Журнал обчислювальної та прикладної математики | 2015 № 2 (119) | Journal of Computational & Applied Mathematics

UDC 519.6

DIFFERENCE METHODS FOR SOLVING INVERSE EIGENVALUE PROBLEM

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Резюме. В роботі розглянуто обернену задачу на власні значення. Для чисельного розв'язування задачі застосовано метод хорд і метод лінійної інтерполяції (метод Курчатова). На відміну від методу Ньютона, ці ітераційні процеси використовують лише значення оператора з двох попередніх ітерацій та не потребують аналітично заданих похідних. Запропоновані методи застосовано для розв'язування обернених задач на власні значення різного типу. Розглянуті ітераційні процеси порівнюються з методом Ньютона за кількістю операцій, потрібних для обчислення першої поділеної різниці та похідної детермінанта.

Abstract. In this paper an inverse eigenvalue problem is considered. Secant method and method of the linear interpolation (Kurchatov's method) are applied for the numerical solution of this problem. Unlike Newton's method, these methods use only values of the operator at two previous iterations and do not require analytical derivatives. Proposed methods are used for solving different types of inverse eigenvalue problems. Considered iterative processes are compared with the Newton's method by the number of operations required to compute the first divided difference and derivative of determinant.

1. Introduction

An inverse eigenvalue problem (IEP) is to determine a matrix from a given spectral data. These problems arise in many applications, including control design, system identification, structure analysis and so on. There are special cases of inverse eigenvalue problems. Let's consider the following problems.

General IEP. Let $A_i = \{a^i_{jk}\}$ be complex $n \times n$ matrices for $i = \overline{0,n}$ and $\lambda = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{C}^n$. Find the vector $p = (p_1, p_2, \ldots, p_n) \in \mathbb{C}^n$, such that matrix

$$A(p) = A_0 + \sum_{i=1}^{n} p_i A_i$$

has eigenvalues $\lambda_1, \ldots, \lambda_n$. This problem involves classic partial cases of additive and multiplicative inverse eigenvalue problems.

Additive IEP. Let A be a given complex $n \times n$ matrix and $\lambda = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{C}^n$. Find the diagonal matrix $D = diag(p_1, p_2, \ldots, p_n)$, $p_i \in \mathbb{C}$, $i = \overline{1,n}$, such that matrix $A + D$ has eigenvalues $\lambda_1, \ldots, \lambda_n$.

Multiplicative IEP. Let A be a given complex $n \times n$ matrix and $\lambda = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{C}^n$. Find the diagonal matrix $D = diag(p_1, p_2, \ldots, p_n)$, $p_i \in \mathbb{C}$, $i = \overline{1,n}$, such that matrix AD has eigenvalues $\lambda_1, \ldots, \lambda_n$.

Key words. Inverse eigenvalue problem, Secant method, Kurchatov's method.

There are a large of literature on conditions for the solvability of inverse eigenvalue problems, different approaches and numerical methods for its solving [1–3,6,8]. We use approach, which calculates the zeros of the nonlinear function

$$F(p) = \begin{bmatrix} \det(A(p) - \lambda_1 I) \\ \vdots \\ \det(A(p) - \lambda_n I) \end{bmatrix}, \tag{1}$$

where $\lambda_1, \ldots, \lambda_n$ are given eigenvalues and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Vector $p = \{p_1, p_2, \ldots, p_n\}^T \in \mathbb{C}^n$ is a solution of the inverse eigenvalue problem if and only if

$$F(p) = 0. \tag{2}$$

In papers [1,8] the Newton's method is used for solving systems of nonlinear equations (2) with $F(p)$ as (1). It is known that the application of Newton's method requires the calculation of the first derivative of determinant at every iteration. To calculate this derivative some authors use Trace-Theorem of Davidenko or LU decomposition of matrix [1,7,8].

In this work we apply difference methods for solving inverse eigenvalue problem, including Secant method and method of linear interpolation (Kurchatov's method), assuming the existence of a solution. These methods do not require analytical derivatives and can be applied to a wider range of problems.

