Журнал обчислювальної та прикладної математики | **2015, №3(120)** | Journal of Computational & Applied Mathematics

УДК 519.233.2
MSC 93BXX

THE MINIMAX APPROACH TO THE ESTIMATION OF SOLUTIONS TO SPECIAL SYSTEMS OF VARIATIONAL EQUATIONS IN HILBERT SPACES AND ITS APPLICATIONS

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МІНІМАКСНИЙ ПІДХІД ДО ОЦІНЮВАННЯ РОЗВ'ЯЗКІВ СПЕЦІАЛЬНИХ СИСТЕМ ВАРІАЦІЙНИХ РІВНЯНЬ В ГІЛЬБЕРТОВИХ ПРОСТОРАХ ТА ЙОГО ЗАСТОСУВАННЯ

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Abstract. We investigate the estimation problems of linear functionals from solutions to systems of variational equations in Hilbert spaces with unknown right-hand sides. The unknown right-hand sides and the unknown correlation operator of estimation error are supposed to belong to the certain sets. It is shown that the linear mean square estimates of the above-mentioned functionals and estimation errors are expressed via solutions to the systems of variational equations of the special type. We elaborate the numerical algorithms for finding such estimates. The obtained results are applied to the case when the system of variational equations is a stationary heat equation with right-hand side satisfying to the linearized Navier-Stokes equation.
Keywords: linear functional, variational equation, observation, mean square estimate.

Резюме. В статті досліджуються задачі оцінювання розв'язків варіаційних рівнянь в спеціальних гільбертових просторах з невідомими правими частинами та похибками спостережень. Припускається, що в праві частини таких рівнянь входять вектори, які в свою чергу, є розв'язками варіаційних рівнянь також із невідомими правими частинами. При умові, що невідомі вектори належать правим частинам вивчаються спочатку питання про оцінки лінійних неперервних функціоналів від розв'язків та правих частин таких рівнянь. Вводяться гарантовані середньоквадратичні лінійні оцінки та гарантовані середньоквадратичні похибки таких оцінок. Показано, що такі оцінки та похибки оцінювання виражаються через розв'язки систем варіаційних рівнянь. Показано також, що гарантовані оцінки лінійних функціоналів дають змогу знайти гарантовані оцінки розв'язківваріаційних рівнянь при

спостереженнях з похибками. Для наближеного знаходження гарантованих оцінок приводиться система лінійних алгебраїчних рівнянь, через яку виражають такі оцінки. Доводиться, що наближені оцінки збігаються до гарантованих. Одержані результати застосовуються у випадку, коли варіаційне рівняння породжене крайовою задачею для стаціонарнарного рівняння переносу з дифузією з правою частиною, що є розв'язком лінеаризованого стаціонарного рівняння Нав'є-Стокса в умовах невизначеності.
Ключові слова: лінійний функціонал, варіаційне рівняння, спостереження, середньоквадратична оцінка.

Introduction

The problems of optimal estimation of solutions to BVPs for partial differential equations with unknown parameters arise in geophysics, optics, acoustics, etc.

In order to reduce the estimation errors, the observation of their solutions in certain points or domains are needed.

Depending on assumptions regarding unknown parameters and observation errors, there is variety of approaches of solving such problems. In many cases these approaches are reduced to the estimation of solutions of variational equations in certain Hilbert spaces.

In the present paper we apply the guaranteed approach for finding linear mean square estimates of a solution to a system of variational equations in Hilbert spaces.

1. Notations

If X is a separable Hilbert space over $\mathbb{R}$ with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$, then by $J_X \in \mathcal{L}(X, X')$ we will denote an operator, called a canonical isomorphism from X onto dual space X', and defined by the equality $(v,u)_X = < v, J_X u >_{X\times X'}$ $\forall u, v \in X$, where $< x, f >_{X\times X'} := f(x)$ for $x \in X$, $f \in X'$, and $\mathcal{L}(X,Y)$ is the set of bounded linear operators mapping X into a Hilbert space Y.

Let D be an open bounded set in $\mathbb{R}^n$ with Lipschitzian boundary Γ. Let $\mathcal{D}(D)$ (or $\mathcal{D}(\bar{D})$) be the space of infinitely differential functions with compact support contained in D (or $\bar{D}$). A continuous linear form on $\mathcal{D}(D)$ is called a distribution on D. We denote by $\mathcal{D}'(D)$ the set of distributions on D. If $T \in \mathcal{D}'(D)$ we denote by $< T, \phi >$ its value on the function $\phi \in \mathcal{D}(D)$.

If $T \in \mathcal{D}'(D)$ the derivative $D_i T = \frac{\partial T}{\partial x_i}$ which coincides with the usual differentiation of continuously differentiable functions, is defined by $< D_i T, \phi > = - < T, D_i \phi >$.

We denote by $L^2(D)$ the space of the real functions defined on D with the second power absolutely integrable for the Lebesgue measure

$dx_1 \dots dx_n$. This is a Hilbert space with the norm

$$\|u\|_{L^2(D)} = \left(\int_D |u(x)|^2 \, dx \right)^{1/2}$$

and inner product

$$(u, v)_{L^2(D)} = \int_D u(x) v(x) \, dx.$$

The Sobolev space $H^1(D)$ is the space of functions in $L^2(D)$ with derivatives of order 1 also belonging to $L^2(D)$. This is a Hilbert space with the norm

$$\|u\|_{H^1(D)} = \left(\|u\|^2_{L^2(D)} + \sum_{j=1}^{n} \|D_j u\|^2_{L^2(D)} \right)^{1/2}$$

and inner product

$$(u, v)_{H^1(D)} = (u, v)_{L^2(D)} + \sum_{j=1}^{n} (D_j u, D_j v)_{L^2(D)}.$$

The closure of $\mathcal{D}(D)$ in $H^1(D)$ is denoted by $H^1_0(D)$.

Denote by γ_0 a bounded linear operator, called the trace operator, which maps the space $H^1(D)$ into the space $L^2(\Gamma)$ such that $\gamma_0 u(x) = u(x)$ for $u \in \mathcal{D}(\bar{D})$. It is known that $H^1_0(D)$ is equal to the kernal of γ_0, i. e. $H^1_0(D) = \{u \in H^1(D) : \gamma_0 u = 0\}$.

We will also use the notation $L^2(D)^n$, $H^1(D)^n$, $H^1_0(D)^n$, $\mathcal{D}(D)^n$ for the spaces consisting of vector functions $\mathbf{u} = (u_1, \dots, u_n)$ whose componets belong to one of the spaces $L^2(D)$, $H^1(D)$, $H^1_0(D)$, $\mathcal{D}(D)$, respectively, with usual product norms and inner products (exept $\mathcal{D}(D)^n$ or $\mathcal{D}(\bar{D})^n$ which are not normed spaces).

For every $v \in \mathcal{D}'(D)$ we put

$$\mathbf{grad}\, v := \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n} \right),$$

which defines the linear differential operator denoted by **grad** from $\mathcal{D}'(D)$ to $\mathcal{D}'(D)^n$.

We define the linear differential operator denoted by div from $\mathcal{D}'(D)^n$ to $\mathcal{D}'(D)$ by

$$\operatorname{div} \mathbf{v} := \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \quad \forall \mathbf{v} = (v_1, \dots, v_n) \in \mathcal{D}'(D)^n$$

and the Laplace operator $\mathbf{\Delta}$ from $\mathcal{D}'(D)^n \to \mathcal{D}'(D)^n$ by

$$\mathbf{\Delta v} = \left(\sum_{i=1}^{n} \frac{\partial^2 v_1}{\partial x_i^2}, \dots, \sum_{i=1}^{n} \frac{\partial^2 v_n}{\partial x_i^2} \right).$$

Set $\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(D)^n,\ \operatorname{div}\mathbf{u} = 0\},\ V =$ closure of $\mathcal{V}$ in $H_0^1(D)^n$.

The space V is a Hilbert space with inner product

$$((\mathbf{u}, \mathbf{v})) = \sum_{i=1}^{n} (D_i\mathbf{u}, D_i\mathbf{v})_{L^2(D)^n}.$$

In [1] it is shown that $V = \{\mathbf{u} \in H_0^1(D)^n,\ \operatorname{div}\mathbf{u} = 0\}$.

By $L^2(\Omega, X)$ we denote the Bochner space composed of random[1] variables $\xi = \xi(\omega)$ defined on a certain probability space $(\Omega, \mathcal{B}, P)$ with values in X such that $\|\xi\|^2_{L^2(\Omega,X)} = \int_\Omega \|\xi(\omega)\|_X^2 dP(\omega) < \infty$. In this case there exists the Bochner integral

$$\mathbb{E}\xi := \int_\Omega \xi(\omega)\, dP(\omega) \in X \tag{1}$$

which is called the mathematical expectation or the mean value of random element $\xi(\omega)$ and satisfies the condition

$$(h, \mathbb{E}\xi)_X = \int_\Omega (h, \xi(\omega))_X\, dP(\omega) \quad \forall h \in X. \tag{2}$$

Being applied to random variable ξ with values in $\mathbb{R}$ this expression leads to a usual definition of its mathematical expectation because the Bochner integral (1) reduces to a Lebesgue integral with probability measure $dP(\omega)$.

