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DYNAMICS OF TERNARY STATISTICAL EXPERIMENTS WITH EQUILIBRIUM STATE

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Резюме. Вивчаються сценарії динаміки моделі тринарних статистичних експериментів. Важливою особливістю моделі є умова балансу і наявність стаціонарного стану рівноваги (ρ). Це дозволяє використовувати різницеві рівняння для приростів ймовірностей для вивчення динаміки моделі. Дається класифікація сценаріїв еволюції моделі, які значно відрізняються один від одного в залежності від області значень основних параметрів моделі V_0 і ρ_0.

Abstract. We study the scenarios of dynamics ternary statistical experiments model. /par An important feature of the model is a balance condition and the presence of steady state (equilibrium). This allows to use difference equations for increments probabilities to study the dynamics of the model. /par We give a classification of scenarios of the model's evolution which are significantly different one from another depending on the domain of the values of the model basic parameters V_0 and ρ_0 (see. Proposition 1).

1. Building a model

We consider *statistical experiments* (SE) with persistent linear regression [1] with additional alternatives.

The basic idea of the model construction is to choose a main factor that determines the essential state of SE, supplemented by additional alternatives in the way that, the aggregation of the principal factor and its complementary alternatives completely describe the dynamics of CE on time.

The *basic characteristic* of the main factor and of the additional alternatives are their probabilities (frequencies): P_0 of the main factor and P_1, P_2 of the additional alternatives, for which the balance condition takes place:

$$P_0 + P_1 + P_2 = 1. \tag{1}$$

The dynamics of SE characteristics is determined by a *linear regression function* which specifies the values of SE characteristics in the next stage of observation, for given value probability at the present stage.

Consider a sequence of SE characteristics values which depends on the stage of observation, or, equivalently, on a discrete time parameter $k \geq 0$:

$$P(k) := (P_0(k), P_1(k), P_2(k)) \ , \quad k \geq 0,$$

and their increments at k-th time instant:

$$\Delta P(k+1) := P(k+1) - P(k) \ , \quad k \geq 0.$$

Key words. Binary statistical experiment, persistent regression, stabilization, stochas-tic approximation, exponential statistical experiment, exponential autoregression process.

Linear regression function of increments is given by a matrix which is generated by directing action parameters:

$$\Delta P(k+1) = -\widehat{\mathbb{V}} P(k) \ , \quad k \geq 0, \tag{2}$$

where

$$\begin{aligned} &\widehat{\mathbb{V}} := [\widehat{V}_{mn} \ ; \quad 0 \leq m, n \leq 2], \\ &\widehat{V}_{mm} = 2V_m \ , \quad \widehat{V}_{mn} = -V_n \ , \quad 0 \leq n \leq 2 \ , \quad n \neq m. \end{aligned} \tag{3}$$

The directing action parameters V_0, V_1, V_2 satisfy the following inequality:

$$|V_m| \leq 1 \ , \quad 0 \leq m \leq 2. \tag{4}$$

An important feature of SE is the presence of steady state ρ (equilibrium), which is determined by zero of the regression function of increments :

$$\widehat{\mathbb{V}} \rho = 0, \tag{5}$$

or in scalar form:

$$\widehat{V}_m \rho := \widehat{V}_{m0}\, \rho_0 + \widehat{V}_{m1}\, \rho_1 + \widehat{V}_{m2}\, \rho_2 = 0 \ , \quad 0 \leq m \leq 2. \tag{6}$$

Of course, the following balance condition takes place:

$$\rho_0 + \rho_1 + \rho_2 = 1. \tag{7}$$

Next, we consider the fluctuations probabilities relative to equilibrium value

$$\widehat{P}_m(k) := P_m(k) - \rho_m \ , \quad 0 \leq m \leq 2. \tag{8}$$

The basic assumption. SE dynamics is determined by a difference equation for the main factor probabilities $\widehat{P}_0(k)$, and by the probabilities of additional alternatives $\widehat{P}_1(k)$ and $\widehat{P}_2(k)$

$$\Delta \widehat{P}(k+1) = -\widehat{\mathbb{V}} \widehat{P}(k) \ , \quad k \geq 0, \tag{9}$$

or in scalar form:

$$\Delta \widehat{P}_m(k+1) = \widehat{V}_{m0}\, \widehat{P}_0(k) + \widehat{V}_{m1}\, \widehat{P}_1(k) + \widehat{V}_{m2}\, \widehat{P}_2(k) \ , \quad 0 \leq m \leq 2 \ , \quad k \geq 0. \tag{10}$$

