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GENERALIZATION OF THE KHOVANSKII'S METHOD FOR SOLVING MATRIX POLYNOMIAL EQUATIONS

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Резюме. Розглянуто алгоритм розв'язування поліноміальних матричних рівнянь. Запропоновані рекурентні формули обчислення наближених розв'язків для рівнянь степеня n. Досліджено збіжність методу для рівнянь другого степеня. Наведено результати чисельних експериментів, що підтверджують справедливість теоретичних викладок.

Abstract. The article deals with the algorithm for solving the polynomial matrix equations. Recurrent formulas for calculating approximate solutions of equations of degree n are proposed. The convergence of the method for equations of the second degree has been researched and the results of the numerical experiments that confirm the validity of the calculations are provided.

1. Introduction

The method reduces itself to the consistent application of a certain matrix operator to the given vector and occupies a special place among various generalizations of continued fractions. In a simpler form, this method has been considered by Euler. He used it to calculate the approximate expression $x^{\frac{p}{q}}$. Here x is a known number, p and q are integers.

Euler's method has also been considered by Laurie, Kraft and Muller. But the possibility of practical use of the method hasn't been considered in these works. Later Khovanskii applied this method to the approximate value of the roots of some degrees and to find approximate solutions of polynomial equations over the field of real numbers.

In particular, the scheme of finding the roots of the equation

$$x^2 = u \tag{1}$$

has been considered in [1].

It has been shown that the solution of equation (1) can be found as the fraction $\frac{P_n}{Q_n}$, where valid values P_n and Q_n are interconnected by relations

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} a & u \\ 1 & a \end{pmatrix} \begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix} \quad (n = 1, 2, \ldots). \tag{2}$$

Here a is a free parameter.

The equation (2) implies that

$$\frac{P_n}{Q_n} = \frac{aP_{n-1} + uQ_{n-1}}{P_{n-1} + aQ_{n-1}},$$

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that is

$$\frac{P_n}{Q_n} = \frac{a\frac{P_{n-1}}{Q_{n-1}} + u}{\frac{P_{n-1}}{Q_{n-1}} + a}. \tag{3}$$

Let limit $\lim\limits_{n\to\infty} \frac{P_{n-1}}{Q_{n-1}}$ exists and is finite. We denote it as x. Than from (3) we receive

$$x = \frac{ax + u}{x + a}$$

or

$$x^2 = u, x = \pm\sqrt{u}.$$

So, if the limit $\lim\limits_{n\to\infty} \frac{P_{n-1}}{Q_{n-1}}$ exists, then it may be equal to $\sqrt{u}$ or to $-\sqrt{u}$. Accordingly, if $u < 0$, then the process (2) diverges.

A similar scheme has been proposed in [1] for solving the quadratic equation $x^2+px+q=0$:

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} a & -q \\ 1 & a+p \end{pmatrix}\begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix} \quad (n = 1, 2, \ldots). \tag{4}$$

From (4) it follows that

$$\frac{P_n}{Q_n} = \frac{a\frac{P_{n-1}}{Q_{n-1}} - q}{\frac{P_{n-1}}{Q_{n-1}} + a + p}. \tag{5}$$

Let the limit $\lim\limits_{n\to\infty} \frac{P_{n-1}}{Q_{n-1}}$ exists and is finite. We denote it as x. Then from (5) we receive

$$x = \frac{ax - q}{x + a + p}$$

or

$$x^2 + px + q = 0, x_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

In [1] the conditions for the convergence of the iterative formulas (3) and (5) have been analysed.

2. The computational scheme of the method

Let us try to generalize the scheme proposed in [1] and apply it to solving the matrix equation

$$A_n X^n + A_{n-1} X^{n-1} + A_{n-2} X^{n-2} + \ldots + A_2 X^2 + A_1 X + A_0 = 0. \tag{6}$$

Here matrices $A_0, A_1, A_2, \ldots, A_{n-2}, A_{n-1}, A_n \in \Re^{m\times m}$ are given coefficients of equation (6) and $X \in \Re^{m\times m}$ is an unknown solution.