In $L^2(\Omega, X)$ one can introduce the inner product

$$(\xi, \eta)_{L^2(\Omega,X)} := \int_\Omega (\xi(\omega), \eta(\omega))_X\, dP(\omega) \quad \forall \xi, \eta \in L^2(\Omega, X). \tag{3}$$

Applying the sign of the mathematical expectation, one can write relationships (1)–(3) as

$$\|\xi\|^2_{L^2(\Omega,X)} = \mathbb{E}\|\xi(\omega)\|_X^2, \tag{4}$$

$$(h, \mathbb{E}\xi)_X = \mathbb{E}(h, \xi(\omega))_X \quad \forall h \in X, \tag{5}$$

$$(\xi, \eta)_{L^2(\Omega,X)} := \mathbb{E}(\xi(\omega), \eta(\omega))_X \quad \forall \xi, \eta \in L^2(\Omega, X). \tag{6}$$

$L^2(\Omega, X)$ equipped with norm (4) and inner product (6) is a Hilbert space.

2. Statement of estimation problem

Let V_1, V_2, H_1, and H_2 be Hilbert spaces such that the following inclusions hold $V_1 \subset H_1$ and $V_2 \subset H_2$ with

$$\|\cdot\|_{H_1} \le c_1 \|\cdot\|_{V_1} \quad \text{and} \quad \|\cdot\|_{H_2} \le c_2 \|\cdot\|_{V_2}, \tag{7}$$

where c_1 and c_2 are positive constants.

[1]Random variable ξ with values in Hilbert space X is considered as a function $\xi : \Omega \to X$ imaging random events $E \in \mathcal{B}$ to Borel sets in X (Borel σ-algebra in X is generated by open sets in X).

Consider the problem: given $f_1 \in H_1$, $f_2 \in H_2$, find $\varphi_1 \in V_1$, $\varphi_2 \in V_2$, such that

$$a_1(\varphi_1, \psi_1) = (B\varphi_2 + f_1, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1, \tag{8}$$

$$a_2(\varphi_2, \psi_2) = (f_2, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2, \tag{9}$$

where $B \in \mathcal{L}(H_2, H_1)$. We make the following assumptions on the involved bilinear forms $a_1 : V_1 \times V_1 \to \mathbb{R}$ and $a_2 : V_2 \times V_2 \to \mathbb{R}$:

$$a_1(\cdot,\cdot) \quad \text{and} \quad a_2(\cdot,\cdot) \quad \text{are continuous, that is}$$

$$\exists M_1 > 0 : \forall \sigma_1, \tau_1 \in V_1 \quad a_1(\sigma_1, \tau_1) \le M_1 \|\sigma_1\|_{V_1} \|\tau_1\|_{V_1},$$

$$\exists M_2 > 0 : \forall \sigma_2, \tau_2 \in V_2 \quad a_2(\sigma_1, \tau_1) \le M_2 \|\sigma_2\|_{V_2} \|\tau_2\|_{V_2}.$$

We also suppose that the following problems: given $g_i \in H_i$, find $\varphi_i \in V_i$ such that

$$a_i(\varphi_i, \psi_i) = (g_i, \psi_i)_{H_i} \quad \forall \psi_i \in V_i, \quad i = 1, 2, \tag{10}$$

are well-posed. It follows from this assumptions that problem (8)–(9) is also well-posed.

We suppose that functions f_1 and f_2 in the right hand sides of equations (8) and (9) are not known exactly. The estimation problem consists in the following: from the observations

$$y = C\varphi_1 + \eta, \tag{11}$$

find optimal in a certain sense estimate of the functional

$$l(\varphi_1) = (l_0, \varphi_1)_{H_1} \tag{12}$$

in the class of estimates linear w.r.t. observations (11),

$$\widehat{l(\varphi_1)} = (y, u)_{H_0} + c \tag{13}$$

under the assumption that errors $\eta = \eta(\omega)$ in observations (11) are realizations of random variables defined on a certain probability space $(\Omega, \mathcal{B}, P)$ with values in a Hilbert space H_0 over $\mathbb{R}$ belonging to the set G_0, and $(f_1, f_2) \in G_1$. Here $C \in \mathcal{L}(H_1, H_0)$ is a linear continuous operator, $u \in H_0$, $c \in \mathbb{R}$,

$$G_1 := \{(\tilde f_1, \tilde f_2) \in H_1 \times H_2 : (Q_1(\tilde f_1 - f_1^0), \tilde f_1 - f_1^0)_{H_1} + (Q_2(\tilde f_2 - f_2^0), \tilde f_2 - f_2^0)_{H_2} \le 1\}, \tag{14}$$

$$G_0 := \{\tilde\eta \in L^2(\Omega, H_0) : \mathbb{E}\tilde\eta = 0,\ \mathbb{E}(Q_0\tilde\eta, \tilde\eta)_{H_0} \le 1\}, \tag{15}$$

$l_0, f_1^0 \in H_1$, and $f_2^0 \in H_2$ are given elements, Q_1, Q_2, and Q_0 are bounded selfadjoint positive definite operators in H_1, H_2, and H_0, respectively, for which there exist bounded inverse operators Q_1^{-1}, Q_2^{-1}, and Q_0^{-1}.

Definition 1. *An estimate*

$$\widehat{\widehat{l(\varphi_1)}} = (y, \hat u)_{H_0} + \hat c \tag{16}$$

is called a minimax estimate of the $l(\varphi_1)$, if elements $\hat{u} \in H_0$ and a number $\hat{c}$ are determined from the condition

$$\inf_{u \in H_0, c \in \mathbb{R}} \sigma(u,c), \tag{17}$$

where $\sigma(u,c) := \sup_{\tilde{f}=(f_1,f_2)\in G_1, \tilde{\eta}, \in G_0} \mathbb{E}[l(\tilde{\varphi}_1) - \widehat{l(\varphi_1)}]^2$, $\tilde{\varphi}_1$ *is a solution to problem* (8),(9) *when* $f_1(x) = \tilde{f}_1(x)$, $f_2(x) = \tilde{f}_2(x)$,

$$\widehat{l(\tilde{\varphi}_1)} = (\tilde{y}, u)_{H_0} + c, \tag{18}$$

and $\tilde{y} = C\tilde{\varphi}_1 + \tilde{\eta}$. *The quantity*

$$\sigma = [\sigma(\hat{u}, \hat{c})]^{1/2} \tag{19}$$

is called the error of the minimax estimation of $l(\varphi_1)$.

Thus, the minimax estimate is an estimate minimizing the maximal mean-square estimation error calculated for the "worst" implementation of perturbations.

3. Representation of minimax estimates and estimation errors

Introduce bilinear forms $a_i^*(\varphi_i, \psi_i)$ in $V_i \times V_i$, adjoint of $a_i(\varphi_i, \psi_i)$, by

$$a_i^*(\varphi_i, \psi_i) = a_i(\psi_i, \varphi_i) \quad \forall \varphi_i, \psi_i \in V_i \times V_i, \quad i = 1, 2. \tag{20}$$

Let $z_1(u) \in V_1$ and $z_2(u) \in V_2$ be a unique solution of the problem

$$a_1^*(z_1(u), \psi_1) = (l_0 - C^* J_{H_0} u, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1, \tag{21}$$

$$a_2^*(z_2(u), \psi_2) = (B^* z_1(u), \psi_2)_{H_2} \quad \forall \psi_2 \in V_2 \tag{22}$$

where $C^* : H_0' \to H_1$ is an operator adjoint of C defined by

$$(p, C^* g)_{H_1} =< Cp, g >_{H_0 \times H_0'} \quad \forall p \in H_1, g \in H_0'$$

and $B^* \in \mathcal{L}(H_1, H_2)$ is a linear operator adjoint of B.[2]

Then the following result holds.

Lemma 1. *The problem of minimax estimation of the functional $l(\varphi_1)$ (i.e. the determination of $\hat{u}$ and $\hat{c}$) is equivalent to the problem of optimal control of the system described by equations (21) – (22) with a cost*

[2] It is easy to see that, owing to our restrictions on the bilinear forms $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$, and on the operator B, problem (21)–(22) is well-posed and, in particular, the following inequality holds

$$\|\tilde{z}_1(u)\|_{V_1} + \|\tilde{z}_2(u)\|_{V_2} \le c_3 \|u\|_{H_0}, \tag{23}$$

where $(\tilde{z}_1(u), \tilde{z}_2(u))$ is a solution of problem (21)–(22) at $l_0 = 0$, $c_3 = \text{const} > 0$.

function

$$I(u) = (Q_1^{-1} z_1(u), z_1(u))_{H_1} + \\ + (Q_2^{-1} z_2(u), z_2(u))_{H_2} + (Q_0^{-1} u, u)_{H_0} \to \inf_{u \in H_0}. \quad (24)$$

Proof. From (71), we have

$$l(\tilde{\varphi}_1) - \widehat{l(\tilde{\varphi}_1)} = (l_0, \tilde{\varphi}_1)_{H_1} - (C\tilde{\varphi}_1 + \tilde{\eta}, u)_{H_0} - c$$
$$= (l_0, \tilde{\varphi}_1)_{H_1} - < C\tilde{\varphi}_1, J_{H_0} u >_{H_0 \times H_0'} - (\tilde{\eta}, u)_{H_0} - c$$
$$= (l_0 - C^* J_{H_0} u, \tilde{\varphi}_1)_{H_1} - (\tilde{\eta}, u)_{H_0} - c. \quad (25)$$

Transform the first term in the r.h.s. Make use of equalities (21)–(22) and (8)–(9) at $f_1 = \tilde{f}_1$ and $f_2 = \tilde{f}_2$ to obtain

$$(l_0 - C^* J_{H_0} u, \tilde{\varphi}_1)_{H_1} = a_1^*(z_1(u), \tilde{\varphi}_1) = a_1(\tilde{\varphi}_1, z_1(u)) = (B\tilde{\varphi}_2 + f_1, z_1(u))_{H_1}$$
$$(f_1, z_1(u))_{H_1} + (B\tilde{\varphi}_2, z_1(u))_{H_1} = (\tilde{f}_1, z_1(u))_{H_1} + (\tilde{f}_2, z_2(u))_{H_2}.$$
$$= (z_1(u), \tilde{f}_1 - f_1^0)_{H_1} + (z_2(u), \tilde{f}_2 - f_2^0)_{H_2} + (z_1(u), f_1^0)_{H_1} + (z_2(u), f_2^0)_{H_2}. \quad (26)$$

