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INVARIANCE AND UNIQUENESS OF SOLUTIONS TO POLYNOMIAL INTERPOLATION PROBLEMS IN EUCLIDEAN SPACE

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Резюме. В роботі розглянуто розв'язання задачі інтерполяції функції багатьох змінних в умовах недовизначеності. Одержано умови інваріантної розв'язуваності та єдиності розв'язку поставленої задачі.

Abstract. In this paper we consider solving of the interpolation problem as applied to many-variable function in the case of under-determinacy. The condition for invariant resolution and uniqueness of this problem is obtained.

1. Introduction

The fundamentals of general theory of operator's interpolation in abstract Hilbert spaces have been established in [1-3]. Then the authors also derived the conditions of invariant solvability for interpolation problems in the event when the solution is available at some or other operator's values in the nodes. The issue of convergence of interpolation processes and estimated accuracy of interpolation for the case of differential operators in Hilbert spaces are considered in [4].

Let X, Y be Hilbert spaces, μ - a Gaussian measure on X such that its first moment is equal to zero, B - the correlation operator of this measure (B belonging to trace-class ones), and $KerB = \emptyset$ [5, 6]. Assume also that Π_n be the set of operator polynomials $P_n : X \to Y$ of n-th power in the form

$$\Pi_n = \{P_n(x) : P_n(x) = L_0 + L_1 x + \cdots + L_n x^n\},$$

where $L_0 \in Y, L_k x^k = L_k(\underbrace{x, x, \ldots, x}_{k})$, and $L_k(x_1, x_2, \ldots, x_k)$ is the k-linear continuous symmetric operator form. Now introduce the scalar product on the set Π_n [2]:

$$(P_n^{(1)}, P_n^{(2)}) = \sum_{k=0}^{n} \int_X \cdots \int_X \Big((L_k^{(1)}(v_1, v_2, \ldots, v_k), \\ L_k^{(2)}(v_1, v_2, \ldots, v_k) \Big)_Y \mu(dv_1)\mu(dv_2)\ldots\mu(dv_k),$$

where $(\cdot,\cdot)_Y$ is the scalar product in the Y-space, while $L_k^{(1)}$ and $L_k^{(2)}$ are k-linear continuous symmetric operator forms corresponding to the polynomials $P_n^{(1)}, P_n^{(2)} \in \Pi_n$ and $\|P_n\| = (P_n, P_n)^{1/2}$.

Key words. Hilbert space, Euclidean space, operator, interpolation polynom, invariance of solution.

2. Formulation and treatment of the interpolation problem in Hilbert space

For the operator $F : X \to Y$ set by its values $F(x_i)$ in the nodes $x_i, i = \overline{1,m}$ we have to find the unique operator polynomial $P_n \in \Pi_n$ that satisfies the interpolation conditions

$$P_n(x_i) = F(x_i), i = \overline{1,m}. \tag{1}$$

Introduce the following notation: $\Gamma = \| \sum_{p=0}^{n}(x_i, x_j)^p\|_{i,j=1}^{m}, 0^0 = 1, (\cdot,\cdot)$ is the scalar product in the X-space, Γ^+ is the Moore-Penrose pseudo-inverse matrix with respect to Γ, and E is identity matrix.

In [1-3], in the event of fulfillment of the necessary and sufficient conditions for solvability of operator interpolation task, such as

$$(E - \Gamma\Gamma^+)\overrightarrow{F} = \overrightarrow{0}, \overrightarrow{F} = \{F(x_i)\}_{i=1}^{m}. \tag{2}$$

the following unique interpolation polynomial of n-th power with minimal norm is constructed:

$$P_n(x) =< \overrightarrow{F}, \Gamma^+ \sum_{p=0}^{n}\{(x, x_i)^p\}_{i=1}^{m} >, \tag{3}$$

where $< \overrightarrow{\alpha}, \overrightarrow{\beta} >= \sum_{i=1}^{m} \alpha_i\beta_i$, $\alpha_i \in Y, \beta_i \in R_1$, i.e. $P_n(x)$ is a solution to the extremum task

$$\|P_n\| = \min \|Q_n\| = (<< \Gamma^+\overrightarrow{P_n}, \overrightarrow{P_n} >>)^{1/2}, Q_n \in \Pi_n^I, \overrightarrow{P_n} = (P_n(x_i))_{i=1}^{m}$$

and Π_n^I is the set of interpolation polynomial of n-th power.

