), \quad k \in \omega, \quad \omega = \{i : (i, j) \in \Omega\}, \tag{5}$$

where Ω is a bounded domain in the coordinate plane. The inner products (4) and (5) are called the Galerkin functionals about equation (1). We denote as $\mathrm{Card}(\Omega)$ the total amount of indexes for (4). Note that in the case of the Fredholm integral operator A the Galerkin functionals (4) and (5) become the Fourier coefficients by basis $\{e_i(t)\}_{i=1}^\infty$ for the kernel and right-hand side correspondingly. In the framework of this paper it is assumed that discrete information about equation (1) is given in the view of sets (4) and (5). Thus the projection methods for solving (1) are more suitable and will be investigated further. The first projection methods for ill-posed problems were proposed in [12] where rectangle $Q_{n,m} = [1, n] \times [1, m]$ was considered as domain Ω. Further this approach was improved by [11] due to the reduction of discretization domain $Q_{n,m}$ (it was replaced by so-called hyperbolic cross) with saving necessary accuracy of approximation. This idea will be used further for constructing an economical projection scheme (see section 3).

Further we call any mapping $\mathcal{P} = \mathcal{P}(\Omega) : L_2 \to L_2$ as projection method that by means of the set of Galerkin functional (4) gives an element $\mathcal{P}(A_\Omega) f_\delta \in L_2$. This elements can be interpreted as approximative solution of (1). In general such mapping can be nonlinear and discontinuous. Let define the error of projection method $\mathcal{P}(\Omega)$ for solving (1) with $A \in \mathcal{H}_\gamma^r$ and $x^\dagger \in M(A)$ in the standard way

$$e_\delta\left(\mathcal{H}_\gamma^r, M(A), \mathcal{P}(\Omega)\right) = \sup_{A\in\mathcal{H}_\gamma^r} \sup_{x^\dagger\in M(A)} \sup_{f_\delta:\|f-f_\delta\|\le\delta} \|x^\dagger - \mathcal{P}(A_\Omega)f_\delta\|.$$

The minimal radius of Galerkin information we set as

$$R_{N,\delta}\left(\mathcal{H}_\gamma^r, M(A)\right) = \inf_{\substack{\Omega, \\ \mathrm{Card}(\Omega)\le N}} \inf_{\mathcal{P}(\Omega)} e_\delta\left(\mathcal{H}_\gamma^r, M(A), \mathcal{P}(\Omega)\right),$$

where N is maximal amount of discrete information (4).

The value $R_{N,\delta}\left(\mathcal{H}_\gamma^r, M(A)\right)$ is very important one and describes the minimal possible error (among the whole projection methods) on all classes of equations under consideration with using not more than N Galerkin functionals. At first the order bounds for minimal radius of Galerkin information for ill-posed problems with Holder-type smooth solutions were found by S.V. Pereverzyev and S.G. Solodky in [8]. Further for different classes of ill-posed problems the similar bounds were established in [16], [10] and others. Among mentioned papers we emphasize [16] where the minimal radius of Galerkin information was found for solving severely ill-posed problems (1) with operators $A \in \mathcal{H}_\gamma^r$ and smooth solutions from (2). In other words, in [16] only a priori case for choosing regularization parameter was considered. In the present paper we extend the set of possible solutions up to (3). Thus, we need to introduce a posteriori way for selecting regularization parameter and correct rule for discretization. Besides we set the goal to save both the order for minimal radius of Galerkin information and the accuracy estimation of the projection methods as it is in [16].