From (25)–(26) it follows that

$$l(\tilde{\varphi}_1) - \widehat{l(\tilde{\varphi}_1)} = (z_1(u), \tilde{f}_1 - f_1^0)_{H_1}$$
$$+ (z_2(u), \tilde{f}_2 - f_2^0)_{H_2} + (z_1(u), f_1^0)_{H_1} + (z_2(u), f_2^0)_{H_2} - (\tilde{\eta}, u)_{H_0} - c. \quad (27)$$

Taking into consideration the relationship $\mathbb{D}\xi = \mathbb{E}(\xi - \mathbb{E}\xi)^2 = \mathbb{E}\xi^2 - (\mathbb{E}\xi)^2$ that couples dispersion $\mathbb{D}\xi$ of the random variable ξ and its expectation $\mathbb{E}\xi$ and equality (5), we obtain from the last formulas

$$\mathbb{E}\left| l(\tilde{\varphi}_1) - \widehat{l(\tilde{\varphi}_1)} \right|^2 = \Big| (z_1(u), \tilde{f}_1 - f_1^0)_{H_1} + (z_2(u), \tilde{f}_2 - f_2^0)_{H_2}$$
$$+ (z_1(u), f_1^0)_{H_1} + (z_2(u), f_2^0)_{H_2} - (\tilde{\eta}, u)_H - c \Big|^2 + \mathbb{E}\left| (\tilde{\eta}, u)_{H_0} \right|^2. \quad (28)$$

Therefore,

$$\inf_{c \in \mathbb{R}} \sup_{\tilde{f} = (f_1, f_2) \in G_1, \tilde{\eta} \in G_0} \mathbb{E} | l(\tilde{\varphi}_1) - \widehat{l(\tilde{\varphi}_1)} |^2$$

$$= \inf_{c \in \mathbb{R}} \sup_{\tilde{f} = (f_1, f_2) \in G_1} \Big| (z_1(u), \tilde{f}_1 - f_1^0)_{H_1} + (z_2(u), \tilde{f}_2 - f_2^0)_{H_2}$$
$$+ (z_1(u), f_1^0)_{H_1} + (z_2(u), f_2^0)_{H_2} - (\tilde{\eta}, u)_{H_0} - c \Big|^2$$
$$+ \sup_{\tilde{\eta} \in G_0} \mathbb{E} \left| (\tilde{\eta}, u)_{H_0} \right|^2. \quad (29)$$

In order to calculate the first term on the right-hand side of (29) make use of the generalized Cauchy–Bunyakovsky inequality and (14). We have

$$\inf_{c\in\mathbb{R}} \sup_{\tilde f=(f_1,f_2)\in G_1} \Big|(z_1(u),\tilde f_1 - f_1^0)_{H_1} + (z_2(u),\tilde f_2 - f_2^0)_{H_2}$$

$$+ (z_1(u), f_1^0)_{H_1} + (z_2(u), f_2^0)_{H_2} - (\tilde\eta, u)_{H_0} - c\Big|^2$$

$$\le \inf_{c\in\mathbb{R}} \sup_{\tilde f=(f_1,f_2)\in G_1} \big[(Q_1^{-1}z_1(u), z_1(u))_{H_1} + (Q_2^{-1}z_2(u), z_2(u))_{H_2}\big]$$

$$\times\big[(Q_1(\tilde f_1 - f_1^0), \tilde f_1 - f_1^0)_{H_1} + (Q_2(\tilde f_2 - f_2^0), \tilde f_2 - f_2^0)_{H_2}\big]$$

$$\le (Q_1^{-1}z_1(u), z_1(u))_{H_1} + (Q_2^{-1}z_2(u), z_2(u))_{H_2}. \tag{30}$$

The direct substitution shows that that inequality (30) is transformed to an equality on the element $\tilde f^{(0)} = (f_1^{(0)}, f_2^{(0)})$, where

$$\tilde f_1^{(0)} := \frac{1}{d} Q_1^{-1} z_1(x;u) + f_1^0,$$

$$\tilde f_2^{(0)} := \frac{1}{d} Q_2^{-1} z_2(x;u) + f_2^0,$$

$$d = \big[(Q_1^{-1}z_1(u), z_1(u))_{H_1} + (Q_2^{-1}z_2(u), z_2(u))_{H_2}\big]^{1/2}$$

Therefore

$$\inf_{c\in\mathbb{R}} \sup_{\tilde f=(f_1,f_2)\in G_1} \Big|(z_1(u),\tilde f_1 - f_1^0)_{H_1} + (z_2(u),\tilde f_2 - f_2^0)_{H_2}$$

$$+ (z_1(u), f_1^0)_{H_1} + (z_2(u), f_2^0)_{H_2} - (\tilde\eta, u)_{H_0} - c\Big|^2$$

$$= (Q_1^{-1}z_1(u), z_1(u))_{H_1} + (Q_2^{-1}z_2(u), z_2(u))_{H_2} \tag{31}$$

with

$$c = (z_1(u), f_1^0)_{H_1} + (z_2(u), f_2^0)_{H_2}.$$

In order to calculate the second term on the right-hand side of (29), note that the Cauchy–Bunyakovsky inequality and (15) yield

$$\sup_{\tilde\eta\in G_0} \mathbb{E}\,|(\tilde\eta_1, u_1)_{H_0}|^2 \le (Q_0^{-1}u, u)_{H_0}. \tag{32}$$

It is easy to see that (32) becomes an equality at

$$\tilde\eta^{(0)} = \frac{\nu Q^{-1} u}{\{(Q_0^{-1}u, u)_{H_0}\}^{1/2}}$$

and ν is random variable with $\mathbb{E}\nu = 0$ and $\mathbb{E}|\nu|^2 = 1$. Therefore

$$\sup_{\tilde\eta\in G_0} \mathbb{E}\,|(\tilde\eta, u)_{H_0}|^2 = (Q_0^{-1}u, u)_{H_0}, \tag{33}$$

which proves the required assertion. The validity of Lemma 2.2 follows now from relationships (29), (31), and (33). □

Theorem 1. *The minimax estimate of the functional $l(\varphi_1)$ has the form*

$$\widehat{\widehat{l(\varphi_1)}} = (y, \hat{u})_{H_0} + \hat{c}, \tag{34}$$

where

$$\hat{c} = (\hat{z}_1, f_1^0)_{H_1} + (\hat{z}_2, f_2^0)_{H_2}, \quad \hat{u} = Q_0 C p_1, \tag{35}$$

and the elements $\hat{z}_1, p_1 \in V_1$ and $\hat{z}_2 \in V_2$ are determined from solution of the following uniquely solvable problem:

$$a_1^*(\hat{z}_1, \psi_1) = (l_0 - C^* J_{H_0} Q_0 C p_1, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1, \tag{36}$$

$$a_2^*(\hat{z}_2, \psi_2) = (B^* \hat{z}_1, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2 \tag{37}$$

$$a_1(p_1, \psi_1) = (Q_1^{-1} \hat{z}_1 + B p_2, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1, \tag{38}$$

$$a_2(p_2, \psi_2) = (Q_2^{-1} \hat{z}_2, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2. \tag{39}$$

Here $p_2 \in V_2$.

The error of estimation σ is given by an expression

$$\sigma = (l(p_1))^{1/2}. \tag{40}$$

Proof. Taking into account the inequalities (7) and (23), one can easily verify that $I(u)$ is a strictly convex lower semicontinuous functional on H_0. Also

$$I(u) \geq (Q_0^{-1} u, u)_{H_0} \geq c \|u\|_{H_0}^2 \quad \forall u \in H_0, \text{ c=const.} \tag{41}$$

Then, by Theorem 1.1 (see [2]), there exists one and only one element $\hat{u} \in H_0$ such that $I(\hat{u}) = \inf_{u \in H_0} I(u)$.

Therefore, for any $\tau \in \mathbb{R}$ and $v \in H_0$, the following relation is valid

$$\frac{d}{d\tau} I(\hat{u} + \tau v)\Big|_{\tau=0} = 0, \tag{42}$$

Since $z_1(\hat{u} + \tau v) = z_1(\hat{u}) + \tau \tilde{z}_1(v)$, $z_2(\hat{u} + \tau v) = z_2(\hat{u}) + \tau \tilde{z}_2(v)$, where $(\tilde{z}_1(v), \tilde{z}_2(v))$ is the unique solution to (21)–(22) at $u = v$ and $l = 0$, the relation (42) yields

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{d\tau} I(\hat{u} + \tau v)|_{\tau=0} \\ &= \lim_{\tau \to 0} \frac{1}{2\tau} \Big\{ \Big[(Q_1^{-1} z_1(\hat{u} + \tau v), z_1(\hat{u} + \tau v))_{H_1} - (Q_1^{-1} z_1(\hat{u}), z_1(\hat{u}))_{H_1} \Big] \\ &\quad \Big[(Q_2^{-1} z_2(\hat{u} + \tau v), z_2(\hat{u} + \tau v))_{H_2} - (Q_2^{-1} z_2(\hat{u}), z_2(\hat{u}))_{H_2} \Big] + \\ &\quad + \Big[(Q_0^{-1} (\hat{u} + \tau v), \hat{u} + \tau v)_{H_0} - (Q_0^{-1} \hat{u}, \hat{u})_{H_0} \Big] \Big\} \\ &= (Q_1^{-1} z_1(\hat{u}), \tilde{z}_1(v))_{H_1} + (Q_2^{-1} z_2(\hat{u}), \tilde{z}_2(v))_{H_2} + (Q_0^{-1} \hat{u}, v)_{H_0}. \end{aligned} \tag{43}$$

