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ARITHMETICAL COMPLEXITY OF MODIFIED FULLY DISCRETE PROJECTION METHOD FOR THE PERIODIC INTEGRAL EQUATIONS

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Резюме. Розглядається задача скорочення обсягу інформаційних затрат при розв'язанні періодичних інтегральних рівнянь з мінімальною похибкою. Для цього пропонується деяка модифікація повністю дискретного проекційного методу. Доведено, що ця модифікація зберігає найкращу точність чисельного методу в метриці соболєвських просторів з обсягом арифметичних дій $N \log N$ за порядком.

Abstract. The reduction of arithmetical operations for the solving of periodic integral equations with minimal error bound is considered. For this some modification of a fully discrete projection method was proposed. It was proved that such modification guarantees the best possible accuracy of the numerical method in the metric of Sobolev spaces with the order of arithmetical operations $N \log N$.

1. Introduction

Periodic integral equations are frequently found in various problems of natural sciences that can be described by a boundary value problems such as Laplace or Helmholz equations. To illustrate this, we rewrite Dirichlet problem for Laplace equation on the simply connected domain Ω. So it takes the form

$$\triangle G(X) = 0, \quad X \in \Omega, \tag{1}$$

$$G(X) = g(X), \quad X \in \Gamma = \partial\Omega, \tag{2}$$

where Γ is a smooth boundary of domain Ω and function g is continuous. As it is well-known (see [8]), the problem (1) has a unique solution under quite natural condition on Γ. Solving (1) by direct method, using the representation of the function $G(X)$, $X \in \Omega$ in the form of a simple-layer potential, we derive to a boundary integral equation

$$Su = g, \tag{3}$$

where S is a single layer operator with logarithmic kernel and $u = \dfrac{\partial G}{\partial n}$ is a normal derivation on the boundary. Note that by so-called Cauchy data $(G|_\Gamma, \dfrac{\partial G}{\partial n}|_\Gamma)$ we can easily find the function $G(X)$ for $X \in \Omega$. Thus for solving boundary value problem (1) it is necessary to solve periodic integral equation of the first kind (3). It is such kind of problem that will be the object of our investigation. Periodic integral equations are well-known and various aspects of their solving in the metric of Sobolev spaces were investigated, for example,

Key words. Periodic integral equations, fully discrete projection method, GMRES.

in [2], [4], [7]. The most widely-used approaches for numerical solving of periodic integral equations are fully discrete collocation and projection methods that applied together with selfregularization principle. In the paper we will consider modification of a fully discrete projection method that was firstly proposed for solving the integral Symm equation (see Example 1) in [4] and extended on the class of pseudodifferentional equation in [12]. Moreover we introduce some additional projection in the method to reduce amount of arithmetical operations.

2. Statement of the problems

In the space $L_2(0,1)$ we consider the following integral equation

$$\mathcal{A}u(t) = f(t), \quad t \in [0,1], \tag{4}$$

where f is 1- periodic function and operator $\mathcal{A}$ has the form

$$\mathcal{A} = \sum_{p=0}^{q} A_p, \quad A_p u(t) = \int_0^1 k_p(t-s) a_p(t,s) u(s) ds. \tag{5}$$

Let's denote by $C^\infty = C^\infty([0,1]^2)$ the space C^∞ of smooth 1-biperiodic functions of both variables. Suppose that $a_p \in C^\infty([0,1]^2)$, $p = 0, \ldots, q$, and

$$a_0(t,t) \neq 0, \forall t \in [0,1]. \tag{6}$$

Moreover assume that $k_p(t)$ is 1 - periodic function with known Fourier coefficients $\hat{k}_p(n)$ by trigonometric basis for each $p = 0, \ldots, q$. Additionally we suppose that for some $\alpha \in \mathbb{R}$ and $\beta > 0$ the following inequalities

$$c_{00}|n|^\alpha \leq |\hat{k}_0(n)| \leq c_0 |n|^\alpha, \quad n \in \mathbb{Z}/0, \tag{7}$$

$$|\hat{k}_0(n) - \hat{k}_0(n-1)| \leq c\underline{n}^{\alpha-\beta}, \quad n \in \mathbb{Z}, \tag{8}$$

$$|\hat{k}_p(n)| \leq c\underline{n}^{\alpha-\beta}, \quad n \in \mathbb{Z}, p = 1, \ldots, q, \tag{9}$$

hold true, where c, c_0, $c_{00} > 0$ and

$$\underline{n} = \begin{cases} |n|, & n \in \mathbb{Z}/0 \\ 1, & n = 0 \end{cases}.$$

Denote by H^{λ_1} and H^{λ_1,λ_2}, $-\infty < \lambda_1, \lambda_2 < \infty$, Hilbert spaces of 1-periodic functions and 1-biperiodic functions with the norm

$$\|u\|_{\lambda_1} := \left(\sum_{n \in \mathbb{Z}} |\underline{n}|^{2\lambda_1} |\hat{u}(n)|^2 \right)^{1/2} < \infty,$$

$$\|a\|_{\lambda_1,\lambda_2} := \left(\sum_{(k,l) \in \mathbb{Z}^2} |\underline{k}|^{2\lambda_1} |\underline{l}|^{2\lambda_2} |\hat{a}(k,l)|^2 \right)^{1/2} < \infty$$

respectively. Here

$$\hat{u}(n) = \int_0^1 e_{-n}(t) u(t) dt, \quad \hat{a}(k,l) = \int_0^1 \int_0^1 e_{-k}(t) e_{-l}(s) a(t,s) dt ds$$

are Fourier coefficients of functions $u(t)$ and $a(t,s)$ by trigonometric basis $\{e_k\}_{k=-\infty}^{+\infty}$, where $e_k(t) = e^{i2\pi kt}$, $t \in [0,1]$.

In general case in the space $H^0 = L_2(0,1)$ operator $\mathcal{A}$ is compact and problem is unstable. But for considered class of equations (4) with (6)-(9) it is possible to choose appropriate pair of spaces to regularized problem. As it was shown in [7, Theorem 6.3.1], operator $\mathcal{A}$ under our assumptions creates isomorphism between H^λ and $H^{\lambda-\alpha}$ for any $\lambda \in \mathbb{R}$. That is why if $f \in H^{\lambda-\alpha}$ the equation (4) has unique solution $u \in H^\lambda$. Let's consider more precisely the structure of (4). Following [7, Ch.6], we rewrite the equation (4) in a such way

$$\mathcal{A}u = Du + \sum_{p=1}^{q} A'_p u = f', \tag{10}$$

where $Du = \int_0^1 k_0(t-s)u(s)ds$, $A'_0 := A'_0 = \frac{1}{a_0(t,t)}\int_0^1 k(t-s)(a_0(t,s) - a_0(t,t))u(s)ds$, $A'_p := A'_p = \frac{A_p}{a_0(t,t)}$ for $p = 1..q$ and $f := f' = \frac{f}{a_0(t,t)}$. Note that $D \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ is performing the isomorphism between the spaces H^λ and $H^{\lambda-\alpha}$ and operators $A_p \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\beta}), p = 0,..,q$ are compact on the pair of spaces H^λ and $H^{\lambda-\alpha}$. Further we will deal with equation (10) instead of (4).

Thereafter for all $\lambda \leq \mu$ there are constants $c'_\lambda, c''_\lambda > 0$, such that for any $v \in H^\lambda$ the following inequality

$$c'_\lambda \|v\|_\lambda \leq \|\mathcal{A}v\|_{\lambda-\alpha} \leq c''_\lambda \|v\|_\lambda \tag{11}$$

holds true.