We call an interpolation task invariantly solvable if it has a solution at arbitrary $\overrightarrow{F}$. Then, obviously, the matrix Γ in (2.2) has to be nonsingular. According to [7], an interpolation problem is invariantly solvable in Hilbert space if the interpolation nodes $x_i, i = \overline{1,m}$ are different and the condition

$$m \leqslant n + 1. \tag{4}$$

is met.

In practice, we often deal with approximation of many-variable functions. When such function is represented by a set of its values, one of approximation methods consists in polynomial interpolation. But there another problem arises: the conditions for existence and uniqueness of the interpolant are to be established.

In the tasks of object's identification based on its responses to input signals, of particular interest is the case when the information available is not sufficient: for example, the number of conditions is less than dimension of the space of polynomials used for seeking the solution in Euclidean space. This problem will be called underdetermined.

This work focuses on treatment of the interpolation problem as applied to many-variable functions in the case of under-determinacy, and on analysis of conditions for invariant resolution and uniqueness of the final result.

3. Solution of the interpolation problem in Euclidean spaces

To begin with, apply the above results of treatment of the interpolation problem to the case of Euclidean space E_2. Consider the interpolation of the function $f : E_2 \to R_1$ set by its values in nodes $\gamma_i = (x_i, y_i), i = \overline{1,m}$. Let us represent the solution in the form of interpolant with minimum norm:

$$P_n(x,y) = < \overrightarrow{f}, \Gamma^+ \sum_{p=0}^{n} \{(x_i x + y_i y)^p\}_{i=1}^m >, \tag{5}$$

where $\overrightarrow{f} = \{f(\gamma_i)\}_{i=1}^m, \Gamma = \| \sum_{p=0}^n (x_i x_j + y_i y_j)^p \|_{i,j=1}^m$. If inequality (2.4) holds and all nodes γ_i are different then $\Gamma^+ = \Gamma^{-1}$ (see [7]). In this work for the Euclidean space we obtain a stronger result for invertibility of the matrix Γ as compared to (2.4).

First we construct the solution to this problem based on the general interpolation theory of multivariable functions [8]. The required interpolation polynomial $P_n(x,y)$ will be written as

$$\begin{aligned} P_n(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots + \\ + a_{n0}x^n + a_{n-1,1}x^{n-1}y + \cdots + a_{0n}y^n, \end{aligned} \tag{6}$$

and $a_{ik} \in R_1, i,k = \overline{0,n}$ are unknown coefficients. Denote by $p = (n+1)(n+2)/2$ the dimension of space of n-th power polynomials defined in E_2. To get the unique solution to the interpolation problem, we have to find the nodes $\gamma_i \in E_2, i = \overline{1,p}$ such that the determinant of the system of linear algebraic equations for a_{ik}

$$P_n(\gamma_i) = f(\gamma_i), i = \overline{1,p} \tag{7}$$

is always nonzero.

As shown in [9], it happens if for interpolation nodes we take the following system of points:

$$\begin{array}{l} (x_0,y_0),(x_1,y_0),\ldots,(x_{n-1},y_0),(x_n,y_0), \\ (x_0,y_1),(x_1,y_1),\ldots,(x_{n-1},y_1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (x_0,y_{n-1}),(x_1,y_{n-1}), \\ (x_0,y_n), \\ x_i \neq x_j, y_i \neq y_j \text{ as } i \neq j. \end{array} \tag{8}$$

Such selection of nodes gives us single-valued a_{ik}, and the interpolation polynomial (3.2) is feasible and unique.