3. Method for solving

A modified projection scheme will be applied for economical discretization of operator A. The point of such scheme is to take as discretization domain Ω the hyperbolic cross of the form

$$\Gamma_{b,n} = \{1\}\times[1;2^{bn}] \cup_n^{k=1} (2^{k-1};2^k]\times[1;2^{bn-k}] \subset [1;2^n]\times[1;2^{bn}],$$

where $1 < b \le 2$, $n \in \mathbb{N}$. For simplicity of our computations we consider bn as the integer number. Then by approximative operator to A we understand the following finite-dimensional mapping

$$A_n = P_1 A P_{2^{bn}} + \sum_{k=1}^{n} (P_{2^k} - P_{2^{k-1}}) A P_{2^{bn-k}}. \tag{6}$$

Denote by N the total amount of integer pairs $(i,j) \in \Gamma_{b,n}$. It is known (see [16]) that $N := \mathrm{Card}(\Gamma_{b,n}) = c' 2^{bn} n$ for $1/2 \le c' \le 3/2$. The approximation properties of (6) for the operator class $\mathcal{H}_\gamma^r$ were investigated in [16] and we rewrite them below. So, for any $A \in \mathcal{H}_\gamma^r$ it holds true

$$\|A_n^* A_n - A^* A\| \le C_1 2^{-brn}, \tag{7}$$

$$\|(P_{2^n}A - A_n)\ln^{-p}(A^*A)^{-1}v\| \le \frac{C_2 2^{-brn}}{(brn\ln 2)^{p-1}}, \tag{8}$$

where

$$C_1 = \gamma_1 \max\{\gamma_1,\gamma_0\}\Big[3 + \frac{2^{2r+1}}{2^r-1}\Big], \quad C_2 = \gamma_1\rho(\ln 2)^{-1}\frac{2^r}{r}\beta(p),$$

$$\beta(p) = \frac{1}{p-1}\left(\frac{(b-1)^{1-p}}{b^{1-p}} - 1\right),$$

for $p \neq 1$, and $\beta(1) = \ln\frac{b}{b-1}$.

Because the problem under consideration is ill-posed we need some regularization method to guarantee stability of approximations. In the framework of the paper we stabilize equation (1) following [1]. So, we construct an inverse operator to (1) by means of so-called generating function $g(\lambda)$. The function $g_\alpha(\lambda)$ is Borel measurable on the interval $[0,\gamma_0^2]$ and the following conditions are satisfied

$$\sup_{0<\lambda\le\gamma_0^2} \sqrt{\lambda}\ \ |g_\alpha(\lambda)| \le \tfrac{\chi_*}{\sqrt{\alpha}}, \tag{9}$$

$$\sup_{0<\lambda\le\gamma_0^2} |1-\lambda g_\alpha(\lambda)|\ln^{-p}\lambda^{-1} \le \chi\ln^{-p}\tfrac{1}{\alpha}, \quad 0<p<p_1, \tag{10}$$

where χ, χ_* are some positive constants independent of α. Then as the approximate solution we take

$$x^\delta_{\alpha,n} = g_\alpha(A_n^*A_n)A_n^*P_{2^n}f_\delta. \tag{11}$$

There are many well-known regularization methods satisfying (9). In particular, we can mention Tikhonov's method (with $g_\alpha(\lambda) = (\alpha+\lambda)^{-1}$), Landweber's method (with $g_\alpha(\lambda) = \lambda^{-1}[1-(1-\mu\lambda)^{1/\alpha}]$, $0<\mu<2$), and Showalter's method (with $g_\alpha(\lambda) = \lambda^{-1}(1-\exp(-\lambda/\alpha))$).

In the paper [16] the error bound for (11) was found. For completeness we rewrite the stretch of proof.

Theorem 1 ([16]). *Let approximate solution has the form (11). Then on the class of equations (1) with $A\in\mathcal{H}^r_\gamma, x^\dagger\in M_p(A)$ for any $p>0$ the following holds true*

$$\|x^\dagger - x^n_{\alpha,\delta}\| \le \tag{12}$$

$$\le \chi\rho\ln^{-p}\tfrac{1}{\alpha} + \tfrac{\chi_*}{\sqrt{\alpha}}\left[\delta + \|(P_{2^n}A-A_n)\ln^{-p}(A^*A)^{-1}v\|\right] + \tag{13}$$

$$+\chi\rho C_3\ln^{-p}\|A^*A - A_n^*A_n\|^{-1}, \tag{14}$$

where $C_3 = \begin{cases} 1, & 0<p\le 1 \\ 1+4(5p)^p, & p>1 \end{cases}.$

Proof. The error for (11) can be divided onto two terms

$$x^\dagger - x^n_{\alpha,\delta} := x^\dagger - g_\alpha(A_n^*A_n)A_n^*P_{2^n}f_\delta =$$

$$= (x^\dagger - g_\alpha(A_n^*A_n)A_n^*P_{2^n}f) + g_\alpha(A_n^*A_n)A_n^*P_{2^n}(f-f_\delta). \tag{15}$$