Introduce functions $p_1 \in V_1$ and $p_2 \in V_2$ as the unique solution of the problem

$$a_1(p_1, \psi_1) = (Q_1^{-1} z_1(\hat{u}) + B p_2, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1, \tag{44}$$

$$a_2(p_2,\psi_2) = (Q_2^{-1} z_2(\hat u), \psi_2)_{H_2} \quad \forall \psi_2 \in V_2. \tag{45}$$

Then from (43) we obtain

$$(Q_1^{-1} z_1(\hat u), \tilde z_1(v))_{H_1} + (Q_2^{-1} z_2(\hat u), \tilde z_2(v))_{H_2} + (Q_0^{-1}\hat u, v)_{H_0}$$
$$= a_1(p_1, \tilde z_1(v)) - (Bp_2, \tilde z_1(v)) + a_2(p_2, \tilde z_2(v)) + (Q_0^{-1}\hat u, v)_{H_0}$$
$$= a_1^*(\tilde z_1(v), p_1) - (Bp_2, \tilde z_1(v))_{H_1} + a_2^*(\tilde z_2(v), p_2) + (Q_0^{-1}\hat u, v)_{H_0}$$
$$= -(p_1, C^* J_H v)_{H_1} - (Bp_2, \tilde z_1(v))_{H_1} + (p_2, B^*\tilde z_1(v))_{H_2} + (Q_0^{-1}\hat u, v)_{H_0}$$
$$= -(Cp_1, v)_{H_1} + (Q_0^{-1}\hat u, v)_{H_0} = 0.$$

Hence $\hat u = Q_0 C p_1$.

Now let us establish the validity of formula (40). From (24) at $u = \hat u$ and (35), it follows

$$\sigma^2 = I(\hat u) = (Q_1^{-1}\hat z_1, \hat z_1)_{H_1} + (Q_2^{-1}\hat z_2, \hat z_2)_{H_2} + (Q_0^{-1}\hat u, \hat u)_{H_0}$$
$$= (Q_1^{-1}\hat z_1, \hat z_1)_{H_1} + (Q_2^{-1}\hat z_2, \hat z_2)_{H_2} + (Cp_1, Q_0 C p_1)_{H_0}.$$

Transform the sum of the first and the second terms in the r.h.s. of the last relation. Make use of equality (38) to obtain

$$(Q_1^{-1}\hat z_1, \hat z_1)_{H_1} + (Q_2^{-1}\hat z_2, \hat z_2)_{H_2} = (A_1 p_1 - Bp_2, \hat z_1)_{H_1} + (A_2 p_2, \hat z_2)_{H_2}$$
$$= a_1(p_1, \hat z_1) - (p_2, B^*\hat z_1)_{H_2} + a_2(p_2, \hat z_2)$$
$$= (p_1, l_0 - C^* J_{H_0} Q_0 C p_1)_{H_1} = (l_0, p_1)_{H_1} - (Cp_1, Q_0 C p_1)_{H_0}.$$

From the latter relations it follows that $\sigma^2 = l(p_1)$. □

Obtain now another representation for the minimax mean square estimate of quantity $l(\varphi_1)$ which is independent of l. To this end, introduce vector-functions $\hat p_1, \hat\varphi_1 \in V_1, \hat p_2, \hat\varphi_2 \in V_2$ as a solution to the problem

$$a_1^*(\hat p_1, \psi_1) = (C^* J_{H_0} Q_0 (y - C\hat\varphi_1), \psi_1)_{H_1} \quad \forall \psi_1 \in V_1, \tag{46}$$
$$a_2^*(\hat p_2, \psi_2) = (B^* \hat p_1, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2, \tag{47}$$
$$a_1(\hat\varphi_1, \psi_1) = (Q_1^{-1}\hat p_1 + B\hat\varphi_2 + f_1^0, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1, \tag{48}$$
$$a_2(\hat\varphi_2, \psi_2) = (Q_2^{-1}\hat p_2 + f_2^0, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2. \tag{49}$$

at realizations y that belong with probability 1 to space H.

Note that unique solvability of problem (46)–(49) at every realization can be proved similarly to the case of (36)–(39). Namely, setting $d = C^* J_{H_0} Q_0 y$, one can show that solutions to the problem of optimal control of the system

$$a_1^*(\hat p_1(v), \psi_1) = (d - C^* J_{H_0} v, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1$$
$$a_2^*(\hat p_2(v), \psi_2) = (B^* \hat p_1(v), \psi_2)_{H_2} \quad \forall \psi_2 \in V_2$$

with the cost function

$$\tilde I(v) = (Q_1^{-1}(\hat p_1(v) - Q_1 f_1^0), \hat p_1(v) - Q_1 f_1^0)_{H_1}$$
$$+(Q_2^{-1}(\hat p_2(v) - Q_2 f_2^0), \hat p_2(v) - Q_2 f_2^0)_{H_2} + (Q_0^{-1} v, v)_{H_0} \to \inf_{v\in H_0}$$

can be reduced to the solution of problem (46)–(49), where the optimal control $\hat{v}$ is expressed via solution to this problem as $v = Q_0 C\hat{\varphi}_1$; the unique solvability of the problem follows from the existence of the unique minimum point $\hat{v}$ of functional $\tilde{I}(v)$.

Theorem 2. *The minimax estimate* $\widehat{\widehat{l(\varphi_1)}}$ *of the functional* $l(\varphi_1)$ *has the form*

$$\widehat{\widehat{l(\varphi_1)}} = l(\hat{\varphi}_1), \tag{50}$$

where element $\hat{\varphi}_1 \in V_1$ *is determined from the solution to the problem (46)–(49). The random fields* $\hat{p}_1$, $\hat{\varphi}_1$ *and* $\hat{p}_2$, $\hat{\varphi}_2$, *whose realizations satisfy problem (46)–(49), belong to the space* $L^2(\Omega, V_1)$ *and* $L^2(\Omega, V_2)$, *respectively.*

Proof. By virtue of (35), (46)–(49), and (36)–(39),

$$\begin{gathered}
\widehat{\widehat{l(\varphi_1)}} = (y, \hat{u})_{H_0} + \hat{c} = (y, Q_0 C p_1)_{H_0} + \hat{c} = \\
(C^* J_{H_0} Q_0 y, p_1)_{H_1} + \hat{c} = a_1^*(\hat{p}_1, p_1) + C^* J_{H_0} Q_0 C\hat{\varphi}_1, p_1)_{H_1} + \hat{c} \\
= a_1(p_1, \hat{p}_1) + (C^* J_{H_0} Q_0 C\hat{\varphi}_1, p_1)_{H_1} + \hat{c} \\
= (Q_1^{-1}\hat{z}_1 + Bp_2, \hat{p}_1)_{H_1} + (C\hat{\varphi}_1, Q_0 C p_1)_{H_0} + \hat{c} \\
= (Q_1^{-1}\hat{p}_1, \hat{z}_1)_{H_1} + (B^*\hat{p}_1, p_2)_{H_2} + (C\hat{\varphi}_1, Q_0 C p_1)_{H_0} + \hat{c} \\
= a_1(\hat{\varphi}_1, \hat{z}_1) - (B\hat{\varphi}_2, \hat{z}_1)_{H_1} - (f_1^0, \hat{z}_1)_{H_1} + a_2^*(\hat{p}_2, p_2) + (C\hat{\varphi}_1, Q_0 C p_1)_{H_0} + \hat{c} \\
= a_1^*(\hat{z}_1, \hat{\varphi}_1) - (\hat{\varphi}_2, B^*\hat{z}_1)_{H_2} - (f_1^0, \hat{z}_1)_{H_1} + a_2(p_2, \hat{p}_2) + (C\hat{\varphi}_1, Q_0 C p_1)_{H_0} + \hat{c} \\
= (l_0 - C^* J_{H_0} Q_0 C p_1, \hat{\varphi}_1)_{H_1} - a_2^*(\hat{z}_2, \hat{\varphi}_2) \\
-(f_1^0, \hat{z}_1)_{H_1} + a_2(p_2, \hat{p}_2) + (C\hat{\varphi}_1, Q_0 C p_1)_{H_0} + \hat{c} \\
= (l_0, \hat{\varphi}_1)_{H_1} - a_2(\hat{\varphi}_2, \hat{z}_2) - (f_1^0, \hat{z}_1)_{H_1} + (Q_2^{-1}\hat{z}_2, \hat{p}_2)_{H_2} + \hat{c} \\
= (l_0, \hat{\varphi}_1)_{H_1} - (Q_2^{-1}\hat{p}_2, \hat{z}_2)_{H_2} - (f_2^0, \hat{z}_2)_{H_2} - (f_1^0, \hat{z}_1)_{H_1} + (Q_2^{-1}\hat{z}_2, \hat{p}_2)_{H_2} + \hat{c} \\
= (l_0, \hat{\varphi}_1)_{H_1} = l(\hat{\varphi}_1).
\end{gathered}$$

□

4. Numerical aspects

Using the Galerkin method for solving the aforementioned equations, we obtain approximate estimates via solutions of linear algebraic equations and show their convergence to the optimal estimates.

Introduce a sequence of finite-dimensional subspaces V^h in V, defined by an infinite set of parameters $h_1, h_2, \dots$ with $\lim_{k\to 0} h_k = 0$.

