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SPECIAL ESTIMATORS FOR CORRECTING SOME SOLUTIONS OF INTEGRAL EQUATIONS

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Резюме. У роботі проведено аналіз чисельного розв'язування двовимірного інтегрального рівняння теорії потенціалу на незамкнених поверхнях. На прикладі аналізу конкретної модельної задачі показано, як, враховуючи специфіку початкових даних, вирішити проблему спеціального зображення самого інтегрального рівняння. Таке зображення дозволяє при побудові відповідної наближеної схеми суттєво спростити використання апріорної інформації про характер поведінки шуканого розв'язку. Останнє відіграє важливу роль у процесі реалізіації різних процедур уточнення отримуваних наближених розв'язків на основі спеціально побудованих оцінювачів. У роботі представлені результати чисельних експериментів.

Abstract. The numerical solution of two-dimensional integral equation on unclosed surfaces is analyzed in present paper. Such equations with weak singularities in the kernels are considered in potential theory. General problem of integral equation solving, and besides that special representation of considered equation, are exemplified by the model task, taking into account the specificity of initial date. In the process of appropriate numerical scheme constructing such a representation gives the possibility to essentially simplify the use of a priori information on desired solution. The last is important for objectifying various correction procedures of obtained results on the basis of special estimators. The results of numerical experiments are presented.

1. Introduction

In previous paper [2] with a similar research object various aspects of numerical schemes construction for solving integral equations of the first kind were considered. In this connection we had to deal with two-dimensional equations in the form as

$$(A\sigma)(M) \equiv \iint\limits_S \sigma(P)|M-P|^{-1}dS_P = U(M),\ M \in S, \tag{1}$$

where, in general case, S is an open Lipschitz surface; M and P are the points of Euclidean space $\mathbb{R}^3$. In present article, by solving one typical model problem, we analyze the proposed schemes adaptive possibilities for maximal taking account of desired solutions specificity in order to receive the results with preassigned accuracy. The equations of type (1) have been used in mathematical modelling of some boundary value problems in electron optics [3]. Ordinary generalization of (1) is an assumption that S is formed by the aggregate of m surfaces, so

Key words. Two-dimensional integral equation, weak singularity, rational representation, numerical scheme construction, correction of obtained results, special estimators.

that $S := \bigcup_{i=1}^{m} S_i$. In this case, we interpret $\sigma(P)$ as a desired total charge distribution density on S, that is $\sigma(P) := \{\sigma_i(P),\ P \in S_i\}_{i=1}^{m}$.

It is possible to research the solvability of integral equation (1) in various functional spaces. However, it should be taken into account the specificity of investigated physical phenomenon. In this connection, the modelling of electrostatic field in the substantially spatial setting foresees the account of desired charge distribution density $\sigma(P)$ behavior near the contour of unclosed surface S. As to right hand side of (1), we consider that $U(M),\ M \in S$, is the given boundary value of potential on an electrode which is actually simulated by a surface S ($U(M) \equiv const$). At last, the solvability of (1) can be expressed by the following inequalities [4, 6]:

$$m_1\|\sigma\|_{H_{00}^{-1/2}(S)} \le \|A\sigma\|_{H^{1/2}(S)} \le m_2\|\sigma\|_{H_{00}^{-1/2}(S)}\ (0 < m_1 \le m_2),$$

where $H^{1/2}(S)$ is a trace space, $H_{00}^{-1/2}(S)$ is dual space with respect to $H_{00}^{1/2}(S)$. Note that S is an open surface treated as a component of some close surface Σ. In addition, $H_{00}^{1/2}(S)$ is different from $H^{1/2}(S)$, and in the case of smooth S, relevant norm may be defined as

$$\|\sigma\|^2_{H_{00}^{1/2}(S)} = \|\sigma\|^2_{H^{1/2}(S)} + \|\rho^{-1/2}\sigma\|^2_{L_2(S)},$$

where $\rho(M)$ is the distance from $M \in S$ to the smooth edge ∂S.