Also the initial values have to be fixed:

$$\widehat{P}(0) = (\widehat{P}_0(0), \widehat{P}_1(0), \widehat{P}_2(0)).$$

Remark 1. *Considering equations (5) - (6) and the balance condition (7), we have explicit formulas for equilibrium:*

$$\begin{aligned} &\rho_m = V_m^{-1} / \overline{V} \ , \quad 0 \leq m \leq 2, \\ &\overline{V} := V_0^{-1} + V_1^{-1} + V_2^{-1}, \end{aligned}$$

or in other form:

$$\begin{aligned} &\rho_0 = V_1 V_2 / V \ , \quad \rho_1 = V_0 V_2 / V \ , \ \rho_2 = V_0 V_1 / V, \\ &V := V_1 V_2 + V_0 V_2 + V_0 V_1. \end{aligned} \tag{11}$$

The validity of the formulas (11) and (12) can be easily confirmed by their substitution in equations (6) - (7). This is obvious the additional condition: $V \neq 0$.

Remark 2. *The dynamics determination, by linear regression function (9) - (10), in regression model of statistic experiments, does not envolves the balance condition (1), and the equilibrium (7) with additional restrictions:*

$$0 \leq P_m(k) \leq 1 \ , \quad 0 \leq m \leq 2 \ , \quad k \geq 0 \ ; \quad 0 \leq \rho_m \leq 1 \ , \quad 0 \leq m \leq 2$$

for solutions of difference equations (12), or, equivalently (20), and equations (5) - (6) for equilibriums.

2. The model interpretation

The model of SE is constructed in several stages. First, the main factor should be chosen, characterized by probability (or frequency, concentration etc.). So there exist supplementary alternatives, whose probabilities are complement to the main factor probability. In particular, having only one alternative, the binary models of SE are considered in the works [1,2,3] (see also [4,5]). The presence of two or more alternatives brings more difficulties in the analysis of SE.

With a full set of characteristics CE, the probabilities of the main factor and of additional alternatives satisfy the balance condition (1) or, equivalently, the balance condition (9), the dynamics of the probability of the main factor P_0, as well as of supplementary factors P_1, P_2 is given by the following difference equations for the probabilities of fluctuations for all $k \geq 0$:

$$\begin{aligned} \Delta\widehat{P}_0(k+1) &= V_1\widehat{P}_1(k) + V_2\widehat{P}_2(k) - 2V_0\widehat{P}_0(k) \ , \\ \Delta\widehat{P}_1(k+1) &= V_0\widehat{P}_0(k) + V_2\widehat{P}_2(k) - 2V_1\widehat{P}_1(k) \ , \\ \Delta\widehat{P}_2(k+1) &= V_0\widehat{P}_0(k) + V_1\widehat{P}_1(k) - 2V_2\widehat{P}_2(k) \ . \end{aligned} \tag{12}$$

The increment of probabilities fluctuations of the main and supplementary factors

$$\Delta\widehat{P}_m(k+1) := \widehat{P}_m(k+1) - \widehat{P}_m(k) \ , \quad 0 \leq m \leq 2 \ , \quad k \geq 0,$$

is determined by the values of guide action parameters ï V_0, V_1, V_2.

Remark 3. *The fluctuations of probabilities in (7) - (8) satisfy the balance condition:*

$$\widehat{P}_0(k) + \widehat{P}_1(k) + \widehat{P}_2(k) = 0 \ , \quad k \geq 0, \tag{13}$$

and by formula (8) one has:

$$\Delta\widehat{P}_m(k) = \Delta P_m(k) \ , \quad 0 \leq m \leq 2 \ , \quad k \geq 0. \tag{14}$$

The equation (12) characterizes two basic principles of alternatives interaction: *stimulation* (positive terms) and *containment* (negative term).

3. The model analysis

The existence of equilibrium point for the fluctuations increments regression function (5) provides the possibility to analyze the dynamics of CE (by $k \to \infty$) in view of the possible guide parameters values which satisfy the constraint (4).