Owing to (9) we estimate the second term as following

$$\|g_\alpha(A_n^*A_n)A_n^*P_{2^n}(f-f_\delta)\| \le \frac{\chi_*\delta}{\sqrt{\alpha}}.$$

The first term we rewrite as

$$\begin{aligned} x^\dagger & -g_\alpha(A_n^*A_n)A_n^*P_{2^n}Ax^\dagger = \\ &= x^\dagger - g_\alpha(A_n^*A_n)A_n^*A_n x + g_\alpha(A_n^*A_n)A_n^*(A_n - P_{2^n}A)x^\dagger = \\ &= [\ln^{-p}(A_n^*A_n)^{-1}v - g_\alpha(A_n^*A_n)A_n^*A_n \ln^{-p}(A_n^*A_n)^{-1}v] + \\ &+(I - g_\alpha(A_n^*A_n)A_n^*A_n)(\ln^{-p}(A^*A)^{-1}v - \ln^{-p}(A_n^*A_n)^{-1}v) + \\ &+g_\alpha(A_n^*A_n)A_n^*(A_n - P_{2^n}A)x^\dagger. \end{aligned} \tag{16}$$

Then by (9) we immediately get

$$\begin{aligned} \|x^\dagger & -g_\alpha(A_n^*A_n)A_n^*P_{2^n}f\| \le \\ &\le \chi\rho\ln^{-p}\tfrac{1}{\alpha} + \tfrac{\chi_*}{\sqrt{\alpha}}\|(A_n - P_{2^n}A)x^\dagger\| + \\ &+\|(I - g_\alpha(A_n^*A_n)A_n^*A_n)(\ln^{-p}(A^*A)^{-1}v - \ln^{-p}(A_n^*A_n)^{-1}v)\| \le \\ &\le \chi\rho\ln^{-p}\tfrac{1}{\alpha} + \tfrac{\chi_*}{\sqrt{\alpha}}\|(P_{2^n}A - A_n)x^\dagger\| + \\ &+\chi\|\ln^{-p}(A^*A)^{-1}v - \ln^{-p}(A_n^*A_n)^{-1}v\|. \end{aligned}$$

Using the following relation (see [5, Theorem 4])

$$\left|\ln^{-p}\frac{1}{s} - \ln^{-p}\frac{1}{t}\right| \le C_3\ln^{-p}|s-t|^{-1},$$

where $|s-t| < e^{-1}$ for $s,t \in (0; e^{-1}]$, we have

$$\begin{aligned} \|x^\dagger & -x_{\alpha,\delta}^n\| \le \\ &\le \chi\rho\ln^{-p}\tfrac{1}{\alpha} + \tfrac{\chi_*}{\sqrt{\alpha}}\left[\delta + \|(P_{2^n}A - A_n)\ln^{-p}(A^*A)^{-1}v\|\right] + \\ &+\chi C_3\rho\ln^{-p}\|A^*A - A_n^*A_n\|^{-1}, \end{aligned}$$

that has to be proved. □

Remark 9. *Let consider the function $\beta(p)$ which is included in the bound (8). The analysis of behavior of $\beta(p)$ shows that it is continuous monotonically increasing function. Thus we have that for all $0 < p \le p_1$ the following inequality holds true*

$$\beta(p) \le \beta(p_1) = \begin{cases} \frac{1}{p_1-1}\left(\frac{(b-1)^{1-p_1}}{b^{1-p_1}} - 1\right), & p_1 \ne 1, \\ \ln\frac{b}{b-1}, & p_1 = 1. \end{cases}$$

To minimize the error bound (12) we take discretization parameter n according to the rule

$$(br\ln 2)n2^{-brn} = \delta. \tag{17}$$

The equality means that as discretization value n we take the number which is rounded up to solution of (17). Taking into account (17) and remark 9 the estimations (7) and (8) can be rewritten in the following way

$$\|A_n^*A_n - A^*A\| \le C_1\delta, \tag{18}$$

$$\|(P_{2^n}A - A_n)\ln^{-p}(A^*A)^{-1}v\| \le C_4\delta, \tag{19}$$

where $C_4 = \gamma_1\rho(\ln 2)^{-1}\frac{2^r}{r}\beta(p_1)$.