2. The numerical scheme for model problem testing

Let us consider the calculation problem of plane-parallel condenser electrostatic field. From mathematical model point of view this condenser can be represented as a surface S, which is an aggregate of two parallel identical plates S_1 and S_2 situated symmetrically with respect to a coordinate plane XY, so that $S := S_1 \bigcup S_2$. The distance between them equals $2h$. Suppose that U_1 and U_2 are the given potential values on S_1 and S_2, respectively. The electrostatic treatment of problem (1) means that U_1 and U_2 are arbitrary constant. As we mentioned in [2], this problem is not trivial, and the results of calculation are especially sensitive with respect to variation of output data.

With a view to analyze integral equation (1) let us use such S_l representation

$$S_l := \Big\{(x,y,z)^\top \in \mathbb{R}^3 \Big| (x,y) \in [-1,1]^2;\ z = (-1)^{l-1}h;\ l = \overline{1,2};\ h > 0\Big\}. \quad (2)$$

According to (2), we can represent S in the form of congruent components combination:

$$S = \bigcup_{l=1}^{2}\Big(\bigcup_{k=1}^{4} S_{lk}\Big).$$

Taking into account subdivision of S_1 and S_2, integral equation (1), in its turn, can be formally represented as

$$\sum_{l=1}^{2}\sum_{k=1}^{4}\iint\limits_{S_{lk}} \sigma_{lk}(P)|P-M|^{-1}dS_p = U(M) = \begin{cases} U_1, & M \in S_1, \\ U_2, & M \in S_2, \end{cases} \quad (3)$$

where $\sigma_{lk}(P) := \sigma_{lk}(x,y)$ is the restriction of $\sigma(P)$ onto S_{lk};

$$M := (x_0, y_0, z_0 = \pm h)^\top;\ (x,y), \quad (x_0,y_0) \in [-1,1]^2.$$

Then, applying in (3) some changes of variables, we realize the conversion from integration over S to integration over its congruent constituent S_{11}. As a result, we get the system of eight linear integral equations with respect to unknown density $\sigma_j(x,y)(j=\overline{1,8})$, according to the chosen group of surface S symmetry:

$$\sum_{j=1}^{8} \iint\limits_{\triangle_1} \sigma_j(x,y) G_{|i-j|+1}(x,y;x_0,y_0;h)dxdy = U(M_i), \quad (i=\overline{1,8}). \tag{4}$$

Here, $\triangle_1 := [0,1]^2$; $M_i := \Big((-1)^{r-1}x_0, (-1)^{s-1}y_0, (-1)^{p-1}h\Big)^\top \in S_{pq}$; in this case $i := 4(p-1)+2(r-1)+s$, and $q := 2(r-1)+s$ with $p,r,s = 1,2$; M_i are the points of collocation; $(x_0,y_0) \in \triangle_1$. The point of integration is

$$P := \Big((-1)^{n-1}x, (-1)^{m-1}y, (-1)^{l-1}h\Big)^\top \in S_{lk};$$

in this case, $j := 4(l-1)+2(n-1)+m$, and $k := 2(n-1)+m$ with $l,n,m = 1,2$; and finally

$$G_{|i-j|+1}(x,y;x_0,y_0;h) := |P - M_i|^{-1}.$$

It is easy to see that the system of integral equations (4) may be written in the form of matrix operator equation

$$A\overline{\sigma} = \overline{U}, \tag{5}$$

where

$$\overline{\sigma} := (\sigma_1(x,y), \sigma_2(x,y), \ldots, \sigma_8(x,y))^\top,$$
$$\overline{U} := (U(M_1), U(M_2), \ldots, U(M_8))^\top;$$

and $A := (A_{ij})_{i,j=1}^8$, in this case, A_{ij} is an integral operator that acts by the rule

$$A_{ij}\sigma_j(M_i) \equiv \iint\limits_{\triangle_1} \sigma_j(x,y) G_{|i-j|+1}(x,y;x_0,y_0;h)dxdy.$$