We say that sequence $\{V^h\}$ is complete in V, if for any $\varphi \in V$ and $\epsilon > 0$ there exists an $\hat{h} = \hat{h}(\varphi, \epsilon) > 0$ such that $\inf_{\psi \in V^h} \|\varphi - \psi\|_V < \varepsilon$ for any $h < \hat{h}$. In other words, the completeness of sequence $\{V^h\}$ means that any element $\varphi \in V$ may be approximated with any degree of accuracy by elements of $\{V^h\}$.

Take an approximate minimax estimate of $l(\varphi_1)$ as

$$\widehat{l^h(\varphi_1)} = (y, \hat{u}^h)_{H_0} + \hat{c}^h,$$

where

$$\hat{c} = (\hat{z}_1^h, f_1^0)_{H_1} + (\hat{z}_2^h, f_2^0)_{H_2}, \quad \hat{u}^h = Q_0 C p_1^h,$$

and elements $\hat{z}_1^h, p_1^h \in V_1^h$, $\hat{z}_2^h \in V_2^h$ are determined from the following uniquely solvable system of variational equalities

$$a_1^*(\hat{z}_1^h, \psi_1) = (l_0 - C^* J_{H_0} Q_0 C p_1^h, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1^h, \tag{51}$$

$$a_2^*(\hat{z}_2^h, \psi_2) = (B^* \hat{z}_1^h, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2^h \tag{52}$$

$$a_1(p_1^h, \psi_1) = (Q_1^{-1} \hat{z}_1^h + B p_2^h, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1^h, \tag{53}$$

$$a_2(p_2^h, \psi_2) = (Q_2^{-1} \hat{z}_2^h, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2^h, \tag{54}$$

where $p_2^h \in V_2^h$. Then the following results hold.

Theorem 3. *Approximate minimax estimate of* $\widehat{l^h(\varphi_1)}$ *of* $l(\varphi_1)$ *tends to a minimax estimate* $\widehat{\widehat{l(\varphi_1)}}$ *of this expression as* $h \to 0$ *in the sense that*

$$\lim_{h \to 0} \mathbb{E} |\widehat{l^h(\varphi_1)} - \widehat{\widehat{l(\varphi_1)}}|^2 = 0, \tag{55}$$

and

$$\lim_{h \to 0} \mathbb{E} |\widehat{l^h(\varphi_1)} - l(\varphi_1)|^2 = \mathbb{E} |\widehat{\widehat{l(\varphi_1)}} - l(\varphi_1)|^2. \tag{56}$$

Let us formulate a similar result in the case when an estimate of the state φ_1 is directly determined from the solution to problem (46)–(49).

Theorem 4. *Let* $\hat{\varphi}_1^h \in V_1^h$ *and* $\hat{\varphi}_2^h \in V_2^h$ *be an approximate estimates of the functions* $\hat{\varphi}_1$ *and* $\hat{\varphi}_2$*, respectively, which are determined from the solution to the variational problem*

$$a_1^*(\hat{p}_1^h, \psi_1) = (C^* J_{H_0} Q_0 (y - C \hat{\varphi}_1^h), \psi_1)_{H_1} \quad \forall \psi_1 \in V_1^h, \tag{57}$$

$$a_2^*(\hat{p}_2^h, \psi_2) = (B^* \hat{p}_1^h, \psi_2)_{H_2} \quad \forall \psi_2 \in V_2^h, \tag{58}$$

$$a_1(\hat{\varphi}_1^h, \psi_1) = (Q_1^{-1} \hat{p}_1^h + B \hat{\varphi}_2^h + f_1^0, \psi_1)_{H_1} \quad \forall \psi_1 \in V_1^h, \tag{59}$$

$$a_2(\hat{\varphi}_2^h, \psi_2) = (Q_2^{-1} \hat{p}_2^h + f_2^0, \psi_2^h)_{H_2} \quad \forall \psi_2 \in V_2^h, \tag{60}$$

where $\hat{p}_1^h \in V_1^h$ *and* $\hat{p}_2^h \in V_2^h$.

Then

$$\|\hat{\varphi}_1 - \hat{\varphi}_1^h\|_{V_1} + \|\hat{\varphi}_2 - \hat{\varphi}_2^h\|_{V_2} \to 0 \quad \text{as} \quad h \to 0.$$

The proofs of Theorem 3 and Theorem 4 are similary to the proof of Theorem 1.3 from [4].

The problem of finding $\hat{z}_1^h, p_1^h \in V_1^h$ and $\hat{z}_2^h, p_2^h \in V_2^h$ from (51)–(54) is equivalent to determination of coefficients in their expansions by basis elements of the spaces V_1^h and V_2^h from the corresponding system of linear algebraic equations.

Introducing the basises in the spaces V_1^h and V_2^h, problem (51)–(54) can be rewritten as a system of liner algebraic equations. To do this, let us denote the elements of the basis in V_1^h by ξ_i^1 $(i_1 = 1, \ldots, N_1)$ and in V_2^h by ξ_i^2 $(i_2 = 1, \ldots, N_2)$, where $N_1 = \dim V_1^h$, $N_2 = \dim V_2^h$. The fact that $\hat{z}_1^h, p_1^h$, and $\hat{z}_2^h, p_2^h$, belong to the spaces V_1^h and V_2^h, respectively, means the existence of constants $\hat{z}_{i_1}^1, p_{i_1}^1$ and $\hat{z}_{i_2}^2, p_{i_2}^2$ such that

$$\hat{z}_1^h = \sum_{i_1=1}^{N_1} \hat{z}_{i_1}^1 \xi_{i_1}^1, \;\; p_1^h = \sum_{i_1=1}^{N_1} p_{i_1}^1 \xi_{i_1}^1, \;\; \hat{z}_2^h = \sum_{i_2=1}^{N_2} \hat{z}_{i_2}^2 \xi_{i_2}^2, \;\; p_2^h = \sum_{i_2=1}^{N_2} p_{i_2}^2 \xi_{i_2}^2. \quad (61)$$

Setting in (51)–(54) $\psi_1 = \xi_{j_1}^1$ $(j_1 = 1, \ldots, N_1)$ and $\psi_2 = \xi_{j_2}^1$ $(j_2 = 1, \ldots, N_2)$, we obtain that finding $\hat{z}_1^h, p_1^h, \hat{z}_2^h, p_2^h$ is equivalent to solving the following system of linear algebraic equations with respect to coefficients $\hat{z}_{i_1}^1$, $p_{i_1}^1$, $\hat{z}_{i_2}^2$, $p_{i_2}^2$ $(i_1 = 1, \ldots, N_1, i_2 = 1, \ldots, N_2)$ of expansions (61):

$$\sum_{i_1=1}^{N_1} a_{j_1 i_1}^1 \hat{z}_{i_1}^1 + \sum_{i_1=1}^{N_1} b_{i_1 j_1} p_{i_1}^1 = g_{j_1}, \quad j_1 = 1, \ldots, N_1,$$

$$\sum_{i_2=1}^{N_2} a_{j_2 i_2}^2 \hat{z}_{i_2}^2 + \sum_{i_1=1}^{N_1} c_{i_1 j_1} \hat{z}_{i_1}^1 = 0, \quad j_2 = 1, \ldots, N_2,$$

$$\sum_{i_1=1}^{N_1} a_{i_1 j_1}^1 p_{i_1}^1 + \sum_{i_1=1}^{N_1} d_{i_1 j_1} \hat{z}_{i_1}^2 + \sum_{i_2=1}^{N_2} e_{i_2 j_1} p_{i_2}^2 = 0, \quad j_1 = 1, \ldots, N_1,$$

$$\sum_{i_2=1}^{N_2} a_{i_2 j_2}^2 p_{i_2}^2 + \sum_{i_2=1}^{N_2} h_{i_2 j_1} \hat{z}_{i_2}^2 = 0, \quad j_2 = 1, \ldots, N_2,$$

where

$$a_{i_1 j_1}^1 = a_1(\xi_{i_1}^1, \xi_{j_1}^1), \quad i_1, j_1 = 1, \ldots, N_1,$$
$$a_{i_2 j_2}^2 = a_2(\xi_{i_2}^2, \xi_{j_2}^2), \quad i_2, j_2 = 1, \ldots, N_2,$$
$$b_{i_1 j_1} = (C^* J_{H_0} Q_0 C \xi_{i_1}^1, \xi_{j_1}^1)_{H_1}, \quad i_1, j_1 = 1, \ldots, N_1,$$
$$c_{i_1 j_2} = -(B^* \xi_{i_1}^1, \xi_{j_2}^2)_{H_2}, \quad i_1 = 1, \ldots, N_1, \quad j_2 = 1, \ldots, N_2,$$
$$d_{i_1 j_1} = -(Q_1^{-1} \xi_{i_1}^1, \xi_{j_1}^1)_{H_1}, \quad i_1, j_1 = 1, \ldots, N_1,$$
$$e_{i_2 j_1} = -(B \xi_{i_2}^2, \xi_{j_1}^1)_{H_1}, \quad i_2 = 1, \ldots, N_2, \quad j_1 = 1, \ldots, N_1,$$
$$h_{i_2 j_2} = -(Q_2^{-1} \xi_{i_2}^2, \xi_{j_2}^2)_{H_2}, \quad i_2, j_2 = 1, \ldots, N_2,$$

and

$$g_{j_1} = (l_0, \xi^1_{j_1})_{H_1}, \quad j_1 = 1, \dots, N_1.$$

Analogously, representing the elements $\hat{p}^h_1, \hat{\varphi}^h_1, \hat{p}^h_2, \hat{\varphi}^h_2$ as