Further we assume that exact solution of equation (4) belongs to some Sobolev spaces, namely $u \in H^\mu$ for some $\mu > \alpha + 1/2$ and $\|u\|_\mu \leq 1$. Then due to conditions (11) we have that $f \in H^{\mu-\alpha}$ and $\|f\|_{\mu-\alpha} \leq c''_\mu$.

Note that classical elliptic pseudodifferential equations are included in the class of equations (4) with conditions (6)- (9) (see for detail [6]). Below we rewrite the examples of some equations that satisfy the conditions (6)- (9).

Example 1. The typical example of equation from the class under consideration is an integral Symm's equation

$$\mathcal{A}u(t) := \int_0^1 k_0(t-s)u(s)ds + \int_0^1 a_1(t,s)u(s)ds = f(t), \tag{12}$$

$$k_0(t-s) = \log|\sin\pi(t-s)|, \tag{13}$$

$$a_1(t,s) = \begin{cases} \log\frac{|\gamma(t)-\gamma(s)|}{|\sin\pi(t-s)|}, & t \neq s \\ \log(|\gamma'(t)/\pi|), & t = s \end{cases}.$$

As it is known, the kernel $a_1(t,s)$ of operator A_1 presents the C^∞-smooth and 1-biperiodic function and Fourier coefficients k_0 have the view

$$\hat{k}_0(n) = \begin{cases} \frac{1}{2|n|}, & n \in \mathbb{Z}/0 \\ \log 2, & n = 0. \end{cases}$$

It is evident that conditions (6)-(9) are satisfied for $a_0(t,s) = k_1(t,s) \equiv 1$, $\alpha = -1$ and any $\beta > 0$.

Example 2. The integral equation

$$\int_0^1 |x(t) - x(s)|^2 \log |x(t) - x(s)| u(s) ds = f(t), \quad t \in [0,1],$$

arises for solving biharmonic Dirichlet problems in the bounded domain with smooth Jordan boundary (see for more detailed information, for example, [1], [7, Ch. 6]). Rewrite the equation in the form

$$\int_0^1 k_0(t-s) a_0(t,s) u(s) ds + \int_0^1 a_1(t,s) u(s) ds = f(t),$$

where

$$a_0(t,s) = \frac{|x(t) - x(s)|^2}{\sin^2 \pi(t-s)} \quad \text{for} \quad t \neq s, \quad a_0(t,t) = \frac{|x'(t)|^2}{\pi^2},$$

$$a_1(t,s) = |x(t) - x(s)|^2 \log \frac{|x(t) - x(s)|}{|\sin \pi(t-s)|} \quad \text{for} \quad t \neq s, \quad a_1(t,t) \equiv 0,$$

$$k_0(t) = \sin^2 \pi t \log |\sin \pi t|.$$

The Fourier coefficients k_0 are known and have the following view $\hat{k}_0(0) = -\frac{1}{2} \log 2 + \frac{1}{4}$,
$\hat{k}_0(\pm 1) = \frac{1}{4} \log 2 - \frac{3}{16}$,

$$\hat{k}_0(n) = \frac{1}{4|n|(n^2-1)}, \quad |n| \geq 2.$$

It is easy to see that conditions (7)-(9) satisfied for $\alpha = -3, \beta = 1$. Thus, the equation under consideration is also included in the investigated class of problems.

To make more precise the smoothness properties of functions a_p, $p = 0, \ldots, q$, we introduce in consideration the space of Gevre function of Roumieu type (see [3, p.112]):

$$\begin{aligned} G_{\eta_1,\eta_2} = &\Big\{ a \in C^\infty : \|a\|^2_{\eta_1,\eta_2} := \\ &:= \sum_{k,l=-\infty}^{\infty} |\hat{a}(k,l)|^2 e^{2\eta_2(|k|^{1/\eta_1} + |l|^{1/\eta_1})} < \infty \Big\}, \quad \eta_1, \eta_2 > 0. \end{aligned} \tag{14}$$

Note that with $\eta_1 = 1$ by (14) it follows that function $a(t,s)$ has analytic continuations in both variables into the strip $\{z : z = t + is, |s| < \frac{\eta_2}{2\pi}\}$ of complex plane. Further suppose that $a_p \in G_{\eta_1,\eta_2}, p = 0, \ldots, q$, for some $\eta_1 \geq 1$ and $\eta_2 > 0$. It should be noted that condition (14) doesn't restrict the class of equation under consideration but allows to take better into account the smoothness of kernels a_p. At first such assumption for a_p was proposed in the paper [4], which considered particular case of mentioned class of periodic integral equations, namely Symm integral equation.

In the paper we state the aim to reduce the amount of arithmetical operations of fully discrete projection method for solving (4) with conditions (7)-(9) and (14). For that we propose modification of the method that should not influence on the best error accuracy of solution for a priori case of choosing regularization parameter.

3. Auxiliary statements

For further presentation of our results we will use the following notations. Let's introduce n-dimensional subspaces of trigonometric polynomials

$$\mathcal{T}_N = \{u_N : u_N(t) = \sum_{k\in Z_N} c_k e_k(t)\},$$

$$Z_N = \left\{k : -\frac{N}{2} < k \le \frac{N}{2}, k = 0, \pm 1, \pm 2, \dots\right\}. \tag{15}$$

Denote by P_N and P_Ω orthogonal projectors

$$P_N u(t) = \sum_{k\in Z_N} \hat{u}(k) e_k(t) \in \mathcal{T}_N,$$

$$P_{\Omega_N} a(t,s) = \sum_{l,k\in\Omega_N} \hat{a}(k,l) e_k(t) e_l(s) \in \mathcal{T}_N \times \mathcal{T}_N,$$

where Ω_N is some domain on coordinate plane restricted by square $(-N/2, N/2] \times (-N/2, N/2]$. Also denote by Q_N and $Q_{N,N}$ interpolation projectors, such that $Q_N u(t) \in \mathcal{T}_N$, $Q_{N,N} a(t,s) \in \mathcal{T}_N \times \mathcal{T}_N$ and on the uniform grid it holds true

$$(Q_N u)(jN^{-1}) = u(jN^{-1}), \quad j = 1, 2, \dots, N,$$
$$(Q_{N,N} a)(jN^{-1}, iN^{-1}) = a(jN^{-1}, iN^{-1}), \quad j, i = 1, 2, \dots, N.$$

It is well-known (see, for example, [7, Ch.8]), that

$$\|u - P_N u\|_\lambda \le \left(\frac{N}{2}\right)^{\lambda-\mu} \|u\|_\mu, \quad \lambda \le \mu, \quad u \in H^\mu, \tag{16}$$

$$\|u - Q_N u\|_\lambda \le c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu, \quad 0 \le \lambda \le \mu, \quad \mu > \frac{1}{2}, \quad u \in H^\mu, \tag{17}$$

where $c_{\lambda,\mu} = \left(\frac{1}{2}\right)^{\lambda-\mu} \gamma_\mu$, and $\gamma_\mu = \left(1 + 2\sum_{j=1}^\infty \frac{1}{j^{2\mu}}\right)^{\frac{1}{2}}$.