Since an initial integral equation has an Abelian eighth order group of symmetry [7], then, we can split (5) into eight independent integral equations $A'\overline{\sigma}' = \overline{U}'$, where $A' := F \cdot A \cdot F^{-1}$, $\overline{\sigma}' := F\overline{\sigma}$, $\overline{U}' := F\overline{U}$. Here, $F := (F_{ij})_{i,j=1}^8$ is known matrix of Fourier transform [2,7]; $A' := (A'_i)_{i=1}^8$, in this case,

$$A'_i\sigma'_i(M_i) \equiv \iint\limits_{\triangle_1} \sigma'_i(x,y) R_i(x,y;x_0,y_0;h)dxdy,$$

$$R_i(x,y;x_0,y_0;h) := \sum_{j=1}^{8} F_{ij} G_{|i-j|+1}(x,y;x_0,y_0;h),$$

$$\sigma'_i(x,y) := \sum_{j=1}^{8} F_{ij}\sigma_j(x,y), \qquad U'(M_i) := \sum_{j=1}^{8} F_{ij}U(x,y).$$

Solving every of independent integral equations, as the final result, it is possible to reproduce $\sigma_j(x,y)$.

Then, without loss of generality let us consider one special case of integral equation (5) presentation. Namely, taking into account the antisymmetry of boundary values of potentials on condenser plates ($U := U_1 = -U_2$), and in accordance with this similar properties of (5) solutions, it is possible to represent (5) in the form as

$$(A\sigma)(x_0,y_0) \equiv \iint\limits_{\triangle_1} \sigma(x,y)\hat{R}(x,y;x_0,y_0;h)dxdy = U(x_0,y_0), \qquad (x_0,y_0) \in (0,1)^2, \tag{6}$$

where

$$\hat{R}(x,y;x_0,y_0;h) := \sum_{l=1}^{2}(-1)^{l-1}\sum_{p=1}^{2}\sum_{k=1}^{2}\Big\{4h^2(l-1)+ +\big[(-1)^{p-1}x+(-1)^{k-1}x_0\big]^2+\big[y+(-1)^k y_0\big]^2\Big\}^{-\frac{1}{2}}.$$

It is easy see that integral equation (6) is an equation with weak singularity in the kernel. In addition, (6) has mentioned singularity only in one item of the sum $\hat{R}(x,y;x_0,y_0;h)$, where $k=l=1,\ p=2$. Moreover, in the process of numerical scheme constructing it is necessary to take into consideration special behavior of desired solution only on S_{11}.

It is known [5] that desired solution $\sigma(x,y)$ has singularities in the neighborhood of S_{11} corner point and at the points which border on a straight edge of S_{11}. In the first case, the charge singularity is proportional to $\rho^{-0,7034}$, and, in the second case, the charge singularity is proportional to $\rho^{-0,5}$, where ρ is the distance from the vertex and straight edge of S_{11}, respectively. These singularities can be expressed by the following weight function

$$\frac{(1-x)^\gamma+(1-y)^\gamma}{\big[(1-x)(1-y)\big]^{1/2}} \qquad (\gamma = 0,2966).$$

This function is applied for mentioned singularities isolation in the notation of charge distribution density $\sigma(x,y)$. But such accounting of desired solution characteristics is rather complicated from practical point of view. So, we apply the different method, based on progressive analysis and correction of obtained results.

Using the collocation method under the condition of piecewise-constant approximation of desired density $\sigma(x,y)$, two-dimensional integral equation (6) was reduced to the following system of linear algebraic equations

$$\sum_{j=1}^{N_x}\sum_{i=1}^{N_y}\sigma_{ij}\int\limits_{x_i-\frac{H_x}{2}}^{x_i+\frac{H_x}{2}}\int\limits_{y_j-\frac{H_y}{2}}^{y_j+\frac{H_y}{2}}\hat{R}(x,y;x_0,y_0,h)dxdy = U(x_0,y_0),$$

where $H_x := N_x^{-1}$, $H_y := N_y^{-1}$ $(N_x,\ N_y \in \mathbb{N})$;

$$x_0 \in \left\{\frac{H_x}{2}(2i-1)\right\}_{i=1}^{N_x}, \quad y_0 \in \left\{\frac{H_y}{2}(2j-1)\right\}_{j=1}^{N_y};$$

σ_{ij} are approximate values of desired density $\sigma(x,y)$ at the points of collocation (x_0, y_0). In this case, we used uniform subdivision of S_{11} onto elements, that is $H_x = H_y$, and $N_x = N_y$.