Due to (18) and (19) the error bound (12) can be represented as follows

$$\|x^\dagger - x^n_{\alpha,\delta}\| \le \chi\rho\ln^{-p}\frac{1}{\alpha} + \chi_*(1+C_4)\frac{\delta}{\sqrt{\alpha}} + \chi C_3\rho\ln^{-p}(C_1\delta)^{-1}. \tag{20}$$

Obviously, that for $\alpha_0 = \ln(\delta^{-1})(C_1\delta)^2$ we have

$$\ln^{-p}(C_1\delta)^{-1} = 2^p\ln^{-p}\left(\ln(\delta^{-1})(\ln(\delta^{-1})C_1^2\delta^2)^{-1}\right) = 2^p\ln^{-p}\left(\ln(\delta^{-1})(\alpha_0)^{-1}\right).$$

In this way for all $\alpha \ge \alpha_0$ it holds true that

$$\ln^{-p}\alpha^{-1} \ge \ln^{-p}\alpha_0^{-1} > \frac{1}{2^p}\ln^{-p}(C_1\delta)^{-1}.$$

Let denote by $\eta_1(\alpha) = C_5\ln^{-p}\frac{1}{\alpha}$ and $\eta_2(\alpha) = C_6\frac{\delta}{\sqrt{\alpha}}$, where $C_5 = \chi\rho + \chi C_3\rho 2^p$ and $C_6 = \chi_*(1+C_4)$. Thus error bound (20) can be rewritten as follows

$$\|x^\dagger - x^n_{\alpha,\delta}\| \le \eta_1(\alpha) + \eta_2(\alpha), \tag{21}$$

where the functions $\eta_1(\alpha)$ and $\eta_2(\alpha)$ for $\alpha \to \infty$ are monotone increasing and decreasing convex functions respectively.

4. A posteriori selection of regularization parameter

Fix some real number $q > 1$ and define by D_M the set of possible values for the parameter α:

$$D_M = \{\alpha_i = \alpha_0(q^2)^i, i = 1, 2, ..., M\},$$

where $\alpha_0 = \ln(\delta^{-1})(C_1\delta)^2$, $M = \left[\frac{\log\alpha_0^{-1}}{2\log q}\right]$. Then according to the balancing principle (see, for example, [9]) selection of index i_+ for parameter α is realized by the rule

$$i_+ = \max\{i : \alpha_i \in D_M^+\}, \tag{22}$$

where

$$D_M^+ = \{\alpha_i \in D_M : \|x^\delta_{\alpha_i,n} - x^\delta_{\alpha_j,n}\| \le 4\eta_2(\alpha_j), \quad j = 1, ..., i\}.$$

Further we introduce the auxiliary values

$$\alpha_* := \max\{\alpha_i \in D_M : \eta_1(\alpha_i) \le \eta_2(\alpha_i)\},$$

$$\hat\alpha = \{\alpha_i \in D_M : \quad \eta_1(\alpha_i) = \eta_2(\alpha_i)\}.$$

Theorem 2. *Let $A \in \mathcal{H}_r^\gamma$ and $x^\dagger \in M(A)$. Then for the projection method (11), (17), (22) the following error bound*

$$\|x^\dagger - x^\delta_{\alpha_+,n}\| \le 6q\eta_1(\hat\alpha) \tag{23}$$

takes place.