$$\hat{p}^h_1 = \sum_{i_1=1}^{N_1} \hat{p}^1_{i_1}\xi^1_{i_1},\ \hat{\varphi}^h_1 = \sum_{i_1=1}^{N_1} \hat{\varphi}^1_{i_1}\xi^1_{i_1},\ \hat{p}^h_2 = \sum_{i_2=1}^{N_2} \hat{p}^2_{i_2}\xi^2_{i_2},\ \hat{\varphi}^h_2 = \sum_{i_2=1}^{N_2} \hat{\varphi}^2_{i_2}\xi^2_{i_2}, \quad (62)$$

we rewrite problem (57)–(60) as the following system of liner algebraic equations with respect to the coefficients $\hat{p}^1_{i_1}, \hat{\varphi}^1_{i_1}, \hat{p}^2_{i_2}, \hat{\varphi}^2_{i_2}$ of the expansions (62):

$$\sum_{i_1=1}^{N_1} a^1_{j_1 i_1}\hat{p}^1_{i_1} + \sum_{i_1=1}^{N_1} b_{i_1 j_1}\hat{\varphi}^1_{i_1} = \tilde{g}_{j_1}, \quad j_1 = 1, \dots, N_1,$$

$$\sum_{i_2=1}^{N_2} a^2_{j_2 i_2}\hat{p}^2_{i_2} + \sum_{i_1=1}^{N_1} c_{i_1 j_1}\hat{p}^1_{i_1} = 0, \quad j_2 = 1, \dots, N_2,$$

$$\sum_{i_1=1}^{N_1} a^1_{i_1 j_1}\hat{\varphi}^1_{i_1} + \sum_{i_1=1}^{N_1} d_{i_1 j_1}\hat{p}^2_{i_1} + \sum_{i_2=1}^{N_2} e_{i_2 j_1}\hat{\varphi}^2_{i_2} = q^1_{j_1}, \quad j_1 = 1, \dots, N_1,$$

$$\sum_{i_2=1}^{N_2} a^2_{i_2 j_2}\hat{\varphi}^2_{i_2} + \sum_{i_2=1}^{N_2} h_{i_2 j_1}\hat{p}^2_{i_2} = q^2_{j_2}, \quad j_2 = 1, \dots, N_2,$$

where

$$q^1_{j_1} = (f^0_1, \xi^1_{j_1})_{H_1}, \quad j_1 = 1, \dots, N_1, \quad q^1_{j_2} = (f^0_2, \xi^2_{j_2})_{H_2}, \quad j_2 = 1, \dots, N_2,$$

$$\tilde{g}_{j_1} = (C^* J_{H_0} Q_0 y, \xi^1_{j_1})_{H_1}, \quad j_1 = 1, \dots, N_1.$$

5. Corollary from the obtained results

Setting in (8)–(9) $H_1 = L^2(D)$, $V_1 = H^1_0(D)$, $H_2 = L^2(D)^n$, $V_2 = V$, $\varphi_1 = T, \psi_1 = \psi \in H^1_0(D)$, $\varphi_2 = \mathbf{v} = (v_1, \dots, v_n)$, $\psi_2 = \mathbf{u} = (u_1, \dots, u_n) \in V$, $f_1 = f \in L^2(D)$, $f_2 = \mathbf{f} \in L^2(D)^n$,

$$a_1(\varphi_1, \psi_1) = a_1(T, \psi)$$

$$= \int_D \left(\sum_{i=1}^n \frac{\partial T}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \sum_{i=1}^n \bar{v}_i(x)\frac{\partial T}{\partial x_i}\psi + \rho(x)T\psi \right) dx \quad \forall \psi \in H^1_0(D),$$

$$a_2(\varphi_2, \psi_2) = a_2(\mathbf{v}, \mathbf{u}) = \nu((\mathbf{v}, \mathbf{u})) \quad \forall \mathbf{u} \in V,$$

$$B : L^2(D)^n \to L^2(D), \quad B\varphi_2 = B\mathbf{v} = \sum_{i=1}^n v_i(x) g_i(x),$$

where $\rho(x)$, $g_1(x), \ldots, g_n(x)$, and $\bar{\mathbf{v}} = (\bar{v}_1(x), \ldots, \bar{v}_n(x))$ are given bounded measurable functions in D, $\nu = \text{const} > 0$,[3] we arrive at the estimation problem of the form: given

$$(f, \mathbf{f}) \in G_1, \quad y = CT + \eta, \tag{64}$$

where observation error η in (64) is a realization of stochastic prosess belonging to G_0, find the minimax estimate of the quantity

$$l(T) = \int_D l_0(x) T(x)\, dx, \tag{65}$$

where function $T \in H_0^1(D)$ satisfies to the following uniquely solvable variational problem:

$$\int_D \left(\sum_{i=1}^n \frac{\partial T}{\partial x_i} \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^n \bar{v}_i(x) \frac{\partial T}{\partial x_i} \psi + \rho(x) T \psi \right) dx$$
$$= \int_D \left(f(x) + \sum_{i=1}^n v_i(x) g_i(x) \right) \psi(x)\, dx \quad \forall \psi \in H_0^1(D), \tag{66}$$

$$\nu((\mathbf{v}, \mathbf{u})) = \big(\mathbf{f}, \mathbf{u}\big)_{L^2(D)^n} \quad \forall \mathbf{u} \in V. \tag{67}$$

Here $l_0 \in L^2(D)$ is a prescribed function, $\mathbf{v} \in V$,

$$G_1 := \{(\tilde{f}, \tilde{\mathbf{f}}) : \tilde{f} \in L^2(D), \tilde{\mathbf{f}} \in L^2(D)^n,$$
$$(Q_1(\tilde{f} - f_0), \tilde{f} - f_0)_{L^2(D)} + (Q_2(\tilde{\mathbf{f}} - \mathbf{f}_0), \tilde{\mathbf{f}} - \mathbf{f}_0)_{L^2(D)^n} \le 1\} \tag{68}$$

$f_0 \in L^2(D)$ and $\mathbf{f}_0 = (f_{01}, \ldots, f_{0n}) \in L^2(D)^n$ are given functions, Q_1 and Q_2 are bounded selfadjoint positive definite operators in $L^2(D)^n$ and $L^2(D)$, respectively, for which there exist bounded inverse operators Q_1^{-1} and Q_2^{-1}.

Definition 1 of the the minimax estimate transforms in the considered case as follows:

[3] the bilinear form $a_1(T, \psi)$ is supposed to be such that homogeneous variational problem

$$\int_D \left(\sum_{i=1}^n \frac{\partial T}{\partial x_i} \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^n \bar{v}_i(x) \frac{\partial T}{\partial x_i} \psi + \rho(x) T \psi \right) dx = 0 \quad \forall \psi \in H_0^1(D)$$

has only the trivial solution $T = 0$. From this suggestion it follows (see [1]) that nonhomogeneous variational problem

$$\int_D \left(\sum_{i=1}^n \frac{\partial T}{\partial x_i} \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^n \bar{v}_i(x) \frac{\partial T}{\partial x_i} \psi + \rho(x) T \psi \right) dx = \int_D g \psi\, dx \quad \forall \psi \in H_0^1(D) \tag{63}$$

has a unique solution $T \in H_0^1(D)$ for any function $g \in L^2(D)$. Sufficient conditions for uniqueness, and hence solvability, of the problem (63) are given by Theorem 2.1 in [3].

An estimate

$$\widehat{\widehat{l(T)}} = (y, \hat{u})_{H_0} + \hat{c} \tag{69}$$

is called a minimax estimate of the $l(T)$, *if elements* $\hat{u} \in H_0$ *and a number* $\hat{c}$ *are determined from the condition*

$$\inf_{u \in H_0, c \in \mathbb{R}} \sigma(u, c), \tag{70}$$

where

$$\sigma(u, c) := \sup_{\tilde{f}=(f,\mathbf{f}) \in G_1, \tilde{\eta}, \in G_0} \mathbb{E}[l(\tilde{T}) - \widehat{l(\tilde{T})}]^2,$$

$\tilde{T}$ *is a solution to problem* (3)–(67) *when* $f(x) = \tilde{f}(x)$, $\mathbf{f}(x) = \tilde{\mathbf{f}}(x)$,

$$\widehat{l(\tilde{T})} = (\tilde{y}, u)_{H_0} + c, \tag{71}$$

and

$$\tilde{y} = C\tilde{T} + \tilde{\eta}.$$

The quantity

$$\sigma = [\sigma(\hat{u}, \hat{c})]^{1/2} \tag{72}$$

is called the error of the minimax estimation of $l(T)$.

Reasoning as in the proof of Lemma 2.1 from [1], we can prove that problem (66)–(67) is equivalent to the following system of heat and Stokes equations: [4]

$$-\Delta T + \sum_{i=1}^{n} \bar{v}_i(x) \frac{\partial T}{\partial x_i} + \rho(x) T = (\mathbf{v}(x), \mathbf{g}(x))_{\mathbb{R}^n} + f(x) \quad \text{в} \quad D, \tag{73}$$

$$T = 0 \quad \text{on} \quad \Gamma, \tag{74}$$

$$-\nu \boldsymbol{\Delta} \mathbf{v} + \mathbf{grad}\, p = \mathbf{f} \quad \text{in} \quad D, \tag{75}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad D, \tag{76}$$

$$\mathbf{v} = 0 \quad \text{on} \quad \Gamma \tag{77}$$

in the sense that if T and $\mathbf{v}$ satisfy (3)–(67) then they satisfy (73)–(77) and vice versa. Here by a solution of the system of equations (73)–(77) we mean a pair of functions $(T, \mathbf{v}) \in H_0^1(D) \times H_0^1(D)^n$ satisfying these equations in the following weak sense: there exists a unique (up to a constant) function $p \in L^2(D)$ such that $-\nu \boldsymbol{\Delta} \mathbf{v} + \mathbf{grad}\, p = \mathbf{f}$ in the distributive sense in D; functions T and $\mathbf{v}$ satisfy (73) and (76) in the distributive sense in D; boundary conditions (74) and (77) are understood as $\gamma_0 T = 0$ and $\gamma_0 \mathbf{v} = 0$.