Suppose, that X is a non singular matrix and let us denote $Y_0 = X^{-1}$. .... After the right multiplication of the equation (6) with X^{-1} we receive

$$A_n X^{n-1} + A_{n-1} X^{n-2} + A_{n-2} X^{n-3} + \ldots + A_2 X + A_1 + A_0 Y_0 = 0. \tag{7}$$

Let $Y_1 = Y_0 X^{-1} = \left(X^{-1}\right)^2$ and let us right multiply the equation (7) with X^{-1} :

$$A_n X^{n-2} + A_{n-1} X^{n-3} + A_{n-2} X^{n-4} + \ldots + A_2 + A_1 Y_0 + A_0 Y_1 = 0.$$

Accordingly, after $(n-2)$ the right multiplication of the equation (6) with X^{-1} we get

$$A_n X^2 + A_{n-1} X + A_{n-2} + A_{n-3} Y_0 + \ldots + A_2 Y_{n-5} + A_1 Y_{n-4} + A_0 Y_{n-3} = 0, \quad (8)$$

where

$$Y_0 = \left(X^{-1}\right)^1, Y_1 = \left(X^{-1}\right)^2, \ldots, Y_{n-3} = \left(X^{-1}\right)^{n-2}.$$

We introduce the parameter, a non singular matrix $L \in \Re^{m\times m}$ and left multiply equation (8) with L :

$$LA_n X^2 + LA_{n-1} X + LA_{n-2} + \ldots + LA_2 Y_{n-5} + LA_1 Y_{n-4} + LA_0 Y_{n-3} = 0. \quad (9)$$

Obviously the equation (9) is equivalent to

$$\begin{gathered} LA_n X^2 + \left(LA_{n-1} + K - K\right) X + LA_{n-2} + \ldots + \\ + LA_2 Y_{n-5} + LA_1 Y_{n-4} + LA_0 Y_{n-3} = 0. \end{gathered}$$

Here $K \in \Re^{m\times m}$ is a non singular matrix.

And it is evident that

$$LA_n X^2 + KX = \left(K - LA_{n-1}\right) X - LA_{n-2} - \ldots - LA_1 Y_{n-4} - LA_0 Y_{n-3}$$

or

$$\left(LA_n X + K\right) X = \left(K - LA_{n-1}\right) X - LA_{n-2} - \ldots - LA_1 Y_{n-4} - LA_0 Y_{n-3}.$$

Then, assuming $\det\left(K - LA_{n-1}\right) \neq 0$ we get

$$\begin{gathered} X = \left(LA_n X + K\right)^{-1} \big(\left(K - LA_{n-1}\right) X - LA_{n-2} - \ldots - \\ - LA_1 Y_{n-4} - LA_0 Y_{n-3}\big). \end{gathered} \quad (10)$$

Now let us consider the obvious equality

$$LA_n X X^{-1} + KY_0 = LA_n + KY_0$$

or

$$\left(LA_n X + K\right) Y_0 = LA_n + KY_0. \quad (11)$$

Then from (11) we get

$$Y_0 = \left(LA_n X + K\right)^{-1} \left(LA_n + KY_0\right). \quad (12)$$

Applying similar transformations, we receive formulas for $Y_1, Y_2, \ldots, Y_{n-3}$ calculation:

$$\begin{array}{l} Y_1 = \left(LA_n X + K\right)^{-1} \left(LA_n Y_0 + KY_1\right); \\ Y_2 = \left(LA_n X + K\right)^{-1} \left(LA_n Y_1 + KY_2\right); \\ \vdots \\ Y_{n-3} = \left(LA_n X + K\right)^{-1} \left(LA_n Y_{n-2} + KY_{n-3}\right). \end{array} \quad (13)$$

Then, on the basis of the formulas (10),(12) and (13) we get an approximate calculation algorithm for solving the polynomial matrix equation (6):

1. Set the accuracy $\varepsilon > 0$;
2. Set the initial approximation, a non singular matrix $X_0 \in \Re^{m\times m}$;
3. Set the counter $n = 1$;