2. Algorithms of difference methods

A well-known simple difference method for solving nonlinear equations is the Secant method

$$p^{(k+1)} = p^{(k)} - F(p^{(k-1)}; p^{(k)})^{-1} F(p^{(k)}) \tag{3}$$

with convergence order $\dfrac{1+\sqrt{5}}{2}$. An other method is the quadratically convergent Kurchatov's method

$$p^{(k+1)} = p^{(k)} - F(2p^{(k)} - p^{(k-1)}; p^{(k-1)})^{-1} F(p^{(k)}). \tag{4}$$

In formulas (3) and (4) $F(x; y)$ is a divided difference of the first order of F at the points x and y. Convergence analysis of difference methods (3) and (4) for solving nonlinear operator equations was conducted by the authors [4,5,9,10].

Let F be a nonlinear operator defined on a subset D of a linear space X with values in a linear space Y and let x, y be two points of D. A linear operator from X into Y, denoted as $F(x; y)$, which satisfies the condition

$$F(x; y)(x - y) = F(x) - F(y)$$

is called a divided difference of the first order of F at the points x and y. In the case of systems of nonlinear equations the divided difference $F(x; y)$ is $n \times n$ matrix. Its elements are calculated by the following formula:

$$F(x; y)_{i,j} = \frac{F_i(x_1, \ldots, x_j, y_{j+1}, \ldots, y_n) - F_i(x_1, \ldots, x_{j-1}, y_j, \ldots, y_n)}{x_j - y_j}, \quad i, j = \overline{1, n}.$$

From (1) and the last formula we see that to calculate the elements of vector F and matrix of divided differences we need to calculate determinants of matrices. To calculate the determinant we apply the LU decomposition of matrix, as in [7, 8]. Let $D(\lambda)$ be a matrix whose elements are functions of λ. Then for a fixed value $\lambda = \lambda_m$ we can calculate $D = LU$ or $PD = LU$, where P is a permutation matrix, $\det P = (-1)^q$, q is a number of permutations and

$$\det D = \det L \det U = \prod_{i=1}^{n} u_{ii} \tag{5}$$

or

$$\det D = \det P \det L \det U = (-1)^q \prod_{i=1}^{n} u_{ii}. \tag{6}$$

Algorithm of the Secant method for solving IEP.

1. Choose initial approximations $p^{(-1)}$ and $p^{(0)}$.
2. For $k = 0$ until convergence, do:
   (a) Compute LU decomposition of matrices $D_i = A(p^{(k)}) - \lambda_i I$, $i = \overline{1,n}$, $D_i^{'} = A(p') - \lambda_i I$, $D_i^{''} = A(p'') - \lambda_i I$, $(i, j = \overline{1,n})$, where
   $$p' = (p_1^{(k-1)}, \dots, p_j^{(k-1)}, p_{j+1}^{(k)}, \dots, p_n^{(k)}),$$
   $$p'' = (p_1^{(k-1)}, \dots, p_{j-1}^{(k-1)}, p_j^{(k)}, \dots, p_n^{(k)}).$$
   (b) Compute $F_i(p^{(k)}) = \det(D_i)$, $i = \overline{1,n}$ by formula (5) or (6) and form vector $F(p^{(k)})$.
   (c) Compute $F_i(p') = \det(D_i^{'})$, $F_i(p'') = \det(D_i^{''})$, $i, j = \overline{1,n}$ by formula (5) or (6) and form matrix $F(p^{(k-1)}; p^{(k)})$, where
   $$F(p^{(k-1)}; p^{(k)})_{i,j} = \frac{F_i(p') - F_i(p'')}{p_j^{(k-1)} - p_j^{(k)}}, (i, j = \overline{1,n}).$$
   (d) Compute $p^{(k+1)}$ by the formula (3).

Algorithm of the Kurchatov's method for solving IEP.