3. A posteriori error estimation of (6) numerical solution under the condition of $\triangle_1$ irregular partition onto elements

In numerical solving of integral equation (6) the problem of obtained results error estimation is actual from practical point of view. Taking into account a priori information of desired density special behavior, the method based on experience proved to be the most acceptable. Let us note that stable results obtaining is also important problem independently of S_{11} uniform or nonuniform partition onto elements.

Let $\sigma_\varepsilon(P)$ be a numerical solution of integral equation (6) that belongs to the chosen approximation space. It generates approximate potential value at arbitrary point Q between charged condenser electrodes simulated by appropriate surfaces:

$$U_\varepsilon(Q) = (A\sigma_\varepsilon)(Q).$$

In addition, general error function e_U of integral equation (6) approximate solution may be represented as [1]

$$e_U = A\sigma - A\sigma_\varepsilon = A(\sigma - \sigma_\varepsilon) = Ae_\sigma,$$

where e_σ is a solution of such integral equation

$$(Ae_\sigma)(M) = U - (A\sigma_\varepsilon)(M), \quad M \in S_{11}. \tag{7}$$

Integral equation (6) solution has irregular behavior near the contour of unclosed surface S (essentially in the neighborhood of its corner points) [5]. Therefore, the reproduction of error function e_U, specified the level of boundary values satisfaction, is established onto elements D^e. These elements appear in the process of surface S sequential nonuniform partition (in the present case, its congruent component S_{11}). On D^e the function e_U may reach maximum values. Moreover, on D^e the function e_σ is approximately equal to its value at checking point T (see fig.1, fig.2):

$$e_\sigma(T) = \frac{U - (A\sigma_\varepsilon)(T)}{\int\limits_{D^e} |T-P|^{-1} dS_P}.$$

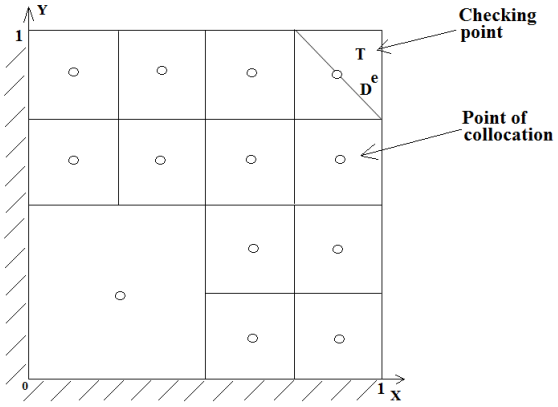

Fig. 1. Nonuniform partition

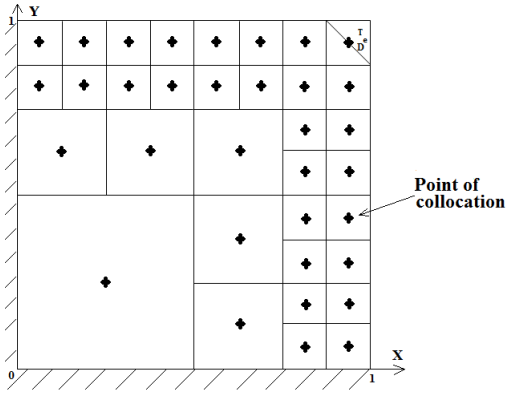

Fig. 2. Nonuniform partition in progress

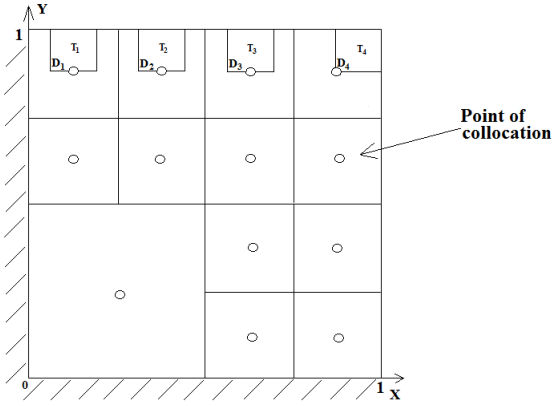

Fig. 3. Checking elements D_i

Selecting the furthest strategies of obtained results correction, it is possible to use various methodologies. Let us consider the method, different from proposed in the paper [2], which is sufficiently effective for two-dimensional integral equations numerical solution. The main idea of this strategy consists of the following. In the process of domain $\triangle_1$ nonuniform subdivision let us consider not