4. Calculate

$$Y_0^{(0)} = \left(X^{(0)^{-1}}\right)^1, Y_1^{(0)} = \left(X^{(0)^{-1}}\right)^2,$$

$$Y_2^{(0)} = \left(X^{(0)^{-1}}\right)^3, \ldots, Y_{n-3}^{(0)} = \left(X^{(0)^{-1}}\right)^{n-2};$$

5. Calculate

$$\begin{array}{l} Y_0^{(n)} = \left(LA_n X^{(n-1)} + K\right)^{-1} \left(LA_n + KY_0^{(n-1)}\right), \\ Y_1^{(n)} = \left(LA_n X^{(n-1)} + K\right)^{-1} \left(LA_n Y_0^{(n)} + KY_1^{(n-1)}\right), \\ Y_2^{(n)} = \left(LA_n X^{(n-1)} + K\right)^{-1} \left(LA_n Y_1^{(n)} + KY_2^{(n-1)}\right), \\ \vdots \\ Y_{n-3}^{(n)} = \left(LA_n X^{(n-1)} + K\right)^{-1} \left(LA_n Y_{n-2}^{(n)} + KY_{n-3}^{(n-1)}\right), \\ X^{(n)} = \left(LA_n X^{(n-1)} + K\right)^{-1} \times \\ \quad \times \left((K - LA_{n-1}) X^{(n-1)} - LA_{n-2} - \ldots - LA_0 Y_{n-3}^{(n)}\right); \end{array} \tag{14}$$

6. Verify the condition $\left\|X^{(n)} - X^{(n-1)}\right\| < \varepsilon$. If this condition is not satisfied, , set the counter $n = n + 1$ and go to step 5, or else return $X^{(n)}$.

3. The convergence of the method for equations of the second power

Let us consider the equation

$$A_2 X^2 + A_1 X + A_0 = 0. \tag{15}$$

Like the equation (6) we left multiply (15) with a non singular diagonal matrix $L = l \cdot E, L \in \Re^{m \times m}$:

$$LA_2 X^2 + (LA_1 + K - K) X + LA_0 = 0$$

or

$$(LA_2 X + LA_1 + K) X = KX - LA_0. \tag{16}$$

Here $K = k \cdot E, K \in \Re^{m \times m}$ is non singular diagonal matrix.

Assuming that $\det (LA_2 X + LA_1 + K) \neq 0$ from (16) we get

$$X = (LA_2 X + LA_1 + K)^{-1} (KX - LA_0)$$

or as a recurrent formula

$$X^{(n)} = \left(LA_2 X^{(n-1)} + LA_1 + K\right)^{-1} \left(KX^{(n-1)} - LA_0\right) \ (n = 1, 2, \ldots). \tag{17}$$

Let A and B be real, square $m \times m$ matrix with $\det B \neq 0$. Further multiplication operation $B^{-1} A$ will be written in the form of a matrix fraction $\frac{A}{B}$.

Inasmuch

$$X = \frac{KX - LA_0}{LA_2 X + LA_1 + K} = \frac{kX - lA_0}{lA_2 X + lA_1 + kE} = \frac{kX - lA_0}{lX + lA_2^{-1} A_1 + kA_2^{-1}} =$$

$$\frac{X-\frac{l}{k}A_0}{X+A_2^{-1}A_1+\frac{k}{l}A_2^{-1}}=\frac{X+A_2^{-1}A_1+\frac{k}{l}A_2^{-1}-\left(A_2^{-1}A_1+\frac{k}{l}A_2^{-1}+\frac{l}{k}A_0\right)}{X+A_2^{-1}A_1+\frac{k}{l}A_2^{-1}},$$

then

$$X=E-\frac{A_2^{-1}A_1+\frac{k}{l}A_2^{-1}+\frac{l}{k}A_0}{X+A_2^{-1}A_1+\frac{k}{l}A_2^{-1}}. \tag{18}$$