1. Choose initial approximations $p^{(-1)}$ and $p^{(0)}$.
2. For $k = 0$ until convergence, do:
   (a) Compute LU decomposition of matrices $D_i = A(p^{(k)}) - \lambda_i I$, $i = \overline{1,n}$, $D_i^{'} = A(p') - \lambda_i I$, $D_i^{''} = A(p'') - \lambda_i I$, $(i, j = \overline{1,n})$, where
   $$p' = (2p_1^{(k)} - p_1^{(k-1)}, \dots, 2p_j^{(k)} - p_j^{(k-1)}, p_{j+1}^{(k-1)}, \dots, p_n^{(k-1)}),$$
   $$p'' = (2p_1^{(k)} - p_1^{(k-1)}, \dots, 2p_{j-1}^{(k)} - p_{j-1}^{(k-1)}, p_j^{(k-1)}, \dots, p_n^{(k-1)}).$$
   (b) Compute $F_i(p^{(k)}) = \det(D_i)$, $i = \overline{1,n}$ by formula (5) or (6) and form vector $F(p^{(k)})$.
   (c) Compute $F_i(p') = \det(D_i^{'})$, $F_i(p'') = \det(D_i^{''})$, $i, j = \overline{1,n}$ by formula (5) or (6) and form matrix $F(2p^{(k)} - p^{(k-1)}; p^{(k-1)})$, where
   $$F(2p^{(k)} - p^{(k-1)}; p^{(k-1)})_{i,j} = \frac{F_i(p') - F_i(p'')}{2(p_j^{(k)} - p_j^{(k-1)})}, (i, j = \overline{1,n}).$$

(d) Compute $p^{(k+1)}$ by the formula (4).

Note, that matrices D_i, $D_i^{'}$, $D_i^{''}$ can coincide with each other. In this case LU decomposition and determinant can be calculated only once and thus the amount of computation is reduced.

Next we consider the computational complexity of proposed algorithms. Let compute the amount of operations (multiplications and division) required to compute divided differences. It is known that to get LU decomposition of matrix and compute its determinant by formula (5) it is need $\dfrac{n^3+2n-3}{3}$ operations [7, 8]. In the same articles it is shown that to compute the first derivative of determinant it is required n^3+n^2-n operations.

To compute divided difference of determinant using LU decomposition it is required $\dfrac{2n^3+4n-3}{3}$ operations for Secant method and $\dfrac{2n^3+7n-3}{3}$ operations for Kurchatov's method.

From these assessments we conclude that the difference methods are more effective than Newton's method by the amount of calculations in one iteration. However, the number of iterations for difference methods usually is greater than for Newton's method, in particular for the Secant method.

3. Numerical experiments

In this section we present results of Secant and Kurchatov's methods and compare with results of Newton's method. We consider inverse eigenvalue problems with distinct eigenvalues. All vectors will be written as row-vectors. To apply the methods (3) and (4) we need to set the additional approximation $p^{(-1)}$. To get good starting values it was chosen in the following way: $p^{(-1)} = p^{(0)} + 10^{-4}$. The iterations of considered iterative processes were stopped when $\|p^{(k+1)} - p^{(k)}\|_\infty < \varepsilon$ or $\|F(p^{(k+1)})\|_\infty < \varepsilon$, $\varepsilon = 10^{-9}$.

Example 3.1 Consider the general inverse eigenvalue problem [1]. Let $n = 5$,

$$A_0 = \begin{pmatrix} 2 & -0.08 & 0 & 0 & 0 \\ -0.03 & 2 & -0.08 & 0 & 0 \\ 0 & -0.03 & 2 & -0.08 & 0 \\ 0 & 0 & -0.03 & 2 & -0.08 \\ 0 & 0 & 0 & -0.03 & 2 \end{pmatrix},$$

$$R = \sum_{i=1}^{n} r_i e_i^T = \begin{pmatrix} 1 & 0 & 0.01 & -0.02 & 0.03 \\ -0.03 & 1 & 0 & 0.01 & -0.02 \\ 0.02 & -0.03 & 1 & 0 & 0.01 \\ -0.01 & 0.02 & -0.03 & 1 & 0 \\ 0 & -0.01 & 0.02 & -0.03 & 1 \end{pmatrix}$$

and $A_i = r_i e_i^T$, $i = 1, \ldots, 5$, where e_i – i-th unit vector. The given eigenvalues are $\lambda = (\delta,\ 1-\delta,\ 2+\delta,\ 3-\delta,\ 4)$.