[4]From physical point of view scalar functions T, p and vector-function $\mathbf{v}$ represent the temperature, the pressure, and the velosity of fluid, respectively, which are defined in the domain D and the positive constant ν is the coefficient of kinematic viscosity.

Taking into account the notations mentioned above, one can easely verify that the following assertions are direct consequences of the previous theorems.

Theorem 5. *The minimax estimate of the functional $l(T)$ has the form*

$$\widehat{\widehat{l(T)}} = (y, \hat{u})_{H_0} + \hat{c}, \tag{78}$$

where

$$\hat{c} = \int_D \hat{z}_1(x) f_0(x)\, dx + \int_D (\hat{\mathbf{z}}_2(x), \mathbf{f}_0(x))_{\mathbb{R}^n}\, dx, \quad \hat{u} = Q_0 C p_1, \tag{79}$$

and the functions $\hat{z}_1$, $p_1 \in H_0^1(D)$, and $\hat{\mathbf{z}}_2 = (\hat{z}_1^{(2)}, \dots, \hat{z}_n^{(2)}) \in V$ are determined from solution of the following uniquely solvable variational problem:

$$\int_D \left(\sum_{i=1}^n \frac{\partial \hat{z}_1}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \hat{z}_1 \frac{\partial \psi_1}{\partial x_i} + \rho \hat{z}_1 \psi_1 \right) dx$$
$$= \int_D (l_0 - C^* J_{H_0} Q_0 C p_1) \psi_1\, dx \quad \forall \psi_1 \in H_0^1(D), \tag{80}$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial \hat{z}_1^{(2)}}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial \hat{z}_2^{(2)}}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \dots + \frac{\partial \hat{z}_n^{(2)}}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx$$
$$= \sum_{i=1}^n \int_D \hat{z}_1 g_i \psi_i^{(2)}\, dx \quad \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \dots, \psi_n^{(2)}) \in V, \tag{81}$$

$$\int_D \left(\sum_{i=1}^n \frac{\partial p_1}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \frac{\partial p_1}{\partial x_i} \psi_1 + \rho p_1 \psi_1 \right) dx$$
$$= \int_D \left(Q_1^{-1} \hat{z}_1 + \sum_{i=1}^n p_i^{(2)} g_i \right) \psi_1\, dx \quad \forall \psi_1 \in H_0^1(D), \tag{82}$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial p_1^{(2)}}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial p_2^{(2)}}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \dots + \frac{\partial p_n^{(2)}}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx$$
$$= \sum_{i=1}^n \int_D (Q_2^{-1} \hat{\mathbf{z}}_2)_i \psi_i^{(2)}\, dx \quad \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \dots, \psi_n^{(2)}) \in V. \tag{83}$$

Here $\mathbf{p}_2 = (p_1^{(2)}, \dots, p_n^{(2)}) \in V$.

Theorem 6. *The minimax estimate* $\widehat{\widehat{l(T)}}$ *of* $l(T)$ *has the form*

$$\widehat{\widehat{l(T)}} = l(\hat{T}), \tag{84}$$

where function $\hat{T} \in H_0^1(D)$ *is determined from the solution to the problem*

$$\int_D \left(\sum_{i=1}^n \frac{\partial \hat{p}_1}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \hat{p}_1 \frac{\partial \psi_1}{\partial x_i} + \rho \hat{p}_1 \psi_1 \right) dx \\ = \int_D C^* J_H Q(y - C\hat{T}) \psi_1 \, dx \quad \forall \psi_1 \in H_0^1(D), \tag{85}$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial \hat{p}_1^{(2)}}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial \hat{p}_2^{(2)}}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \cdots + \frac{\partial \hat{p}_n^{(2)}}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx \\ = \sum_{i=1}^n \int_D \hat{p}_1 g_i \psi_i^{(2)} \, dx \quad \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \dots, \psi_n^{(2)}) \in V, \tag{86}$$

$$\int_D \left(\sum_{i=1}^n \frac{\partial \hat{T}}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \frac{\partial \hat{T}}{\partial x_i} \psi_1 + \rho \hat{T} \psi_1 \right) dx \\ = \int_D \left(Q_1^{-1} \hat{p}_1 + \sum_{i=1}^n \hat{v}_i^{(2)} g_i + f_0 \right) \psi_1 \, dx \quad \forall \psi_1 \in H_0^1(D), \tag{87}$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial \hat{v}_1}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial \hat{v}_2}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \cdots + \frac{\partial \hat{v}_n}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx \\ = \sum_{i=1}^n \int_D \left((Q_2^{-1} \hat{\mathbf{p}}_2)_i + f_{0i} \right) \psi_i^{(2)} \, dx \quad \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \dots, \psi_n^{(2)}) \in V. \tag{88}$$

Here $\hat{p}_1 \in H_0^1(D)$ $\hat{\mathbf{p}}_2 = (\hat{p}_1^{(2)}, \dots, \hat{p}_n^{(2)}) \in V$, $\hat{\mathbf{v}} = (\hat{v}_1, \dots, \hat{v}_n) \in V$.

If we put approximate minimax estimate of $l(T)$ as

$$\widehat{l^h(T)} = (y, \hat{u}^h)_{H_0} + \hat{c}^h,$$

where

$$\hat{c} = \int_D \hat{z}_1^h f_0 \, dx + \sum_{i=1}^n \int_D \hat{z}_{2,i}^h f_{0i} \, dx, \quad \hat{u}^h = QCp_1^h,$$

and elements $\hat{z}_1^h, p_1^h \in \{H_0^1(D)\}^h$, $\hat{\mathbf{z}}_2^h = \left(\hat{z}_{2,1}^h, \ldots, \hat{z}_{2,n}^h\right) \in V^h$ are determined from the following uniquely solvable system of variational equalities:

$$\int_D \left(\sum_{i=1}^n \frac{\partial \hat{z}_1^h}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \hat{z}_1^h \frac{\partial \psi_1}{\partial x_i} + \rho \hat{z}_1^h \psi_1 \right) dx$$
$$= \int_D (l_0 - C^* J_H Q C p_1^h) \psi_1 \, dx \quad \forall \psi_1 \in \{H_0^1(D)\}^h, \quad (89)$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial \hat{z}_{2,1}^h}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial \hat{z}_{2,2}^h}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \cdots + \frac{\partial \hat{z}_{2,n}^h}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx$$
$$= \sum_{i=1}^n \int_D \hat{z}_1^h g_i \psi_i^{(2)} \, dx \quad \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \ldots, \psi_n^{(2)}) \in V^h, \quad (90)$$

$$\int_D \left(\sum_{i=1}^n \frac{\partial p_1^h}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \frac{\partial p_1^h}{\partial x_i} \psi_1 + \rho p_1^h \psi_1 \right) dx$$
$$= \int_D \left(Q_1^{-1} \hat{z}_1^h + \sum_{i=1}^n p_i^{(2)h} g_i \right) \psi_1 \, dx \quad \forall \psi_1 \in \{H_0^1(D)\}^h, \quad (91)$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial p_{2,1}^h}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial p_{2,2}^h}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \cdots + \frac{\partial p_{2,n}^h}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx$$
$$= \sum_{i=1}^n \int_D (Q_2^{-1} \hat{z}_2^h)_i \psi_i^{(2)} \, dx \quad \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \ldots, \psi_n^{(2)}) \in V^h. \quad (92)$$

where $\mathbf{p}_2^h = \left(p_{2,1}^h, \ldots, p_{2,n}^h\right) \in V^h$ then the following result is valid.

Theorem 7. *Approximate minimax estimate of $\widehat{l^h(T)}$ of $l(T)$ tends to a minimax estimate $\widehat{\widehat{l(T)}}$ of this expression as $h \to 0$ in the sense that*

$$\lim_{h \to 0} \mathbb{E} |\widehat{l^h(T)} - \widehat{\widehat{l(T)}}|^2 = 0, \quad (93)$$

and

$$\lim_{h \to 0} \mathbb{E} |\widehat{l^h(T)} - l(T)|^2 = \mathbb{E} |\widehat{\widehat{l(T)}} - l(T)|^2. \quad (94)$$

Let us formulate a similar result in the case when an estimate of the state $(T, \mathbf{v})$ is directly determined from the solution to problem (85)–(88).