Now apply the system of nodes (3.4) to set up the interpolant (3.1). Since the solution to the problem in this case is unique, interpolation polynomials in (3.2) and in minimum norm (3.1) are coincident. Consider next the entries of the matrix Γ

$$\sum_{p=0}^{n}(\gamma_i,\gamma_j)^p = \sum_{p=0}^{n}(\overline{x}_i\overline{x}_j+\overline{y}_i\overline{y}_j)^p =$$
$$= 1+\overline{x}_i\overline{x}_j+\overline{y}_i\overline{y}_j+(\overline{x}_i\overline{x}_j)^2+2\overline{x}_i\overline{x}_j\overline{y}_i\overline{y}_j+(\overline{y}_i\overline{y}_j)^2+$$
$$+\cdots+(\overline{x}_i\overline{x}_j)^n+n(\overline{x}_i\overline{x}_j)^{n-1}\overline{y}_i\overline{y}_j+\cdots+n\overline{x}_i\overline{x}_j(\overline{y}_i\overline{y}_j)^{n-1}+(\overline{y}_i\overline{y}_j)^n,$$

where $(\overline{x}_i,\overline{y}_j)$ are the points of set (3.4). Introduce a set of vectors s_i defined as follows:

$$s_i = (1,\overline{x}_i,\overline{y}_i,\overline{x}_i^2,\sqrt{2}\overline{x}_i\overline{y}_i,\overline{y}_i^2),\ldots,\overline{x}_i^n,\sqrt{n}\overline{x}_i^{n-1}\overline{y}_i,\ldots,\sqrt{n}\overline{x}_i\overline{y}_i^{n-1},\overline{y}_i^n), \quad i=\overline{1,p} \tag{9}$$

and, in conformity to [9], are linearly independent. Then the matrix Γ takes the form of Gram's matrix

$$\Gamma = \begin{pmatrix} (s_1,s_1) & \ldots & (s_1,s_p) \\ \ldots & \ldots & \ldots \\ (s_p,s_1) & \ldots & (s_p,s_p) \end{pmatrix} \tag{10}$$

which is nonsingular. Since any subsystem of vectors (3.5) is also linearly independent and the matrix Γ is invertible, our interpolation task will be invariantly solvable and have a single solution in the form of an interpolating polynomial with minimum norm (3.1), where $\Gamma^+ = \Gamma^{-1}$. Based on the above, the following theorem may be suggested.

Theorem 1. *Let the function $f : E_2 \to R_1$ be set by its values $f(\gamma_i), i=\overline{1,m}$. If the interpolation nodes $\gamma_i, i=\overline{1,m}$ are so selected that the subsystem of vectors from (3.5) is linearly independent (representing, for example,a subset of points (3.4)), then an interpolation problem with two-dimensional function is invariantly solvable and has a single solution with minimum norm under the condition $m \le p$, where p is the dimension of space of polynomials in n-th power defined in E_2.*

Thus, with Theorem 3.1 taking into account, for the function $f : E_2 \to R_1$ we obtained better results compared to inequality (2.4) (see [7]).

Example.Consider the derivation of an interpolational polynomial with minimum norm (3.1) of the second power $P_2(x,y)$. The interpolation nodes are selected from the set of points (3.4), so that

$$\gamma_1=(0,0),\gamma_2=(1,0),\gamma_3=(-1,0),$$
$$\gamma_4=(0,1),\gamma_5=(1,1),$$
$$\gamma_6=(0,-1)$$

Based on formula (3.5), the vectors s_i will be written as

$$\begin{aligned} &s_1=(1,0,0,0,0,0), s_2=(1,1,0,1,0,0), s_3=(1,-1,0,1,0,0),\\ &s_4=(1,0,1,0,0,1), s_5=(1,1,1,1,\sqrt{2},1), s_6=(1,0,-1,0,0,1)\end{aligned} \tag{11}$$

Since the vectors $s_i, i=\overline{1,m}$ are linearly independent, the matrix Γ defined by formula (3.6) is invertible. So we come to the conclusion that in order to construct the interpolant (3.1) we may select any subsystem of vectors (3.7), meaning that the interpolation problem is invariantly solvable and has a unique solution in the event when $m \leq 6$ (m is the number of nodes from set (3.4)). Compared to inequality (2.4), where $m \leq 3$, we obtain a better result.

As noted above, in practice we may encounter problems where the number of interpolation nodes and the function values in these nodes are less than p. In this case the interpolation task treated in classical manner [8] has nonunique solution.