Moreover, for any $v_N \in \mathcal{T}_N$ according to inverse Bernshtein inequality it holds

$$\|v_N\|_\mu \le \left(\frac{N}{2}\right)^{\mu-\lambda} \|v_N\|_\lambda, \quad \lambda \le \mu. \tag{18}$$

4. Discretization of operator A_p, $p = 0, \dots, q$

Note that operator D has simple structure and doesn't need any additional discretization. Thus we need to discretize only operators A_p for each $p = 0, \dots, q$,. This will be done further.

Let's consider the following domain of coordinate plane

$$D_M^{\eta_1} = \{(k,l) : |k|^{1/\eta_1} + |l|^{1/\eta_1} < \left(\frac{M}{2}\right)^{\frac{1}{\eta_1}}, k, l = 0, \pm 1, \pm 2 \dots\} \tag{19}$$

Note that $D_M^{\eta_1} \subseteq D_M^1$ for all $\eta_1 \ge 1$.

Assume that the discrete information about kernels $a_p(t,s)$ and right hand side f is given in the knots of uniform grids $\left(\frac{j_1}{M}, \frac{j_2}{M}\right)$, where $j_1, j_2 = 1..M$.

Let's approximate the kernels a_p in the following way

$$a_{p,M} = P_{D_M^{\eta_1}} Q_{M,M} a_p, \tag{20}$$

where $P_{D_M^{\eta_1}}$ is ortoprojector on span of vectors $\{e_i, e_j\}$ such that $(i,j) \in D_M^{\eta_1}$.

Then the operators $A_{p,M}$ can be approximate by

$$A_{p,M}u(t) = \int_0^1 k_p(t-s) a_{p,M}(t,s) u(s) ds. \tag{21}$$

where function $a_{p,M}$ has the form (20). To find the approximative properties of operator (21) we state the following auxiliary lemmas.

Lemma 1. *Let $a \in G_{\eta_1,\eta_2}$ for $\eta_1 \geq 1$, then for $\forall \lambda_1, \lambda_2$ and*

$$M > 2\left(\frac{\max\{\lambda_1,\lambda_2\}\eta_1}{\eta_2}\right)^{\eta_1}$$

it holds true

$$\|a - P_{D_M^{\eta_1}} a\|_{\lambda_1,\lambda_2} \leq \left(\frac{M}{2}\right)^{\lambda_1+\lambda_2} e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|_{\eta_1,\eta_2}.$$

Proof. We rewrite the norm of element $a - P_{D_M^{\eta_1}} a$ in the following way

$$\begin{aligned}\|a - P_{D_M^{\eta_1}} a\|^2_{\lambda_1,\lambda_2} &\leq \|\sum_{|k|>0} \sum_{l:(k,l)\notin D_M^{\eta_1}} \hat{a}(k,l) e_k(t) e_l(s)\|^2_{\lambda_1,\lambda_2} = \\ &= \sum_{|k|>0} \sum_{l:(k,l)\notin D_M^{\eta_1}} |k|^{2\lambda_1} |l|^{2\lambda_2} |\hat{a}(k,l)|^2 = \\ &= \sum_{|k|>0} \sum_{l:(k,l)\notin D_M^{\eta_1}} |k|^{2\lambda_1} |l|^{2\lambda_2} |\hat{a}(k,l)|^2 e^-_{k,l} e^+_{k,l} =: S_1,\end{aligned}$$

where $e^{\pm}_{k,l} = e^{\pm 2\eta_2(|k|^{1\eta_1} + |l|^{1/\eta_1})}$. Further it is worth to estimate the norm of S_1 depending on values k and l.

At first we consider the case then $|k| < \frac{M}{2}$, $|l| < \frac{M}{2}$ and $(k,l) \notin D_M^{\eta_1}$. In the view of fact that $\max_{k,l \notin D_M^{\eta_1}} |k|^{2\lambda_1} |l|^{2\lambda_2} e^-_{k,l} = (\frac{M}{2})^{2(\lambda_1+\lambda_2)} e^{-4\eta_1\left(\frac{M}{2}\right)^{1/\eta_2}}$ we have

$$\begin{aligned}S_1 &= \sum_{|k|<\frac{M}{2}} \sum_{|l|<\frac{M}{2}:(k,l)\notin D_M^{\eta_1}} |k|^{2\lambda_1} |l|^{2\lambda_2} |\hat{a}(k,l)|^2 e^-_{k,l} e^+_{k,l} = \\ &= \left(\frac{M}{2}\right)^{2(\lambda_1+\lambda_2)} e^{-4\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|^2_{\eta_1,\eta_2}.\end{aligned}$$

Let consider the element S_1 for the case $|k| < \frac{M}{2}$, $|l| \geq \frac{M}{2}$ and $(k,l) \notin D_M^{\eta_1}$, then

$$S_1 = \sum_{|k|<\frac{M}{2}} |k|^{2\lambda_1} \sum_{|l|\geq\frac{M}{2}:(k,l)\notin D_M^{\eta_1}} |l|^{2\lambda_2} |\hat{a}(k,l)|^2 e^-_{k,l} e^+_{k,l}.$$

Since the function $x^{2\nu}e^{-2\eta_2 x^{\frac{1}{\eta_1}}}$ has the maximum in the point $x_1 = \left(\frac{\nu\eta_1}{\eta_2}\right)^{\eta_1}$, then for all

$$|l| > \frac{M}{2} \geq \left(\frac{\lambda_2\eta_1}{\eta_2}\right)^{\eta_1}$$

it holds true

$$|l|^{2\lambda_2} e^{-2\eta_2|l|^{1/\eta_1}} < \left(\frac{M}{2}\right)^{2\lambda_2} e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}.$$

With account of this we have

$$S_1 = \sum_{|k|<\frac{M}{2}} |k|^{2\lambda_1} e^{-2\eta_2|k|^{1/\eta_1}} \sum_{|l|\geq\frac{M}{2}:(k,l)\notin D_M^{\eta_1}} |l|^{2\lambda_2}|\hat{a}(k,l)|^2 e^-k, le^+_{k,l} \leq$$
$$\leq \left(\frac{M}{2}\right)^{2(\lambda_1+\lambda_2)} e^{-4\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|^2_{\eta_1,\eta_2}.$$

For the third case when $|k| > \frac{M}{2}$, $l < \frac{M}{2}$, $(k,l) \notin D_M^{\eta_1}$ the estimation of S_1 can be found similar to the second one, namely we get

$$S_1 = \sum_{|k|>\frac{M}{2}} |k|^{2\lambda_1} e^{-2\eta_2|k|^{1/\eta_1}} \sum_{0<|l|<\frac{M}{2}:(k,l)\notin D_M^{\eta_1}} |l|^{2\lambda_2} e^{2\eta_2|l|^{1/\eta_1}} |\hat{a}(k,l)|^2 e^+_{k,l}$$

$$\leq \left(\frac{M}{2}\right)^{2(\lambda_1+\lambda_2)} e^{-4\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|^2_{\eta_1,\eta_2}.$$

And in the last case when $|k| > \frac{M}{2}$, $|l| > \frac{M}{2}$, the element S_1 can be easily estimated as in the cases above, namely we have

$$S_1 = \sum_{|k|\geq\frac{M}{2}} \sum_{|l|\geq\frac{M}{2}} |k|^{2\lambda_1}|l|^{2\lambda_2} e^-_{k,l}|\hat{a}(k,l)|^2 e^+_{k,l}$$

$$\leq \left(\frac{M}{2}\right)^{2(\lambda_1+\lambda_2)} e^{-4\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|^2_{\eta_1,\eta_2}$$

for $M \geq \left(\frac{\max\{\lambda_1,\lambda_2\}\eta_1}{\eta_2}\right)^{\eta_1}$.