Let $P=A_2^{-1}A_1+\frac{k}{l}A_2^{-1}$ and $Q=\frac{l}{k}A_0$. Then (18) can be written as

$$X=E-\frac{P+Q}{X+P},$$

or as an infinite matrix continued fraction

$$X=E-\frac{P+Q}{P+E-\frac{P+Q}{P+E-\dots}}. \tag{19}$$

The matrix continued fraction (18) also can be presented in a compact Prynhsheym's form

$$X=E-\frac{P+Q|}{|P+E}-\frac{P+Q|}{|P+E}-\frac{P+Q|}{|P+E}-\dots \tag{20}$$

Let us consider the continued fraction with real elements. It is evident that

$$\begin{aligned}&\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\frac{a_3|}{|b_3}+\dots+\frac{a_n|}{|b_n}+\dots=\\&=\frac{\frac{a_1}{b_1}\Big|}{|1}+\frac{\frac{a_2}{b_1}\Big|}{|b_2}+\frac{a_3|}{|b_3}+\dots+\frac{a_n|}{|b_n}+\dots\\&=\frac{\frac{a_1}{b_1}\Big|}{|1}+\frac{\frac{a_2}{b_1b_2}\Big|}{|1}+\frac{\frac{a_3}{b_2b_3}\Big|}{|1}+\dots+\frac{\frac{a_n}{b_{n-1}b_n}\Big|}{|1}+\dots\end{aligned} \tag{21}$$

Suppose that the matrix $(P+E)$ is non singular and in (20) we perform transformations similar to (21):

$$\begin{aligned}X&=E-\frac{P+Q|}{|P+E}-\frac{P+Q|}{|P+E}-\frac{P+Q|}{|P+E}-\dots-\frac{P+Q|}{|P+E}-\dots=\\&=E-\frac{(P+E)^{-1}(P+Q)\big|}{|E}-\frac{(P+E)^{-1}(P+Q)\big|}{|P+E}-\\&\quad-\frac{P+Q|}{|P+E}-\dots-\frac{P+Q|}{|P+E}-\dots=\\&=E-\frac{(P+E)^{-1}(P+Q)\big|}{|E}-\frac{(P+E)^{-2}(P+Q)\big|}{|E}-\\&\quad-\frac{(P+E)^{-2}(P+Q)\big|}{|E}-\dots-\frac{(P+E)^{-2}(P+Q)\big|}{|E}-\dots\end{aligned} \tag{22}$$

In [2] Vorpitskyi's sufficient convergence sign has been generalized. It can be used to analyse the convergence of matrix continued fractions of the form (22):

Theorem 1. *Matrix branch continued fraction*

$$\sum_{k_1=1}^{n}\frac{A_{k_1}|}{|E}+\sum_{k_2=1}^{n}\frac{A_{k_1k_2}|}{|E}+\dots+\sum_{k_1=1}^{n}\frac{A_{k_1k_2\dots k_l}|}{|E}+\dots$$

is absolutely convergent if the condition

$$\|A_{k_1k_2\dots k_i}\|\le\frac{1}{4n}\,(i=1,2,3,\dots;k_i=1,2,\dots,n)$$

is true.

Let us apply Theorem 1 to the continued fraction (22). It is obvious that the branched continued fraction (22) will be convergent, if the condition

$$\left\|(P+E)^{-2}(P+Q)\right\| \le \frac{1}{4} \tag{23}$$

is satisfied.

Substituting the values of P and Q in the formula (23) we get sufficient condition for the convergence of the matrix continued fraction (22):

$$\left\|\left(A_2^{-1}A_1 + \frac{k}{l}A_2^{-1} + E\right)^{-2}\left(4A_2^{-1}A_1 + \frac{4k}{l}A_2^{-1} + \frac{4l}{k}A_0\right)\right\| \le 1.$$

4. Computational experiments

To test the effectiveness of the practical application of recurrent formula (14), a series of numerical experiments has been conducted in the FreeMat environment.