Let $\delta = 0$ and $p^{(0)} = (-2, -1, 0, 1, 2)$. Then Newton's method converge to a solution

$$p^* = (1.99279,\ 1.00257,\ 0.00237,\ -0.99786,\ -1.99987).$$

Using the same starting point $p^{(0)}$, we found a different solution

$$p^* = (-2.00240,\ -0.99800,\ 0.00236,\ 1.00271,\ 1.99533)$$

by Secant and Kurchatov's methods.

Let $\delta = 0.441$. Then Newton's method, methods (3) and (4) converge to a solution

$$p^* = (-1.56910,\ -1.43181,\ 0.49205,\ 0.51127,\ 1.99758)$$

with the starting point $p^{(0)} = (-2,\ -1,\ 0,\ 1,\ 2)$. The received results are displayed in the Table 1.

TABL. 1. The numerical results for example 3.1

	Iterations, k	$\|p^{(k)} - p^{(k-1)}\|_\infty$	$\|F(p^{(k)})\|_\infty$
Newton's method	10	5.73238×10^{-10}	8.07568×10^{-15}
Kurchatov's method	10	9.88872×10^{-11}	4.12121×10^{-15}
Secant method	14	2.31415×10^{-11}	8.07565×10^{-15}

Example 3.2 Consider an additive inverse eigenvalue problem with distinct eigenvalues [3]. Here n = 8,

$$A_0 = \begin{pmatrix} 0 & 4 & -1 & 1 & 1 & 5 & -1 & 1 \\ 4 & 0 & -1 & 2 & 1 & 4 & -1 & 2 \\ -1 & -1 & 0 & 3 & 1 & 3 & -1 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & -1 & 4 \\ 1 & 1 & 1 & 1 & 0 & 1 & -1 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & -1 & 6 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{pmatrix}, \quad A_i = e_i e_i^T, \ i = \overline{1,8}.$$

The eigenvalues of the problem 3.2 are $\lambda^* = (10,\ 20,\ 30,\ 40,\ 50,\ 60,\ 70,\ 80)$.

TABL. 2. The numerical results for example 3.2

	Secant method	Kurchatov's method
k	$\|p^{(k)} - p^*\|_\infty$	$\|p^{(k)} - p^*\|_\infty$
0	8.68150	8.68150
1	2.31065	2.31079
2	1.10171	0.59989
3	0.23738	0.05708
4	0.02958	0.00171
5	0.00085	5.26419×10^{-6}
6	4.20674×10^{-6}	5.07569×10^{-10}
7	6.34913×10^{-10}	

Proposed methods converge to a solution

$$p^* = (11.907888,\ 19.705522,\ 30.545498,\ 40.062657,\ 51.587140,\ 64.702131,\ 70.170676,\ 71.318499)$$

with the starting point $p^{(0)} = (10, 20, 30, 40, 50, 60, 70, 80)$. The result was obtained in 8 (Secant method) and 7 (Kurchatov's method) iterations. The nature of the convergence of the considered numerical methods is shown in Table 2.

Applying difference methods (3) and (4) to this problem with the starting point $p^{(0)} = (10, 80, 70, 50, 60, 30, 20, 40)$ we find the following solution in 7 iterations:

$$p^* = (11.461354, 78.880829, 68.353400, 49.878330,$$
$$59.168918, 30.410470, 24.834324, 37.012374).$$

So, difference methods can be applied for solving inverse eigenvalue problems. Also these methods are simple in program implementation.

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Received 10.06.2015