Summarizing all cases considered above, we arrive to statement of lemma.

Lemma 2. *Let $a \in G_{\eta_1,\eta_2}$ for $\eta_1 \geq 1$, then for $\lambda_1, \lambda_2 > 1/2$ and*

$$M > 2\left(\frac{\max\{\lambda_1,\lambda_2\}\eta_1}{\eta_2}\right)^{\eta_1}$$

it holds true

$$\|a - P_{D_M^{\eta_1}} Q_{M,M} a\|_{\lambda_1,\lambda_2} \leq c_1 \left(\frac{M}{2}\right)^{\lambda_1+\lambda_2} e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|_{\eta_1,\eta_2},$$

where $c_1 = z_1 + 1$, $z_1 := z_1(\lambda_1,\lambda_2) = \gamma_{\lambda_1} + \gamma_{\lambda_2} + \gamma_{\lambda_1}\gamma_{\lambda_2}$.

Proof. Due to simple transformation we have

$$\|a - P_{D_M^{\eta_1}} Q_{M,M} a\|_{\lambda_1,\lambda_2} \le \|P_{D_M^{\eta_1}}(a - Q_{M,M}a)\|_{\lambda_1,\lambda_2} + \|(I - P_{D_M^{\eta_1}})a\|_{\lambda_1,\lambda_2}.$$

For the further estimation we need the previous result, that was obtained in Lemma 2 [10]. Namely, for $\lambda_1, \lambda_2 > 1/2$ it holds true

$$\|a - Q_{M,M}a\|_{\lambda_1,\lambda_2} \le z_1 \left(\frac{M}{2}\right)^{\lambda_1+\lambda_2} e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|_{\eta_1,\eta_2}.$$

Using inequality above and lemma 1 we have

$$\|a - P_{D_M^{\eta_1}} Q_{M,M} a\|_{\lambda_1,\lambda_2} \le \|a - P^{\eta_1}_{D_M} a\|_{\lambda_1,\lambda_2} + \|P_{D_M^{\eta_1}}(Q_{M,M}a - a)\|_{\lambda_1,\lambda_2} \le$$

$$z_1 \left(\frac{M}{2}\right)^{\lambda_1+\lambda_2} e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|_{\eta_1,\eta_2} + \left(\frac{M}{2}\right)^{\lambda_1+\lambda_2} e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|_{\eta_1,\eta_2} \le$$

$$\le c_1 \left(\frac{M}{2}\right)^{\lambda_1+\lambda_2} e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} \|a\|_{\eta_1,\eta_2},$$

what was to be proved.

For the further analysis we need following results

Proposition 2. *[7, Lemma 6.1.3] Let $k(t)$ be 1 - periodic function such that*

$$|\hat{k}(n)| \le c_0 \underline{n}^{\alpha} \quad n \in \mathbb{Z}. \tag{22}$$

Then for any $\lambda > \frac{1}{2}$ it fulfils

$$\left\| \int_0^1 k(t-s)v(t,s)ds \right\|_{\lambda-\alpha} \le c_0 2^{\lambda-\alpha+1} \gamma_{\lambda-\alpha} \|v\|_{\lambda,\lambda-\alpha},$$

where c_0 is some constant and $v(t,s)$ is 1-biperiodic function in Sobolev space $H^{\lambda,\lambda-\alpha}$.

Proposition 3. *[7, Lemma 6.1.1] For any $\lambda_1, \lambda_2 \ge \frac{1}{2}$, $u, a \in H^{\lambda_1,\lambda_2}$ it holds true*

$$\|au\|_{\lambda_1,\lambda_2} \le z_2 \|a\|_{\lambda_1,\lambda_2} \|u\|_{\lambda_1,\lambda_2},$$

where $z_2 := z_2(\lambda_1, \lambda_2) = 2^{\lambda_1+\lambda_2+2} \gamma_{\lambda_1} \gamma_{\lambda_2}$.

Further we need the following additional bounds. Namely using the propositions 2, 3 and integral representation of A_p it is easy to find that for any $\lambda_1 > 1/2$ and $\lambda_2 > 1/2$

$$\|A_p\|_{\lambda_1,\lambda_2} \le z_3 \|a_p\|_{\lambda_1,\lambda_2}, \tag{23}$$

where $z_3 := z_3(\lambda_1, \lambda_2) = 2^{\lambda_1+1} \gamma_{\lambda_1} z_2(\lambda_1, \lambda_2)$ is some increasing function. Now we are ready to prove the error of approximation for the operator $A_p \in \mathcal{L}(H^{\lambda}, H^{\lambda-\alpha})$ by $A_{p,M}$. The corresponding result is formulated in the lemma 3.

Lemma 3. *Let A_p has the form (5) for all $p = 0, \dots, q$ and the conditions (6)-(9) are fulfilled. Moreover we assume that $a_p \in G_{\eta_1,\eta_2}$, $p = 0..q$ for $\eta_1 \geq 1$ and $\eta_2 > 0$. Then for all $\lambda > \max\{\frac{1}{2}, \frac{1}{2} + \alpha\}$ and $M > 2\left(\frac{\eta_1}{\eta_2}\max\{\lambda, \lambda - \alpha\}\right)^{\eta_1}$ it holds true*

$$\|A_p - A_{p,M}\|_{\lambda,\lambda-\alpha} \leq c_2\|a\|_{\eta_1,\eta_2}\left(\frac{M}{2}\right)^{2\lambda-\alpha} e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}},$$

where $c_2 = c_1 c_0 2^{\lambda-\alpha+1}\gamma_{\lambda-\alpha}z_2$.

Proof. Taking into account Lemma 1, the Propositions 2 and 3, we have

$$\begin{aligned}\|(A - A_{p,M})\|_{\lambda-\alpha} &= \|\int_0^1 k_p(t-s)(a_p - P_{D_m^{\eta_1}}Q_{M,M}a_p)(t,s)u(s)ds\|_{\lambda-\alpha} \leq \\ &\leq c_0 2^{\lambda-\alpha+1}\gamma_{\lambda-\alpha}\|(a_p - P_{D_m^{\eta_1}}Q_{M,M}a_p)(t,s)u(s)\|_{\lambda-\alpha} \leq \\ &\leq c_0 2^{\lambda-\alpha+1}\gamma_{\lambda-\alpha}z_2\|a_p - P_{D_m^{\eta_1}}Q_{M,M}a_p\|_{\lambda,\lambda-\alpha}\|u(s)\|_\lambda \leq \\ &\leq c_2\left(\frac{M}{2}\right)^{2\lambda-\alpha} e^{-2\eta_2\left(\frac{M}{2}\right)^{\frac{1}{\eta_1}}}\|a_p\|_{\eta_1,\eta_2}\|u(s)\|_\lambda,\end{aligned}$$

which was to be proved.

Corollary 3. *From Lemma 3 follows that*

$$\|\sum_{p=0}^q A_p - A_{p,M}\|_{\lambda,\lambda-\alpha} \leq c_2(q+1)\max_p\{\|a_p\|_{\eta_1,\eta_2}\}\left(\frac{M}{2}\right)^{2\lambda-\alpha} e^{-2\eta_2\left(\frac{M}{2}\right)^{\frac{1}{\eta_1}}}$$

Now we are ready to propose fully discrete method for solving equations under consideration.