only one special element D^e but some set of elements where the desired function errors are inadmissible. Taking into account the symmetry of obtained results it is advisable to select such elements not far from the part of plate contour (for example, on the last horizontal layer). Let $D_1, D_2, \ldots, D_N$ $(N \in \mathbb{N})$ be above mentioned elements (see fig. 3). Then, if we use piece-wise approximation of e_σ and equation (7), it is possible to find solution error on every element D_i, $(i = \overline{1, N})$:

$$e_\sigma(T_i) = \frac{U - (A\sigma_\varepsilon)(T_i)}{\int_{D^e} |T_i - P|^{-1} dS_P}.$$

Let us denote by e_k the solution error e_σ on the element D_k $(k = \overline{1, N})$, that is $e_k = e_\sigma(T_k)$. Then, it needs to calculate the value ξ:

$$\xi = \sqrt{\frac{\sum\limits_{k=1}^{N} \|e_k\|^2}{N}}.$$

At that time for the completeness of domain $\triangle_1$ subdivision process the following condition must be fulfilled

$$\frac{\|e_k\|}{\xi} \cdot 100\% < TOL \quad \forall e_k,\ k = \overline{1, N}. \tag{8}$$

If the condition (8) is fulfilled only for certain elements D_k and appropriate errors e_k, then it is needed later on to eliminate such elements out of previous defined checking. Let us note that the disposition of elements D_k does not strictly allocate, so its sampling must be realized in various ways. In this connection, it is always necessary to control the obtained results of calculation.

4. The analysis of numerical experiments

Example 1. Illustration of calculation stability and analysis of results reliability Using piece-wise approximation of $\sigma(P)$ (charge distribution density) for $N_x = N_y = 40$ (the number of collocation points is 1600) we obtained the following results:

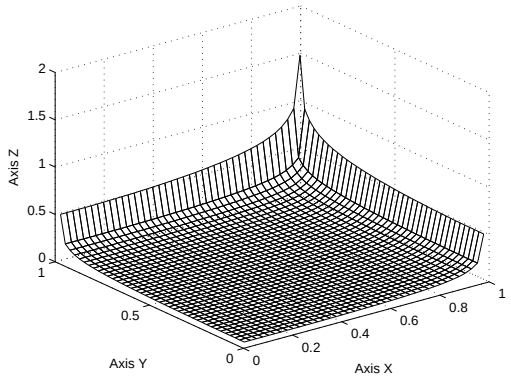

Fig. 4. Charge distribution density. $N_x = N_y = 40$

Let us note, that uniform subdivision of S_{11} can be selected so that there exists a point of collocation which will be present at the next division area. For example, the following divisions of $N_x = N_y = 6$, $N_x = N_y = 18$ contain the collocation point with coordinates (0.75 0.75). Justification of approximation schemes stability and hence the approximate solution results of integral equation solving are shown in the Tabl. 1. Approximative values of density $\sigma(x, y)$ at the checking points are not much different from the values which were obtained in the previous step of division.

Tabl. 1. Charge distribution density. Illustration of calculation stability

Point of collocation (x,y)	$N_x = N_y$		
	6	18	54
(0.250,0.250)	0.0531427	0.0519844	0.0515285
(0.250,0.917)	0.2005168	0.1459617	0.1482432
(0.917,0.250)	0.2005168	0.1459617	0.1482432
(0.917,0.917)	0.3955893	0.2588432	0.2699086

Absolute error e_U of reproduced boundary values for $N_x = N_y = 40$ is represented in the following figure:

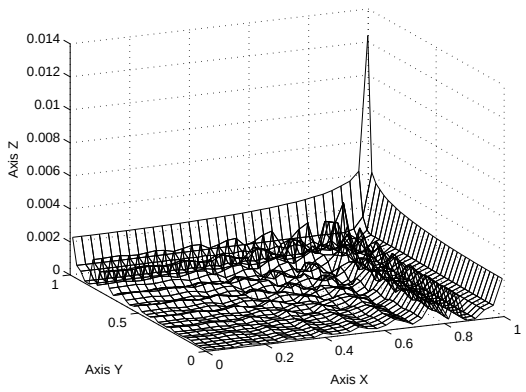

Fig. 5. Absolute error of boundary values. $N_x = N_y = 40$

Example 2. Illustration of nonuniform partition approach. The comparison between approaches Nonuniform partition is applied for better approximation of charge distribution density function and decreasing error function, especially near the contour of unclosed surface. Two parameters are important for this approach: the first one is initial partition of the surface, and the second is the number or steps of nonuniform partition; these parameters affect to the results of calculation. Absolute error of reprodused boundary values is shown in the Fig. 6, in the case when initial partition is $N_x = N_y = 2$. The number of iterations (steps) for nonuniform partition is 9.

The results in this figure reflect the impact of initial partition parameter to the error function: error was reduced near the contour of surface but was not decreased onto others elements. So, next figure displays the results of calculation with different initial partition.

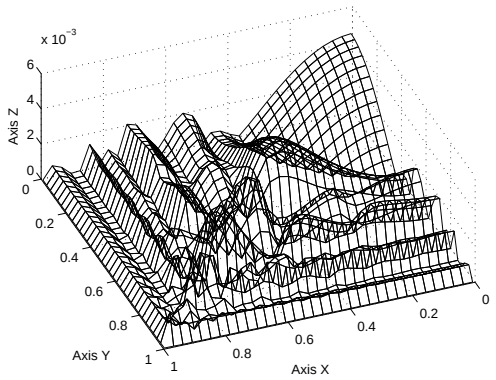

Fig. 6. Absolute error of reproduced boundary values. Nonuniform partition

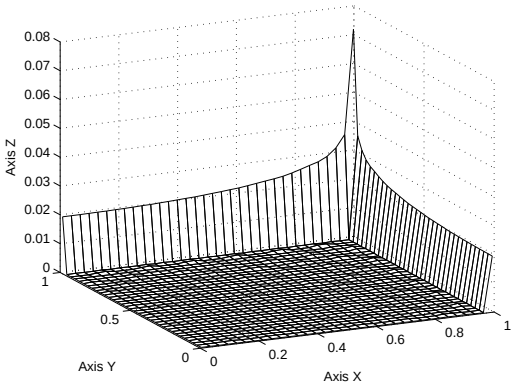

Fig. 7. Absolute error of boundary values. First partition ($N_x = N_y = 8$)

The Fig. 7 presents an absolute error of boundary values for the first partition $N_x = N_y = 8$ and the number of iteration for nonuniform division is 6.

The following two tables represent comparing of surface partition approaches (uniform and nonuniform) and summarize obtained results. The tables contain values of error function at checking points near the contour and comparison of these tables concludes that nonuniform partition is more effective for solving integral equations of such type.

Tabl. 2. Uniform partition

y/x	0.85	0.95	0.995	0.9995
0.85	0.00303	0.00124	0.08051	0.09443
0.95	-	0.02228	0.10008	0.11801
0.995	-	-	0.15808	0.17186
0.9995	-	-	-	0.18295

The Tabl. 2 represents $N_x = N_y = 8$. The number of collocation points is 64.

In the Tabl. 3 initial partition $N_x = N_y$ is equal to 2. The number of steps for nonuniform partition is 4. The number of collocation points is 79.

Tabl. 3: Nonuniform partition

y/x	0.85	0.95	0.995	0.9995
0.85	0.00385	0.01239	0.04937	0.06633
0.95	-	0.02971	0.04802	0.07266
0.995	-	-	0.110306	0.13008
0.9995	-	-	-	0.14557

So, by the example of the concrete model problem solving it is shown how, taking into account the specificity of initial data, to solve the problem of integral equation special representation. In the process of appropriate numerical scheme constructing such a representation gives the possibility to essentially simplify the use of a priori information on desired solution. The last is important for objectifying various correction procedures of obtained results on the basis of special estimators. With the help of proposed estimators the effective solution of initial integral equation were received.

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