Proof. Let check that $\alpha_* \le \alpha_+$. Due to (21) it holds true that for all $\alpha \le \alpha_*$

$$\|x^\delta_{\alpha,n} - x^\delta_{\alpha_*,n}\| \le \|x^\dagger - x^\delta_{\alpha,n}\| + \|x^\dagger - x^\delta_{\alpha_*,n}\| \le \eta_1(\alpha) + \eta_2(\alpha) + \eta_1(\alpha_*) + \eta_2(\alpha_*) \le$$
$$\le 2\eta_2(\alpha) + 2\eta_2(\alpha_*) \le 4\eta_2(\alpha).$$

Consequently $\alpha_* \in D^+_M$ and $\alpha_* \le \alpha_+$.

Taking into account definitions of α_* and α_+, from (22) and (21) we have

$$\|x^\dagger - x^\delta_{\alpha_+,n}\| \le \|x^\dagger - x^\delta_{\alpha_*,n}\| + \|x^\delta_{\alpha_*,n} - x^\delta_{\alpha_+,n}\| \le 6\eta_2(\alpha_*).$$

It is evident that $\alpha_* \le \hat\alpha \le q^2\alpha_*$ then we find

$$\|x^\dagger - x^\delta_{\alpha_+,n}\| \le 6\eta_2(\alpha_*) = 6q\eta_2(\alpha_* q^2) \le 6q\eta_2(\hat\alpha) = 6q\eta_1(\hat\alpha),$$

which was to be proved. □

Theorem 3. *Let $A \in \mathcal{H}^\gamma_r$ and $x^\dagger \in M(A)$. Then error bound for the projection method (11), (17), (22) is the following*

$$\|x^\dagger - x^\delta_{\alpha_+,n}\| \le 6q\kappa_p \ln^{-p}\delta^{-1}, \tag{24}$$

where κ_p is some constant that does not depend on δ.

Proof. It is easy to find that

$$\hat\alpha \le \left(\frac{C_6}{C_5}\delta\right)^{\frac{2}{1+2p}},$$

then from (23) we have

$$\|x^\dagger - x^\delta_{\alpha_+,n}\| \le 6q\ln^{-p}\hat\alpha^{-1} \le 6q\ln^{-p}\left(\frac{c_6}{c_5}\delta\right)^{-\frac{2}{1+2p}} = 6q\kappa_p \ln^{-p}\delta^{-1}. \quad \square$$

Remark 10. *It is well-known (see, for instance [17]) that for severely ill-posed problems any approximation method guaranteing accuracy $O(\ln^{-p}\delta^{-1})$ is optimal by the order on the whole set of solutions (3). Thus, theorem 3 shows that our method (11), (22), (17) saves optimal order of accuracy.*

5. Minimal radius of Galerkin information

Now we are ready to prove the upper bound for $R_{N,\delta}\left(\mathcal{H}^r_\gamma, M(A)\right)$.

Theorem 4. *Let $A \in \mathcal{H}^\gamma_r$ and $x^\dagger \in M(A)$. The parameters n and α for (11) are chosen according to (17) and (22) respectively. Then for sufficiently small δ the following inequality*

$$R_{N,\delta}\left(\mathcal{H}^r_\gamma, M(A)\right) \le c_p \ln^{-p} N^{2r}$$

holds true where $c_p = 6q\kappa_p\left(\frac{r(1-\mu)-\mu}{2r}\right)^{-p}$ and $\forall\mu : r(1-\mu) - \mu > 0$.