5. Fully Discrete Projection Method

Taking into account representation (10), we approximate $\mathcal{A}$ as follows

$$\mathcal{A}_M = D + P_l\sum_{p=0}^q A_{p,M}P_l, \tag{24}$$

where $l = N^\tau$, for some $0 < \tau < 1$. Note that our approximate variant of $\mathcal{A}$ is distinguished from respective approximation from [12] by using additional projections P_l and $P_{D_m^{\eta_1}}$. Such projection helps to bound the amount of arithmetical operations. The right-hand side of equation (4) we approximate as following

$$f_N := Q_N f,$$

where $N > M$. The main idea of the fully discrete projection method (FDPM) for equation (4) consists in solving the equation

$$\mathcal{A}_M u_N := Du_N + P_l\sum_{p=0}^q A_{p,M}P_l u_N = Q_N f, \tag{25}$$

where $A_{p,M}$ has the view (21) and $u_N \in \mathcal{T}_N$ is considered as approximate solution of (4). Note that by virtue of (7) and (8), it holds true $A_{p,M} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\beta})$, $p = 0, \dots, q$.

Lemma 4. *Let the conditions of Lemma 3 be satisfied and $f \in H^{\mu-\alpha}$. Moreover operator $\mathcal{A}_M$ has the form (24). Then for all $l \sim N^{\tau}$, $\tau \in [\frac{\mu-\lambda}{\mu-\lambda+\beta}, 1)$ and $\max\{\alpha+1/2, 1/2\} < \lambda < \mu$ it holds true*

$$\|(\mathcal{A}-\mathcal{A}_{\mathcal{M}})\|_{\lambda,\lambda-\alpha} \leq c_3 \left(\frac{N}{2}\right)^{\lambda-\mu} + c_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{2\lambda-\alpha},$$

where

$$c_3 := 2(q+1)\max_p\{\|a_p\|_{\mu,\mu+\beta-\alpha}\} z_3(\mu, \mu+\beta-\alpha),$$

$$c_4 = c_2(q+1)\max_p\{\|a_p\|_{\eta_1,\eta_2}\}.$$

Proof. Due to simple transformation we have

$$\begin{aligned} \mathcal{A}-\mathcal{A}_M = (I-P_l)\sum_{p=0}^{q} A_p u + \\ + P_l\left(\sum_{p=0}^{q} A_{p,M} - \sum_{p=0}^{q} A_p\right) P_l + P_l \sum_{p=0}^{q} A_p (I-P_l). \end{aligned} \tag{26}$$

Consider each summand separately.

By virtue of the fact that $A_p \in \mathcal{L}(H^{\lambda}, H^{\lambda-\alpha+\beta})$ for $p = 0..q$ and taking into account (16) and (23) we find that

$$\|(I-P_l)\sum_{p=0}^{q} A_p u\|_{\lambda-\alpha} \leq \left(\frac{l}{2}\right)^{\lambda-\mu-\beta} \|\sum_{p=0}^{q} A_p u\|_{\mu-\alpha+\beta} \leq$$

$$\leq \left(\frac{l}{2}\right)^{\lambda-\mu+\beta} (q+1)\max_p\{\|a_p\|_{\mu,\mu+\beta-\alpha}\} z_3(\mu, \mu+\beta-\alpha).$$

Because of $l = N^{\tau}$ and $N^{\tau(\lambda-\mu-\beta)} \leq N^{\lambda-\mu}$ for $\tau \in \left[\frac{\mu-\lambda}{\mu-\lambda+\beta}; 1\right)$, one can derive the estimate

$$\|(I-P_l)\sum_{p=0}^{q} A_p u\|_{\lambda-\alpha} \leq (q+1)\max_p\{\|a_p\|_{\mu,\mu+\beta-\alpha}\} z_3(\mu, \mu+\beta-\alpha)\left(\frac{N}{2}\right)^{\lambda-\mu}.$$

Similar estimate holds for third summand from (26), namely

$$\|P_l(\sum_{p=0}^{q} A_p(I-P_l)u)\|_{\lambda-\alpha} \leq \|\sum_{p=0}^{q} A_p\|_{\lambda-\beta,\lambda-\alpha}\|(I-P_l)u\|_{\lambda-\alpha} \leq$$

$$\leq \left(\frac{l}{2}\right)^{\lambda-\mu-\beta} (q+1)\max_p\{\|a_p\|_{\lambda-\beta,\lambda-\alpha}\} z_3(\lambda-\beta, \lambda-\alpha) \leq$$

$$\leq \left(\frac{N}{2}\right)^{\lambda-\mu} (q+1)\max_p\{\|a_p\|_{\lambda-\beta,\lambda-\alpha}\} z_3(\lambda-\beta, \lambda-\alpha).$$

The second summand from (26) we estimate with help of Lemma 3:

$$\|P_l(\sum_{p=0}^{q} A_{p,M} - \sum_{p=0}^{q} A_p)P_l\|_{\lambda,\lambda-\alpha} \leq$$
$$\leq c_2(q+1)\max_p\{\|a_p\|_{\eta_1,\eta_2}\}e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha}.$$

Combing the corresponding bounds we get the statement of the lemma.

Lemma 5. *Let the conditions of Lemma 3 are fulfilled. Then for any* $\lambda \in (\max\{\alpha+1/2, 1/2\}, \mu)$ *and for sufficiently small* N *and* M *such that*

$$c_3\left(\frac{N}{2}\right)^{\lambda-\mu} + c_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha} < \frac{c'_\lambda}{2}$$

it holds true

$$\|v\|_\lambda \leq d_\lambda \|\mathcal{A}_M v\|_{\lambda-\alpha},$$

where $d_\lambda = \frac{2}{c'_\lambda}$.

Proof. Using the inequality (11) and lemma 4 we have

$$\|v\|_\lambda \leq \frac{1}{c'_\lambda}\|\mathcal{A}v\|_{\lambda-\alpha} \leq \frac{1}{c'_\lambda}(\|\mathcal{A}_M v\|_{\lambda-\alpha} + \|(\mathcal{A}-\mathcal{A}_M)v\|_{\lambda-\alpha}) \leq$$
$$\leq \frac{1}{c'_\lambda}\frac{\|\mathcal{A}_M v\|_{\lambda-\alpha}}{1-\frac{1}{c'_\lambda}\left(c_3\left(\frac{N}{2}\right)^{\lambda-\mu} + c_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha}\right)} \leq \frac{2}{c'_\lambda}\|\mathcal{A}_M v\|_{\lambda-\alpha},$$

which was to be proved.

The estimation of accuracy for FDPM on the class of problems (4)-(9) with nonperturbed input data is established in the following assertion (see for detail [10]).

Theorem 1. *Let the conditions (6)- (9) are fulfilled, and operator* $\mathcal{A}_M$ *has the form (24). Then for any* $\lambda \in (\max\{1/2+\alpha, 1/2\}, \mu)$, $\mu > \alpha + 1/2$ *and for all*

$$\begin{gathered} M, N : M > 2\left(\frac{\eta_1}{\eta_2}\max\{\lambda, \lambda-\alpha\}\right)^{\eta_1}, \\ c_3\left(\frac{N}{2}\right)^{\lambda-\mu} + c_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha} < \frac{c'_\lambda}{2} \end{gathered} \tag{27}$$

it holds true

$$\|u-u_N\|_\lambda \leq c_5\left(\frac{N}{2}\right)^{\lambda-\mu} + c_6 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha}, \tag{28}$$

where $c_5 = 1 + d_\lambda c''_\lambda + d_\lambda c_3 + d_\lambda c''_\mu \gamma_{\mu-\alpha}$, $c_6 = d_\lambda c_4$.