Example 1. Let us consider the polynomial matrix equation

$$A_2X^2 + A_1X + A_0 = 0, \tag{24}$$

with

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}, A_0 = \begin{pmatrix} -13 & -13 & -14 \\ -16 & -18 & -18 \\ -20 & -21 & -23 \end{pmatrix}.$$

Put $l = 1, k = 1$ and initial value

$$X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then using the recurrent formula (17) we obtain the following results

Tabl. 1. Example 1

ε	Number of iterations, n	Approximate solution, X_n	Norm of residual
0.1	15	$\begin{pmatrix} -8.9203 & -9.9203 & -9.9203 \\ -0.5083 & 0.4917 & -0.5083 \\ 8.9038 & 8.9038 & 9.9038 \end{pmatrix}$	0.0848
0.01	19	$\begin{pmatrix} -8.9079 & -9.9079 & -9.9079 \\ -0.5065 & 0.4935 & -0.5065 \\ 8.8948 & 8.8948 & 9.8948 \end{pmatrix}$	0.0056
0.001	22	$\begin{pmatrix} -8.9069 & -9.9069 & -9.9069 \\ -0.5064 & 0.4936 & -0.5064 \\ 8.8941 & 8.8941 & 9.8941 \end{pmatrix}$	0.0007

These results show convergence of the iterative process (17) to the solution of equation (24),

$$X = \begin{pmatrix} -8.9070 & -9.9070 & -9.9070 \\ -0.5064 & 0.4936 & -0.5064 \\ 8.8942 & 8.8942 & 9.8942 \end{pmatrix}$$

with a decrease of ε.

Example 2. Now let us consider the polynomial matrix equation

$$A_2X^2 + A_1X + A_0 = 0, \tag{25}$$

with coefficients

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; A_1 = \begin{pmatrix} -1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & -5 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} -8 & -8 & -10 & -9 \\ -9 & -11 & -9 & -11 \\ -11 & -11 & -16 & -12 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

Let $l = 1, k = 1$ and initial value

$$X_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and we use the recurrent formula (17). We get the results from Table 2.

Tabl. 2. Example 2

ε	Number of iterations, n	Approximate solution, X_n	Norm of residual
0.1	12	$\begin{pmatrix} -8.3232 & -9.3232 & -9.3232 & -9.3232 \\ 5.7750 & 6.7750 & 5.7750 & 5.7750 \\ 2.4210 & 2.4210 & 3.4210 & 2.4210 \\ -0.2323 & -0.2323 & -0.2323 & 0.7677 \end{pmatrix}$	0.0610
0.01	15	$\begin{pmatrix} -8.3335 & -9.3335 & -9.3335 & -9.3335 \\ 5.7775 & 6.7775 & 5.7775 & 5.7775 \\ 2.4216 & 2.4216 & 3.4216 & 2.4216 \\ -0.2288 & -0.2288 & -0.2288 & 0.7712 \end{pmatrix}$	0.0070
0.001	18	$\begin{pmatrix} -8.3323 & -9.3323 & -9.3323 & -9.3323 \\ 5.7773 & 6.7773 & 5.7773 & 5.7773 \\ 2.4216 & 2.4216 & 3.4216 & 2.4216 \\ -0.2293 & -0.2293 & -0.2293 & 0.7707 \end{pmatrix}$	0.0008

These results show convergence of the iterative process (17) to the solution of equation (25),

$$X = \begin{pmatrix} -8.3325 & -9.3325 & -9.3325 & -9.3325 \\ 5.7773 & 6.7773 & 5.7773 & 5.7773 \\ 2.4216 & 2.4216 & 3.4216 & 2.4216 \\ -0.2292 & -0.2292 & -0.2292 & 0.7708 \end{pmatrix}$$

with a decrease of ε.

5. Conclusions

The article deals with the modification of the method that was proposed by A.N. Khovanskii [1] for solving polynomial equations defined over the set of real numbers. Obtained computational scheme allows us to construct approximate solutions of the equation (6), that are considered over the ring of non commutative matrices. Sufficient conditions for the convergence of the iterative process for the equation of the second degree and software implementation of the method were presented. A number of numerical experiments confirm the applicability of the proposed scheme were conducted.

Bibliography

1. Khovanskii A. N. The Application of continued fractions and their generalizations to problems in approximation theory / A. N. Khovanskii, P. Noordhoff. – 1963 – 212 p.
2. Bodnar D. I. Branched Continued Fractions / D. I. Bodnar. – K: Naukova Dumka, 1986. – 176 p. (in Russian).

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