Proof. By virtue of (17) we have

$$br\ln 2(2^{bn}n)^{-r}n^{r+1} = \delta.$$

Using the relation $N = c'2^{bn}n$ we get

$$(c')^{-r}(br\ln 2)^{-1}N^r n^{-r-1} = \delta^{-1}. \tag{25}$$

By evident relation
$$n \le \frac{\ln N}{b \ln 2}$$
and (25) we have
$$\delta^{-1} = \frac{(c')^{-r} N^r}{(\ln 2) b r n^{r+1}} \ge \frac{(c')^{-r} N^r (b \ln 2)^{r+1}}{b r \ln 2 (\ln N)^{r+1}} = \frac{(c')^{-r} N^r (b \ln 2)^r}{r (\ln N)^{r+1}}. \quad (26)$$
Starting with some N it is holds true that $\ln N \le N^\mu$, then for any $\mu > 0$ we have
$$\frac{N^r (b \ln 2)^r}{(c')^r r (\ln N)^{r+1}} \ge \frac{N^r (c' b \ln 2)^r}{(c')^r r N^{\mu(r+1)}} = N^{r - r\mu - \mu} \frac{(c' b \ln 2)^r}{(c')^r r} =$$
$$= N^{r \frac{r(1-\mu)-\mu}{r}} \frac{(c' b \ln 2)^r}{(c')^r r}.$$
Taking into account the relation above from (26) we have
$$N^{r \frac{r(1-\mu)-\mu}{r}} \frac{(b \ln 2)^r}{(c')^r r} \le \delta^{-1}.$$
Without loss of generality we suppose that $\mu : r(1-\mu) - \mu > 0$. Then taking logarithm from inequality above one can find
$$\frac{r(1-\mu)-\mu}{2r} \ln N^{2r} \le \ln \delta^{-1}.$$
Hence, the error estimation (24) takes the form
$$\|x^\dagger - x^\delta_{\alpha_+, n}\| \le 6 q \kappa_p \ln^{-p} \delta^{-1} \le c_p \ln^{-p} N^{2r}.$$
Due to definition for $R_{N,\delta}\left(\mathcal{H}^r_\gamma, M(A)\right)$ we get
$$R_{N,\delta}\left(\mathcal{H}^r_\gamma, M(A)\right) \le c_p \ln^{-p} N^{2r},$$
which was to be proved. □

Theorem 5. *Let $A \in \mathcal{H}^\gamma_r$ and $x^\dagger \in M(A)$, then*
$$\frac{1}{2^{p+1}} \ln^{-p} N^{2r} \le R_{N,\delta}\left(\mathcal{H}^r_\gamma, M(A)\right) \le c_p \ln^{-p} N^{2r},$$
where $N \asymp \delta^{-\frac{1}{r}} \ln^{\frac{r+1}{r}} \delta^{-1}$. Indicated order $O(\ln^{-p} N^{2r})$ is achieved in the framework of projection method (11) , (17), (22).

Proof. It is known (see, for instance [16]) that for all $p > 0$ it fulfills $R_{N,\delta}\left(\mathcal{H}^r_\gamma, M_p(A)\right) \ge \tilde{c}_p \ln^{-p} N^{2r}$, where $\tilde{c}_p = 2^{-p-1}$. By virtue of definition for the sets $M_p(A)$ and $M(A)$ the following inequality holds true
$$R_{N,\delta}\left(\mathcal{H}^r_\gamma, M_p(A)\right) \le R_{N,\delta}\left(\mathcal{H}^r_\gamma, M(A)\right).$$
Due to Theorem 4 we immediately get statement of the theorem. □

Remark 11. *From Theorem 5 it follows that our approach gives optimal error bound with amount of discrete information in the form of Galerkin functionals (4).*

Remark 12. *Let consider the set* $M'(A) = \cup_{p\in[1,p_1]} M_p(A) \subset M(A)$. *If we assume that* $x^\dagger \in M'(A)$, *then the relation (17) should be replaced by the following*

$$(\ln 2) br 2^{-brn} = \delta, \tag{27}$$

with saving bounds (18) and (19). As we can see below, such selection of discretization parameter allows to reduce amount of discrete information by logarithmic multiplier.

Theorem 6. *Let* $A \in \mathcal{H}_r^\gamma$ *and* $x^\dagger \in M'(A)$. *The parameters* n *and* α *for (11) are chosen according to (27) and (22) respectively. Then for sufficiently small* δ *it holds true*

$$\frac{1}{2^{p+1}} \ln^{-p} N^{2r} \le R_{N,\delta}\left(\mathcal{H}_\gamma^r, M'(A)\right) \le c_p \ln^{-p} N^{2r},$$

where $N \asymp \delta^{-\frac{1}{r}} \ln \delta^{-1}$.