Proof. Using the inequality (16) and $\|u\|_\mu \leq 1$ we find

$$\|u-u_N\|_\lambda \leq \|u-P_N u\|_\lambda + \|P_N u - u_N\|_\lambda \leq \left(\frac{N}{2}\right)^{\lambda-\mu} + \|P_N u - u_N\|_\lambda \tag{29}$$

Using Lemma 5 it is easy to find the bounds for second summand in (29), namely

$$\|P_N u - u_N\|_\lambda \leq d_\lambda \|\mathcal{A}_M(P_N u - u_N)\|_{\lambda-\alpha} \leq$$

$$\leq d_\lambda(\|(\mathcal{A}-\mathcal{A}_M)P_N u\|_{\lambda-\alpha}+\|\mathcal{A}_M u_N-\mathcal{A}P_N u\|_{\lambda-\alpha})\leq$$
$$\leq d_\lambda(\|(\mathcal{A}-\mathcal{A}_M)P_N u\|_{\lambda-\alpha}+\|Q_N f-f\|_{\lambda-\alpha}+\|\mathcal{A}(P_N u-u)\|_{\lambda-\alpha}).$$

Taking into account the lemma 4, inequalities (11), (16), (17) and the fact that $\|u\|_\mu \leq 1$ we have

$$\|P_N u-u_N\|_\lambda \leq d_\lambda\Big(c_3\left(\frac{N}{2}\right)^{\lambda-\mu}+c_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha}+$$
$$+c''_\mu \gamma_{\mu-\alpha}\left(\frac{N}{2}\right)^{\lambda-\mu}+c''_\lambda\left(\frac{N}{2}\right)^{\lambda-\mu}\Big).$$

Substituting the bound above in (29) we obtain the desired estimation.

Corollary 4. *As follows from (16), the optimal error of recovering the elements from $u\in H^\mu$, $\lambda<\mu$ is the following*

$$\|u-u_n\|_\lambda \leq n^{\lambda-\mu}\|u\|_\mu,$$

where $u_n \in \mathcal{T}_n$ is some approximation. From Theorem 1 follows that for $M \asymp \log^{\eta_1} N$ we have $\|u-u_N\|_\lambda \asymp \left(\frac{N}{2}\right)^{\lambda-\mu}$, that establish optimality of the method.

6. Calculation of arithmetical operations

Let construct the matrix corresponding to the element $P_l A_{p,M} P_l u_N(t)$. Using the fact that $\int k_0(t-s)e_i(s)ds=\hat{k}_0(i)e_i(t)$ we have

$$\begin{aligned} P_l A_{p,M} P_l u_N(t) &= P_l\int_0^1 k_p(t-s)Q_{M,M}a_p(t,s)P_l u_N(s)ds=\\ &=P_l\int_0^1 k_p(t-s)\sum_{m,k\in D_M^{\eta_1}}\widehat{Q_{M,M}a_p}(m,k)e_m(t)e_k(s)\sum_{i\in\mathbb{Z}_l}\hat{u}(i)e_i(s)ds=\\ &=P_l\sum_{m,k\in D_M^{\eta_1},i\in\mathbb{Z}_l}\widehat{Q_{M,M}a_p}(m,k)\hat{u}(i)e_m(t)\int_0^1 k_p(t-s)e_{k+i}(s)ds=\\ &=P_l\sum_{m,k\in D_M^{\eta_1},i\in\mathbb{Z}_l}\widehat{Q_{M,M}a_p}(m,k)\hat{k}_p(k+i)\hat{u}(i)e_{m+k+i}(t)= \end{aligned} \tag{30}$$

To obtain the matrix form of FDPM (25) one can make the following substitution

$$\left|\begin{array}{c} m+k+i\to m\\ k+i\to k \end{array}\right|$$

and as the result get

$$P_l A_{p,M} P_l u_N(t)=\sum_{m\in\mathbb{Z}_l}\left[\sum_{i\in\mathbb{Z}_l}\Lambda_{m,i}^{p,\eta_1}\hat{u}(i)\right]e_m(t),$$

where

$$\Lambda_{m,i}^{p,\eta_1}=\sum_{(m-k,k-i)\in D_M^{\eta_1},k\in\mathbb{Z}_{M+l}}\widehat{Q_{M,M}a_p}(m-k,k-i)\hat{k}_p(k).$$

Thus, the equation (25) can be rewritten as the system of linear equations

$$D\bar{u} + \sum_{p=0}^{q} \Lambda^{p,\eta_1} \bar{u} = \bar{f}, \tag{31}$$

where $\bar{u} = \{\hat{u}(i)\}_{i\in\mathbb{Z}_N}$ is Fourier coefficient of desired solution, $\bar{f} = \{\hat{f}(i)\}_{i\in\mathbb{Z}_N}$ is Fourier coefficient for right-hand side and $\Lambda^{p,\eta_1} = \{\Lambda^{p,\eta_1}_{m,i}\}_{m,i\in\mathbb{Z}_l}$.

Proposition 4. *Calculation of matrix Λ^{p,η_1} requires $N\log N$ arithmetical operations (a.o.) by the order.*

Proof. Since $D^{\eta_1}_M \subset D^1_M$ for $\eta_1 \geq 1$, then the biggest amount of arithmetical operations is needed for calculation of matrix $\Lambda^{p,1}$ and we consider this case below. Since $(m-k, k-i) \in D^1_M$, then by the definition of the set D^1_M we have that $m - i \in \mathbb{Z}_M$. Let $l = m - i$ and calculate the amount of a.o. for element $\Lambda^{p,1}_{m,i}$ near diagonal l. For that rewrite the element $\Lambda^{p,1}_{m,i}$ in the following way

$$\Lambda^{p,1}_{m,m-l} = y_m = \sum_{\mathbb{Z}_{M+l}} \widehat{Q_{M,M}a_p}(m-k, l-(m-k))\hat{k}_p(k) = \sum_{k\in\mathbb{Z}_{M+l}} \alpha(m-k)k_p(k).$$

Using FFT, we can construct the element $\Lambda^{p,1}_{m,m-l}$ for all $m \in \mathbb{Z}_{M+l}$ with $(M+l)\log(M+l)$ a.o. by the order. Because of $l \in \mathbb{Z}_M$, the total amount of a.o. for constructing elements of matrix $\Lambda^{p,1}$ is $M(M+l)\log(M+l)$. Taking into account the fact that $l\log l \sim N$ for $\tau \in [\frac{\mu-\lambda}{\mu-\lambda+\beta}, 1)$ we arrive to the required result.

Let's calculate the amount of arithmetical operations that is necessary to construct all the elements from equation (31).

- For the element $\widehat{Q_{M,M}a_p}(i,j)$ we apply the relation

$$\widehat{Q_{M,M}a_p}(i,j) = \frac{1}{M^2}\sum_{l_1=1}^{M}\sum_{l_2=1}^{M} a_p(l_1M^{-1}, l_2M^{-1})e_i(l_1M^{-1})e_j(l_2M^{-1})$$

  that can be calculated for all $i, j \in \mathbb{Z}_M$ with the help of FFT by $M^2\log M$ arithmetical operations.
- the elements of the vector $\bar{f}$ can be calculated by the relation

$$\hat{f}(i) = \frac{1}{N}\sum_{=1}^{N} f(lN^{-1})e_i(lN^{-1})$$

  with the help of FFT by $N\log N$ a.o.
- the elements of Λ^{p,η_1} for $l = N^\tau$ can be calculated by $(N\log N)$ a.o. (see proposition 4).