The dynamics of the main factor probability is described by *several scenarios.*

Proposition 1. *The main factor probability $P_0(k)$, $k \geq 0$, determined by the solution of the difference equation (12), as well as by the basic assumption (9), with equilibrium (11), changes with increasing $k \to \infty$ by the following scenarios:*

Attractive equilibrium: $V_0 > 0,\ 0 < \rho_0 < 1$:

$$\lim_{k\to\infty} P_0(k) = \rho_0; \tag{15}$$

Repulsive equilibrium: $V_0 < 0,\ 0 < \rho_0 < 1$:

$$\lim_{k\to\infty} P_0(k) = \begin{cases} 1 & npu \quad P_0(0) > \rho_0; \\ 0 & npu \quad P_0(0) < \rho_0. \end{cases} \tag{16}$$

Dominant equilibrium: $\rho_0 \notin (0,1),\ V_0 < 0$:

$$\lim_{k\to\infty} P_0(k) = 1; \tag{17}$$

Degenerate equilibrium: $\rho_0 \notin (0,1),\ V_0 > 0$:

$$\lim_{k\to\infty} P_0(k) = 0; \tag{18}$$

Remark 4. *Of course, the main factor dynamics scenarios can be formulated by domain of values of the guide parameters V_0, V_1, V_2.*

Remark 5. *Similar scenarios for additional alternative dynamics take place by considering the values of parameters V_1, ρ_1 або V_2, ρ_2.*

4. Annexes

	$V_0 > 0$	$V_0 < 0$
$0 < \rho_0 < 1$	*Attractive equilibrium:* $P_0(k) \to \rho_0$, $k \to \infty$	*Repulsive equilibrium:* $P_0(k) \to \begin{cases} 1, & P_0(0) > \rho_0 , \\ 0, & P_0(0) < \rho_0 , \end{cases}$ $k \to \infty$
$\rho_0 > 1$	*Repulsive degradation:* $P_0(k) \to 0$, $k \to \infty$	*Attractive domination:* $P_0(k) \to 1$, $k \to \infty$
$\rho_0 < 0$	*Attractive degradation:* $P_0(k) \to 0$, $k \to \infty$	*Repulsive domination:* $P_0(k) \to 1$, $k \to \infty$

Fig. 1. Table of scenarios

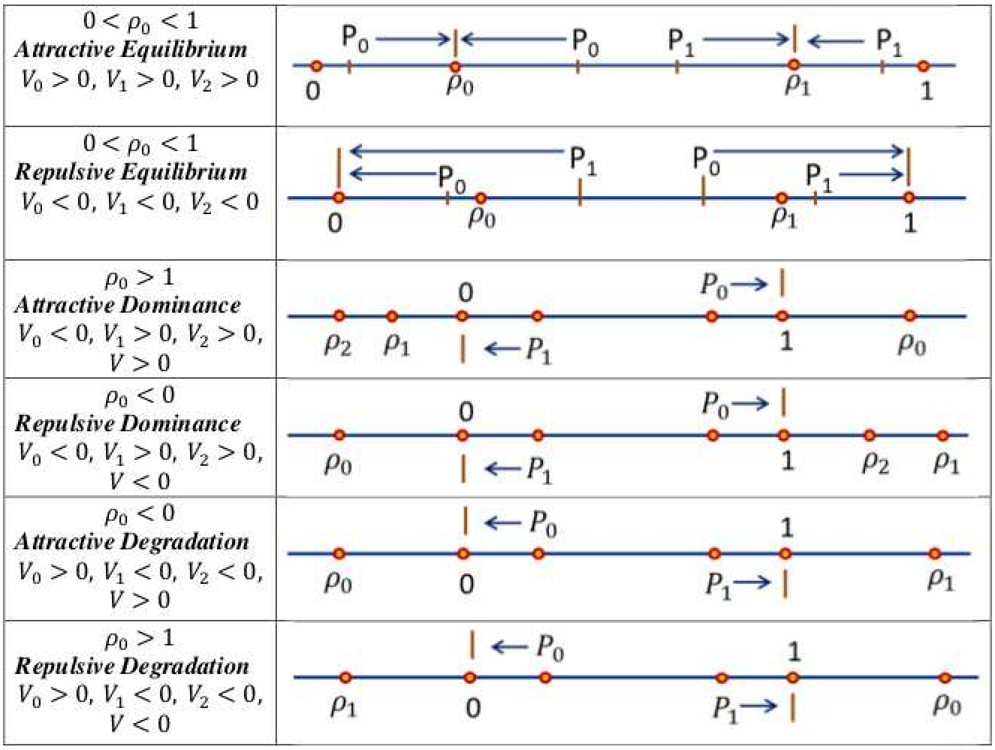

FIG. 2. Illustration of P_i limit behaviour

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