If for solving this problem (at $m \leq p$) we use an interpolant with minimum norm from [1-3] and take the subsystem of vectors s_i from (3.5) for construction of the matrix Γ, then the solution will be invariant and unique. For our example we take $m = 4$ and the subsystem s_1, s_2, s_3, s_4 from (3.7). In this case the matrix Γ is invertible, the interpolation polynomial $P_2(\gamma)$ will be written as

$$P_2(\gamma) = P_2(x,y) =< \overrightarrow{f}, \Gamma^{-1} \sum_{p=0}^{2} \{(x_i x + y_i y)^p\}_{i=1}^{4} >= \sum_{i=1}^{4} l_i(\gamma) f(\gamma_i)$$

that satisfies the conditions $P_2(\gamma_i) = f(\gamma_i)$, where $l_i(\gamma) = l_i(x,y)$ are Lagrange fundamental polynomials of the second power, $l_i(\gamma_j) = \delta_{ij}$, δ_{ij} is the Kronecker symbol, $i, j = \overline{1,4}$, $l_1(x,y) = 1 - x^2 - 1/2y - 1/2y^2$, $l_2(x,y) = 1/2x + 1/2x^2$, $l_3(x,y) = -1/2x + 1/2x^2$, $l_4(x,y) = 1/2y + 1/2y^2$.

Now let us perform comparative analysis of the structure with two interpolants: that corresponding to the classical approach [8], and that suggested here for $m = p$. We choose the system of nodes from the set of points (3.4). In constructing the polynomial (3.2), the problem transforms into search for solutions of linear algebraic equations (3.3) with inverse matrix of general form. In the first case for the solution we use the Gauss method requiring $Q(m) = \frac{2}{3}m^3 + O(m^2)$ arithmetical operations. In the other case for constructing the polynomial (3.1) we have to define the vector

$$\Gamma^{-1} \sum_{p=0}^{n} \{(\overline{x}_i x + \overline{y}_i y)^p\}_{i=1}^{m} = z$$

which is equivalent to solving the system

$$\Gamma z = \sum_{p=0}^{n} \{(\overline{x}_i x + \overline{y}_i y)^p\}_{i=1}^{m} = l(x,y) \tag{12}$$

where $l(x,y)$ is the two-variable polynomial of n-th power. The solution to system (3.8) with its symmetric nonsingular matrix Γ will be sought by the square-root method demanding $Q(m) = \frac{1}{3}m^3 + O(m^2)$ arithmetic operations - with the constant at m^3 twice less than by the Gauss method.

Thus, when comparing the two methods for constructing the interpolation polynomial for the function $f : E_2 \to R_1$ we may conclude that when $m = p$ (m is the number of nodes, and p - dimension of the space of second-power polynomials in E_2) and the interpolation nodes selected correspond to system (3.4), then interpolants (3.1) and (3.2) are coincident, but the polynomial with minimum norm is preferable due to less number of arithmetic operations, so that its formula is easier for applications.

If $m < p$ then for construction (3.2) under conditions (3.3) with nodes (3.4) the classic approach [8] does not ensure uniqueness of solution. On the other hand, polynomial interpolation (3.1) is invariant and unique. In fact, we have obtained a consistent formula making it possible to construct the interpolant of rather simple configuration.

The above results can be extended to the function of many variables $f : E_k \to R_1$, where E_k is k-dimensional Euclidean space. Let the solution of interpolation problem be sought in the space Π_{kn} where Π_{kn} is the space of k-variable polynomials of n-th power. Then, as noted in [8], we always can (find a system of nodes $(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \in E_k$ such that the task of interpolation of multivariable function will have a single solution while the system of vectors s_i can be written as

$$s_i = \left\{ \left(\frac{j!}{j_1! j_2! \cdots j_k!} \right)^{1/2} x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_k}^{j_k}, j_1 + j_2 + \cdots + j_k = j, 0! = 1 \right\}_{j=0}^{n}, \quad i = \overline{1, p} \tag{13}$$

where $p = (n + k)!/n!k!$. Then we may speak of generalization of Theorem 3.1.

Theorem 2. *Let the function $f : E_k \to R_1$ be given its values $f(\gamma_i), i = \overline{1, m}$. If the interpolation nodes γ_i choose so that the relevant subsystem from vectors (3.9) are linearly independent then in the space Π_{kn} interpolation problem of k-variables function with the condition $P_n(\gamma_i) = f(\gamma_i), i = \overline{1, m}$, $P_n \in \Pi_{kn}$ is invariantly solvable and its has a unique solution with minimum norm under the condition $m \le p$, where p - the dimension of the space Π_{kn}.*

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