Summarizing all items above, we can conclude that the total amount of a.o. for constructing all elements from (25) is $N\log N$ by the order.

7. Perturbed input data

Following [7], suppose that instead of functions $a_p(t,s), p = 0,\ldots,q$ and $f(t)$ we are given only some their pertubations $a_{p,\varepsilon}(t,s), p = 0,\ldots,q$, and $f_\delta(t)$ is such that in the points of uniform grids it fulfils

$$M^{-2}\left(\sum_{i,j=1}^{M}|a_{p,\varepsilon}(iM^{-1},jM^{-1})-a_p(iM^{-1},jM^{-1})|\right)^{\frac{1}{2}}\leq\varepsilon,\quad p=0,\dots,q,$$

$$N^{-1}(\sum_{j=1}^{N}|f_\delta(jN^{-1})-f(jN^{-1})|^2)^{1/2}\leq\delta\|f\|_{\mu-\alpha}.$$

It is easy to show (see, for example, [7, p.100]), that mentioned estimations are equivalent to

$$\|Q_{M,M}(a_p-a_{p,\varepsilon})\|_{0,0}\leq\varepsilon,\quad p=0,\dots,q, \tag{32}$$

$$\|Q_N(f_\delta-f)\|_0\leq\delta\|f\|_{\mu-\alpha} \tag{33}$$

respectively. Then taking into account perturbation of input data the FDPM for equation (10) becomes

$$\mathcal{A}_{M,\varepsilon}u_{N,\varepsilon,\delta}=Du_{N,\varepsilon}+P_l\sum_{p=0}^{q}A_{p,M,\varepsilon}P_lu_{N,\varepsilon,\delta}=Q_Nf_\delta, \tag{34}$$

where $A_{p,M,\varepsilon}v(s)=\int_0^1k_p(t-s)P_{D_M^{\eta_1}}Q_{M,M}a_{p,\varepsilon}(t,s)v(s)ds$ and $u_{N,\varepsilon,\delta}\in\mathcal{T}_N$ is approximate solution.

We pose the problem to solve equations (4) and (10) with perturbed input data as (32) and (33) with minimal amount of discrete information (i.e. set of values for functions $f_\delta(t)$ and $a_{p,\varepsilon}(t,s)$ in the points of uniform grid). At the same time arithmetical expenses should be less in comparison with methods known earlier (see, for example, [7] and [12]).

To achieve the aim of our investigation at first we state some auxiliary estimations.

Lemma 6. *Let estimation (32) is satisfied then for any* $\lambda\geq\max\{1/2,\alpha+1/2\}$ *it holds true*

$$\|\mathcal{A}_M-\mathcal{A}_{M,\varepsilon}\|_{\lambda,\lambda-\alpha}\leq c_7\left(\frac{M}{2}\right)^{2\lambda-\alpha}\varepsilon,$$

where $c_7=c_02^{\lambda-\alpha+1}\gamma_{\lambda-\alpha}z_2(\lambda,\lambda-\alpha)(q+1)$.

It is easy to find that

$$(\mathcal{A}_M-\mathcal{A}_{M,\varepsilon})u=P_l(\sum_{p=0}^{q}A_{p,M}-A_{p,M,\varepsilon})P_lu.$$

Using Proposition 2, 3, inequalities (18) and (32) we have

$$\|(\mathcal{A}_M-\mathcal{A}_{M,\varepsilon})v\|_{\lambda-\alpha}\leq$$

$$\leq\|\sum_{p=0}^{q}P_l\int_0^1k(t-s)P_{D_{D_M^{\eta_1}}}Q_{M,M}(a_{p,\varepsilon}-a_p)(t,s)P_lv(s)ds\|_{\lambda-\alpha}\leq$$

$$\leq c_02^{\lambda-\alpha+1}\gamma_{\lambda-\alpha}z_2(\lambda,\lambda-\alpha)\sum_{p=0}^{q}\|Q_{M,M}(a_{p,\varepsilon}-a_p)\|_{\lambda,\lambda-\alpha}\|P_lv\|_\lambda\leq$$

$$\leq (q+1)c_0 2^{\lambda-\alpha+1}\gamma_{\lambda-\alpha} z_2(\lambda,\lambda-\alpha)\left(\frac{M}{2}\right)^{2\lambda-\alpha}\varepsilon\|v\|_\lambda,$$

which is the required result.

Lemma 7. *Let estimation (32) is satisfied and $\mathcal{A}_{M,\varepsilon}$ has the form (34). Then for M such that*

$$d_\lambda c_7\left(\frac{M}{2}\right)^{2\lambda-\alpha}\varepsilon\leq\frac{1}{2} \tag{35}$$

operator $\mathcal{A}_{M,\varepsilon}$ is invertible between spaces H^λ and $H^{\lambda-\alpha}$ and the following holds true

$$\|u\|_\lambda\leq 2d_\lambda\|\mathcal{A}_{M,\varepsilon}u\|_{\lambda-\alpha}. \tag{36}$$

The lemma can be proved in a similar way as lemma 5 by using the statements of lemmas 5 and 6.

Lemma 8. *Let the conditions (6)-(9) and (32), (33) fulfil and $a\in G_{\eta_1,\eta_2}$, $\eta_1\geq 1,\eta_2>0$. Then for all $\lambda\in(\max\{1/2,\alpha+1/2\},\mu)$ it holds true*

$$\|u_N-u_{N,\delta,\varepsilon}\|_\lambda\leq c_8\left(\frac{N}{2}\right)^{\lambda-\alpha}\delta+c_9\left(\frac{M}{2}\right)^{2\lambda-\alpha}\varepsilon,$$

where $c_8=2d_\lambda c''_\mu$ and $c_9=c_{10}c_7 2d_\lambda$ with $c_{10}\leq 2+d_\lambda(c''_\lambda+\frac{c'_\lambda}{2}+c''_\mu\gamma_{\mu-\alpha})$.

Proof. Using Lemmas 7 and 6, inequality (18) and (33) we find

$$\begin{aligned}\|u_N-u_{N,\delta,\varepsilon}\|_\lambda &\leq 2d_\lambda\|\mathcal{A}_{M,\varepsilon}(u_N-u_{N,\delta,\varepsilon})\|_{\lambda-\alpha}\leq\\ &\leq 2d_\lambda\|\mathcal{A}_M u_N-\mathcal{A}_{M,\varepsilon}u_N\|_{\lambda-\alpha}++2d_\lambda\|Q_N f-Q_N f_\delta\|_\lambda\leq\\ &\leq 2d_\lambda\left(\left(\frac{N}{2}\right)^{\lambda-\alpha}\delta\|f\|_{\mu-\alpha}+c_7\left(\frac{M}{2}\right)^{2\lambda-\alpha}\varepsilon\|u_N\|_\lambda\right).\end{aligned} \tag{37}$$

Using (28) and (27) we bound the norm of element u_N as follows:

$$\|u_N\|_\lambda\leq\|u\|_\lambda+\|u-u_N\|_\lambda\leq$$
$$\leq\|u\|_\lambda+c_5\left(\frac{N}{2}\right)^{\lambda-\mu}+c_6 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha}\leq c_{10}.$$

Substituting the estimation above in (37) and taking into account (11) we derive desired estimation.