Theorem 8. *Let* $\hat{T}^h \in \{H_0^1(D)\}^h$ *and* $\hat{\mathbf{v}}^h = (\hat{v}_1^h, \ldots, \hat{v}_n^h) \in V^h$ *be an approximate estimates of the functions* $\hat{T} \in H_0^1(D)$ *and* $\hat{\mathbf{v}} \in V$ *determined from the solution to the variational problem*

$$\int_D \left(\sum_{i=1}^n \frac{\partial \hat{p}_1^h}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \hat{p}_1^h \frac{\partial \psi_1}{\partial x_i} + \rho \hat{p}_1^h \psi_1 \right) dx$$
$$= \int_D C^* J_H Q(y - C\hat{T}^h)\psi_1 \, dx \quad \forall \psi_1 \in \{H_0^1(D)\}^h, \quad (95)$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial \hat{p}_{2,1}^h}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial \hat{p}_{2,2}^h}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \cdots + \frac{\partial \hat{p}_{2,n}^h}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx$$
$$= \sum_{i=1}^n \int_D \hat{p}_1^h g_i \psi_i^{(2)} \, dx \quad \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \ldots, \psi_n^{(2)}) \in V^h, \quad (96)$$

$$\int_D \left(\sum_{i=1}^n \frac{\partial \hat{T}^h}{\partial x_i} \frac{\partial \psi_1}{\partial x_i} + \sum_{i=1}^n \bar{v}_i \frac{\partial \hat{T}^h}{\partial x_i} \psi_1 + \rho \hat{T}^h \psi_1 \right) dx$$
$$= \int_D \left(Q_1^{-1} \hat{p}_1^h + \sum_{i=1}^n \hat{v}_i^h g_i + f_0 \right) \psi_1 \, dx \quad \forall \psi_1 \in \{H_0^1(D)\}^h, \quad (97)$$

$$\nu \sum_{i=1}^n \int_D \left(\frac{\partial \hat{v}_1^h}{\partial x_i} \frac{\partial \psi_1^{(2)}}{\partial x_i} + \frac{\partial \hat{v}_2^h}{\partial x_i} \frac{\partial \psi_2^{(2)}}{\partial x_i} + \cdots + \frac{\partial \hat{v}_n^h}{\partial x_i} \frac{\partial \psi_n^{(2)}}{\partial x_i} \right) dx$$
$$= \sum_{i=1}^n \int_D \left((Q_2^{-1} \hat{\mathbf{p}}_2^h)_i + f_{0,i} \right) \psi_i^{(2)} \, dx \ \ \forall \psi_2 = (\psi_1^{(2)}, \psi_2^{(2)}, \ldots, \psi_n^{(2)}) \in V^h. \quad (98)$$

where $\hat{p}_1^h \in \{H_0^1(D)\}^h$ *and* $\hat{\mathbf{p}}_2^h = (\hat{p}_{2,1}^h, \ldots, \hat{p}_{2,n}^h)$*. Then*

$$\|\hat{T} - \hat{T}^h\|_{H_0^1(D)} + \|\hat{\mathbf{v}} - \hat{\mathbf{v}}^h\|_V \to 0 \quad \text{as} \quad h \to 0.$$

Denoting the elements of the basis in $\{H_0^1(D)\}^h$ by and V^h by $\xi_{i_1}^1$ $(i_1 = 1, \ldots, N_1)$ and in V^h by $\boldsymbol{\xi}_{i_2}^2 = (\xi_{i_2,1}^2, \ldots, \xi_{i_2,n}^2)$ $(i_2 = 1, \ldots, N_2)$, $N_1 = \dim \{H_0^1(D)\}^h$, $N_2 = \dim V^h$, respectively, we obtain the following expansions

$$\hat{z}_1^h = \sum_{i_1=1}^{N_1} \hat{z}_{i_1}^1 \xi_{i_1}^1, \quad p_1^h = \sum_{i_1=1}^{N_1} p_{i_1}^1 \xi_{i_1}^1, \quad \hat{\mathbf{z}}_2^h = \sum_{i_2=1}^{N_2} \hat{z}_{i_2}^2 \boldsymbol{\xi}_{i_2}^2, \quad \mathbf{p}_2^h = \sum_{i_2=1}^{N_2} p_{i_2}^2 \boldsymbol{\xi}_{i_2}^2.$$

and

$$\hat{p}_1^h = \sum_{i_1=1}^{N_1} \hat{p}_{i_1}^1 \xi_{i_1}^1, \quad \hat{T}^h = \sum_{i_1=1}^{N_1} \hat{T}_{i_1}^1 \xi_{i_1}^1, \quad \hat{\mathbf{p}}_2^h = \sum_{i_2=1}^{N_2} \hat{p}_{i_2}^2 \boldsymbol{\xi}_{i_2}^2, \quad \hat{\mathbf{v}}^h = \sum_{i_2=1}^{N_2} \hat{v}_{i_2} \boldsymbol{\xi}_{i_2}^2.$$

for solutions of problems (89)–(92) and (95)–(98) in which constants $\hat{z}^1_{i_1}$, $p^1_{i_1}$, $\hat{z}^2_{i_2}$, $p^2_{i_2}$ and $\hat{p}^1_{i_1}$, $\hat{T}^1_{i_1}$, $\hat{p}^2_{i_2}$, $\hat{v}_{i_2}$ are found from systems of linear algebraic equations

$$\sum_{i_1=1}^{N_1} a^1_{j_1 i_1}\hat{z}^1_{i_1} + \sum_{i_1=1}^{N_1} b_{i_1 j_1} p^1_{i_1} = g_{j_1}, \quad j_1 = 1,\dots,N_1,$$

$$\sum_{i_2=1}^{N_2} a^2_{j_2 i_2}\hat{z}^2_{i_2} + \sum_{i_1=1}^{N_1} c_{i_1 j_1}\hat{z}^1_{i_1} = 0, \quad j_2 = 1,\dots,N_2,$$

$$\sum_{i_1=1}^{N_1} a^1_{i_1 j_1} p^1_{i_1} + \sum_{i_1=1}^{N_1} d_{i_1 j_1}\hat{z}^2_{i_1} + \sum_{i_2=1}^{N_2} e_{i_2 j_1} p^2_{i_2} = 0, \quad j_1 = 1,\dots,N_1,$$

$$\sum_{i_2=1}^{N_2} a^2_{i_2 j_2} p^2_{i_2} + \sum_{i_2=1}^{N_2} h_{i_2 j_1}\hat{z}^2_{i_2} = 0, \quad j_2 = 1,\dots,N_2,$$

and

$$\sum_{i_1=1}^{N_1} a^1_{j_1 i_1}\hat{p}^1_{i_1} + \sum_{i_1=1}^{N_1} b_{i_1 j_1}\hat{T}^1_{i_1} = \tilde{g}_{j_1}, \quad j_1 = 1,\dots,N_1,$$

$$\sum_{i_2=1}^{N_2} a^2_{j_2 i_2}\hat{p}^2_{i_2} + \sum_{i_1=1}^{N_1} c_{i_1 j_1}\hat{p}^1_{i_1} = 0, \quad j_2 = 1,\dots,N_2,$$

$$\sum_{i_1=1}^{N_1} a^1_{i_1 j_1}\hat{T}^1_{i_1} + \sum_{i_1=1}^{N_1} d_{i_1 j_1}\hat{p}^2_{i_1} + \sum_{i_2=1}^{N_2} e_{i_2 j_1}\hat{v}^2_{i_2} = q^1_{j_1}, \quad j_1 = 1,\dots,N_1,$$

$$\sum_{i_2=1}^{N_2} a^2_{i_2 j_2}\hat{v}^2_{i_2} + \sum_{i_2=1}^{N_2} h_{i_2 j_1}\hat{p}^2_{i_2} = q^2_{j_2}, \quad j_2 = 1,\dots,N_2,$$

respectively, where

$$a^1_{i_1 j_1} = \int_D \left(\sum_{i=1}^{n} \frac{\partial \xi^1_{i_1}}{\partial x_i}\frac{\partial \xi^1_{j_1}}{\partial x_i} + \sum_{i=1}^{n} \bar{v}_i \frac{\partial \xi^1_{i_1}}{\partial x_i}\xi^1_{j_1} + \rho \xi^1_{i_1}\xi^1_{j_1} \right) dx, \quad i_1, j_1 = 1,\dots,N_1,$$

$$a^2_{i_2 j_2} = \nu \sum_{i=1}^{n} \int_D \left(\frac{\partial \xi^2_{i_2,1}}{\partial x_i}\frac{\partial \xi^2_{j_2,1}}{\partial x_i} + \frac{\partial \xi^2_{i_2,2}}{\partial x_i}\frac{\partial \xi^2_{j_2,2}}{\partial x_i} + \ldots + \frac{\partial \xi^2_{i_2,n}}{\partial x_i}\frac{\partial \xi^2_{j_2,n}}{\partial x_i} \right) dx, i_2, j_2 = 1,\ldots,N_2,$$

$$b_{i_1 j_1} = \int_D C^* J_H Q C \xi^1_{i_1}(x)\xi^1_{j_1}(x)\, dx, \quad i_1, j_1 = 1,\dots,N_1,$$

$$c_{i_1 j_2} = -\sum_{i=1}^{n} \int_D g_i \xi^1_{i_1}\xi^2_{j_2,i}\, dx, \quad i_1 = 1,\dots,N_1, \quad j_2 = 1,\dots,N_2,$$

$$d_{i_1 j_1} = -\int_D Q_1^{-1}\xi^1_{i_1}(x)\xi^1_{j_1}(x)\, dx, \quad i_1, j_1 = 1,\dots,N_1,$$

$$e_{i_2 j_1} = -\sum_{i=1}^{n} \int_D \xi^2_{j_2,i} g_i \xi^1_{j_1}\, dx, \quad i_2 = 1, \ldots, N_2, \quad j_1 = 1, \ldots, N_1,$$

$$h_{i_2 j_2} = -\sum_{i=1}^{n} \int_D (Q_2^{-1} \boldsymbol{\xi}^2_{i_2})_i \xi^2_{j_2,i}\, dx, \quad i_2, j_2 = 1, \ldots, N_2,$$

and

$$g_{j_1} = \int_D l_0 \xi^1_{j_1}\, dx, \quad j_1 = 1, \ldots, N_1,$$

$$q^1_{j_1} = \int_D f_0 \xi^1_{j_1}\, dx, \quad j_1 = 1, \ldots, N_1,$$

$$q^1_{j_2} = \sum_{i=1}^{n} \int_D f_{0,i} \xi^2_{j_2,i}\, dx, \quad j_2 = 1, \ldots, N_2,$$

$$\tilde{g}_{j_1} = \int_D C^* J_H Q y(x) \xi^1_{j_1}(x)\, dx, \quad j_1 = 1, \ldots, N_1.$$

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Received 10.11.2015