Proof. The proving of the theorem completely repeats as ones for Theorems 4 and 5. □

Remark 13. *Comparing Theorems 5 and 6 we can conclude that due to restriction of the set of possible solutions we obtain reduction of amount of discrete information by logarithmic multiplier (compare the values* N *in both theorems).*

Bibliography

1. Bakushinski A. B. A general method of constructing regularizing algorithms for a linear ill-posed equation in Hilbert Space / A. B. Bakushinski // USSR Comput. Math. Math. Phys. – 1967. – Vol. 7. – P. 672-677.
2. Bakushinski A. B. Iterative methods for solution of ill-posed operator equations with smooth operators / A. B. Bakushinski, M. Yu. Kokurin. – Moscow: Editorial URSS, 2002.
3. Hohage T. Regularization of exponentially ill-posed problems / T. Hohage // Numer. Funct. Anal. Optim. – 2000. – Vol. 21. – P. 439-464.
4. Mair B. A. Tikhonov regularization for finitely and infinitely smoothing operators / B. A. Mair // SIAM J. Math. Anal. – 1994. – Vol. 25. – P. 135-147.
5. Mathe P. Moduli of continuity for operator valued functions /P. Mathe, S.V. Pereverzev // Numer. Funct. Anal. Optim. – 2002. – Vol. 23. – P. 623-631.
6. Mathe P. Discretization strategy for ill-posed problems in variable Hilbert scales / P. Mathe, S. V. Pereverzev // Inverse Problems. – 2003. – Vol. 19. – P. 1263-1277.
7. Pereverzev S. V. Optimization of methods for approximate solution of operator equations / S. V. Pereverzev. – N-Y: Nova Science Publishers, 1996.
8. Pereverzev S. V. The minimal radius of Galerkin information for the Fredholm problems of first kind / S. V. Pereverzev, S. G. Solodky // J. Complexity. – 1996. – Vol. 12. – P. 401-415.
9. Pereverzev S. V. On the adaptive selection of the parameter in regularization of ill-posed problems / S. V. Pereverzev, E. Schock // SIAM J. Numer. Anal. – 2005. – Vol. 43. – P. 2060-2076.
10. Solodky S. G. The optimal approximations for solving linear ill-posed problems / S. G. Solodky // J. Complexity. – 2001. – Vol. 17. – P. 98-116.
11. Pereverzev S. V. Optimization of Projection Methods for Solving Ill-Posed Problems / S. V. Pereverzev // Computing. – 1999. – Vol. 55. – P. 113-124.
12. Plato R. On the Regularization of Projection Methods for solving Ill-posed Problems / R. Plato, G. Vainikko // Numer. Math. – 1990. – Vol. 57. – P. 63-79.

13. Solodky S. G. A generalized projection scheme for solving ill-posed problems / S. G. Solodky // J. Inverse Ill-Posed Probl. – 1999. – Vol. 7. – P. 185-200.
14. Solodky S. G. Bounds of information expenses in constructing projection methods for solving ill-posed problems / S. G. Solodky, E. V. Lebedeva // Comp. Method Appl. Math. – 2006. – Vol. 6. – P. 87-93.
15. Solodky S. G. On the a posteriori choice of regularization parametr in the solution of severelly ill-posed problems / S. G. Solodky, A. V. Grushevaya // Journal of Mathematical Science. – 2012. – Vol. 181. – P. 98-105.
16. Solodky S. G. The minimal radius of Galerkin information for severely ill-posed problems / S. G. Solodky, G. L. Myleiko // Journal of Inverse and Ill-Posed Problems. – 2014. – Vol. 5. – P. 735-757.
17. Tautenhahn U. Optimality for ill-posed problems under general source condition / U. Tautenhahn // Numer. Funct. Anal. and Optimiz. – 1998. – Vol. 19. – P. 377-398.
18. Traub J. F. Information-Based Complexity / J. F. Traub, G. Wasilkowski, H. Wozniakowski. – Boston: Academic Press, 1988.
19. Wozniakowski H. Information based complexity / H. Wozniakowski // Ann. Rev. Comput. Sci. – 1986. – Vol. 1. – P. 319-380.

S. G. Solodky, E. V. Semenova,
Institute of Mathematics, National Academy of Sciences,
3, Tereschenkivs'ka Str., Kyiv, 01601, Ukraine;

Received 10.06.2015