8. Selection of the discretization levels

Generalizing the results of the previos section we rewrite general estimation of error for FDPM. By virtue of Theorem 1 and Lemma 7, the accuracy of method (34) is estimated as

$$\begin{aligned}\|u-u_{N,\delta,\varepsilon}\|_\lambda &\leq\|u-u_N\|_\lambda+\|u_N-u_{N,\delta,\varepsilon}\|_\lambda\leq\\ &\leq c_5\left(\frac{N}{2}\right)^{\lambda-\mu}+c_6 e^{-\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}}\left(\frac{M}{2}\right)^{2\lambda-\alpha}+\\ &+c_8\left(\frac{N}{2}\right)^{\lambda-\alpha}\delta+c_9\left(\frac{M}{2}\right)^{2\lambda-\alpha}\varepsilon.\end{aligned} \tag{38}$$

Further following the paper [12] we consider the problem to select such levels of discretization N and M that minimize the error bound (38). Here we consider only the case then smoothness of parameters μ is known precisely (a priori case).

1. **A priori selection of parameter.** The problem of a priori selection of discretization levels was described in detail in [12] for class of equations under consideration. Here we slightly modify FDPM. However, as we can see below, it doesn't influence on the best accuracy of the method.

Further we denote by $[q]$ the integer part of number q and formulate the theorem that establishes a priori rule for choosing discretization parameter.

Theorem 2. *Let the conditions (6)-(9) fulfil and input data are perturbed as (33) and (32). Then for any* $\lambda \in (\max\{1/2, \alpha + 1/2\}, \mu)$, $\mu > \alpha + 1/2$ *with choosing the discretization parameters according to rule*

$$\bar{M} = \left[2 \left(\frac{1}{2\eta_2} \log \frac{c_{13}}{\varepsilon} \right)^{\eta_1} \right], \tag{39}$$

$$\bar{N} = \left[2 \left(\frac{c_8 \delta}{c_5} \right)^{\frac{1}{\alpha - \mu}} \right] \tag{40}$$

the error bound of the method (34) has the form

$$\|u - u_{N,\delta,\varepsilon}\|_\lambda \le c_{11} \delta^{\frac{\mu-\lambda}{\mu-\alpha}} + c_{12} \varepsilon \log^{\eta_1(2\lambda - \alpha)} \frac{c_{13}}{\varepsilon}, \tag{41}$$

where

$$c_{11} = (c_8)^{\frac{\lambda-\mu}{\alpha-\mu}} c_5^{\frac{\lambda-\alpha}{\mu-\alpha}}, \quad c_{12} = \frac{c_6}{c_{13}} \left(\frac{1}{2\eta_2} \right)^{\eta_1(2\lambda-\alpha)}$$

and

$$c_{13} = \frac{c_1}{c_{10}} \max_p \{\|a\|_{\eta_1,\eta_2}\}.$$

Proof. Direct substitution (39) and (40) in (38) gives the statement of theorem.

Remark 4. *It is evident that condition (35) fulfils with choosing M according (39) for sufficiently small* ε. *Let's check that condition (27) also holds true. From (39) it follows that*

$$c_{13} e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} = \varepsilon.$$

Then taking into account the relation above and (40) we can conclude that condition (27) takes place sor sufficiently small ε.

2. **Fast solving of FDPM (34).** Following [6] for fast solving (34), we propose to use GMRES. Such approach for solving problem under consideration has been detailed in [6] and here we only rewrite main points. Denote by

$$S_N := D + P_l \sum_{p=0}^{q} A_{p,M,\varepsilon} P_l.$$

It is evident that S_N is invertable operator (see lemma 7) that acts in $\mathcal{T}_N$. Thus according to theory we can apply GMRES with operator S_N and right-hand side f_N with respect to the space H^α. The procedure concludes in constructing sequence u_{n_N} that satisfies the condition for $n = 1, 2, \dots$

$$\|S_N u_{n_N} - f_N\|_\alpha = \min_{u \in \mathcal{K}_n(S_N, f_N)} \|S_N u - f_N\|_\alpha,$$

where $\mathcal{K}_n(S_N, f_N)$ is well-known Krylov space. As the stopping rule we consider the discrepancy principle

$$\|S_N u_{n_N} - f_N\|_\alpha \le c\delta \|f_N\|_\alpha, \tag{42}$$

where u_{n_N} is n-iteration of GMRES that we consider as approximation for u_N.

Now we are ready to establish the accuracy of GMRES approximation for our class of problems.

Theorem 3. *Suppose that $N, M \to 0$. Let n be the first number for which the condition (42) fulfils. Then the accuracy of GMRES applied to equation (34) is the following*

$$\|u_{N,\delta,\varepsilon} - u_{N_n}\|_\lambda \le 2 d_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta \|f_N\|_\alpha. \tag{43}$$

Moreover we have that $n = O(\log(N))$.

Proof. Using Lemma 6 we have that

$$\|u_{N,\delta,\varepsilon} - u_{N_n}\|_\lambda \le d_\lambda \|\mathcal{A}_{M,\varepsilon}(u_{N,\delta,\varepsilon} - u_{N_n})\|_{\lambda-\alpha} \le d_\lambda \|f_N - \mathcal{A}_{M,\varepsilon} u_{N_n}\|_{\lambda-\alpha}.$$

Further applying the inequalities (18) and (42) one can obtain

$$\|u_{N,\delta,\varepsilon} - u_{N_n}\|_\lambda \le 2 d_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta \|f_N\|_\alpha,$$

what was to be proved.

Remark 5. *As we can see from Theorem 3 the accuracy of FDPM method in combination with GMRES is the following*

$$\|u - u_{N_n}\|_\lambda \le O(\delta^{\frac{\mu-\lambda}{\mu-\alpha}} + \varepsilon \log^{\eta_1(2\lambda-\alpha)} \frac{1}{\varepsilon}).$$

Such accuracy of FDPM in the case of $\varepsilon = 0$ is optimal by the order (see [11]).

Remark 6. *For the realization of GMRES we need at every iteration to compute a matrix-vector product $S_N f_N$. Due to the structure of S_N as (34) and relation (30), the calculation can be performed by $l \cdot M^2$ operations. Since $M = O(\log N)$ (see corollary 4), then due to $N = l \log l$ for $l = N^\tau, \tau \in [\frac{\mu-\lambda}{\mu-\lambda+\beta}, 1)$ we have that constructing of matrix-vector product $S_N f_N$ requires $N \log N$ a.o. Moreover, as it is known, for realization of GMRES $O(nl)$ floating-point operations must be computed at the n-th iteration, i.e on the n-th step we need $O(N \log N)$ a.o. Thus total amount of a.o. for solving (10) is limited by $O(N \log N)$ by the order.*

Remark 7. *Let us suppose that $\varepsilon \ge c\delta$ and calculate the amount of necessary discrete information for equation (4) to implement the proposed method (34) with the accuracy (41). It is evident that in that case M does not exceed the magnitude $O(\log(N))$. So, for the discretization of $A_{p,\varepsilon}$ less than $O(\log^2 N)$*

values of kernels $a_{p,\varepsilon}(t,s)$ *in the points of the uniform grid should be used. Note, that in the monograph [7] for the realization of the fully discrete projection method (34) at* $M = N$ *the order of discrete information was estimated as* $O(N \log N)$ *.*

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