Журнал обчислювальної та прикладної математики | 2016 № 2 (122) | Journal of Computational & Applied Mathematics

UDC 519.6

EXPONENTIALLY CONVERGENT METHOD FOR DIFFERENTIAL EQUATION IN BANACH SPACE WITH A BOUNDED OPERATOR IN NONLOCAL CONDITION

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Резюме. Розглядається двоточкова нелокальна задача для диференціального рівняння першого порядку з необмеженим операторним коефіцієнтом в банаховому просторі X. В нелокальній умові міститься обмежений операторний коефіцієнт. Побудовано та обґрунтовано новий експоненціально збіжний метод у випадку, коли операторний коефіцієнт A у рівнянні є секторіальним і виконанні умови існування та єдиності розв'язку. Цей метод ґрунтується на зображенні операторних функцій за допомогою інтеграла Данфорда-Коші вздовж гіперболи, що охоплює спектр оператора A, та відповідній квадратурній формулі, що містить невелику кількість резольвент. Ефективність запропонованого методу демонструється за допомогою чисельних розрахунків.

Abstract. The two-pointed nonlocal problem for the first order differential equation with an unbounded operator coefficient in a Banach space X is considered. The nonlocal condition involves a bounded operator coefficient. A new exponentially convergent method is proposed and justified in the case when the operator coefficient A in equatuion is strongly positive and some existence and uniqueness conditions are fulfilled. This method is based on representations of operator functions by a Dunford-Cauchy integral along a hyperbola enveloping the spectrum of A and on the proper quadratures involving short sums of resolvents. The efficiency of proposed method is demonstrated by numerical examples.

1. Introduction

Problems with nonlocal conditions arise in many applications particulary in the theory of physics of plasma [12], nuclear physics [9], waveguides [7] etc. The nonlocal problems for a differential equation with various nonlocal conditions are also interesting from theoretical point of view and are ones of the important topics in the study of differential equations.

Differential equations with operator coefficients in some Hilbert or Banach space can be considered as meta-models for systems of partial or ordinary differential equations and are suitable for investigations using tools of the functional analysis (see e.g. [8, 11]). Nonlocal problems often are considered within this framework [1–3, 18, 19].

Key words. Nonlocal problem; differential equation with an operator coefficient in Banach space; operator exponential; exponentially convergent methods.

In this work we consider the following nonlocal two-pointed problem:

$$\begin{aligned} &u'_t + Au = f(t), \quad t \in [0,T] \\ &u(0) + Bu(T) = u_0, \quad 0 < T, \end{aligned} \tag{1}$$

where $B : X \to X$ is a bounded operator, $f(t)$ is a given vector-valued function with values in Banach space X, $u_0 \in X$. The operator A with domain $D(A)$ in Banach space X is assumed to be a densely defined strongly positive (sectorial) operator, i.e. its spectrum $\Sigma(A)$ lies in a sector of the right half-plane with the vertex at the origin and with a resolvent that decays inversely proportional to $|z|$ at the infinity (see estimate (2) below).

Discretization methods for differential equations in Banach and Hilbert spaces were intensively studied in the last decade (see e.g. [5, 10, 13, 14, 16, 17] and the references therein). Methods from [5, 10, 14, 16, 17] possess an exponential convergence rate, i.e. the error estimate in an appropriate norm is of the type $\mathcal{O}(\mathrm{e}^{-N^{\alpha}})$, $\alpha > 0$ with respect to a discretization parameter $N \to \infty$. For a given tolerance ε such methods provide optimal or nearly optimal computational complexity [4]. One of the possible ways to obtain exponentially convergent approximations to abstract differential equations is based on a representation of the solution through the Dunford-Cauchy integral along a parametrized path enveloping the spectrum of the operator coefficient and choosing a proper quadrature for this integral. In such way we obtain a short sum of resolvents. Since the treatment of such resolvents is usually the most time consuming part of any approximation this leads to a low-cost naturally parallelization techniques. Parameters of the algorithms from [5, 10, 14] were optimized in [20, 21] to improve the convergence rate.

Exponentially convergent method was constructed recently for nonlocal m-point problem for the first order differential equation with an unbounded coefficient in Banach space in [3]. But unlike this work there were considered the case of scalar coefficients in nonlocal condition. The aim of this paper is to construct an exponentially convergent approximation to the problem for a differential equation with two-pointed nonlocal condition with a bounded operator in abstract setting (1). The paper is organized as follows. In Section 2 we discuss the existence and uniqueness of the solution as well as its representation through input data. A numerical method for the homogeneous problem (1) is proposed in section 3. The main result of this section is theorem 1 about the exponential convergence rate of the proposed discretization.

2. Existence and representation of the solution

Let the operator A in (1) be a densely defined strongly positive (sectorial) operator in a Banach space X with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ lies in the sector. Additionally outside the sector and on its boundary Γ_Σ the following estimate for the resolvent holds true

$$\left\|(zI - A)^{-1}\right\| \le \frac{M}{1 + |z|}. \tag{2}$$

Let us assume that operator B is bounded in Banach space X, i.e. $\|B\| \leq c < \infty$.

The hyperbola

$$\Gamma_0 = \{z(\xi) = \rho_0 \cosh \xi - i b_0 \sinh \xi : \ \xi \in (-\infty, \infty), \ b_0 = \rho_0 \tan \varphi\} \tag{3}$$

is called a spectral hyperbola. It has a vertex at $(\rho_0, 0)$ and asymptotes which are parallel to the rays of the spectral angle Σ. The numbers ρ_0, φ are called the spectral characteristics of A.

A convenient representation of operator functions is the one through the Dunford-Cauchy integral (see e.g. [8, 11]) where the integration path plays an important role. We choose the following hyperbola

$$\Gamma_I = \{z(\xi) = a_I \cosh \xi - i b_I \sinh \xi : \ \xi \in (-\infty, \infty)\}, \tag{4}$$

as the integration contour which envelopes the spectrum of A.

One can reduce problem (1) to homogeneous using the following way. Let $u = v + w$, where v is a solution to the problem

$$\begin{aligned} &v'_t + Au = f(t), \quad t \in [0, T] \\ &u(0) = 0. \end{aligned}$$

Namely it has a representation

$$v(t) = \int_0^t \mathrm{e}^{-A(t-\tau)} f(\tau) d\tau. \tag{5}$$

Then for $w(t)$ we obtain the problem

$$\begin{aligned} &w'_t + Aw = 0, \quad t \in [0, T] \\ &w(0) + Bw(T) = u_0 - B \int_0^T \mathrm{e}^{-A(T-\tau)} f(\tau) d\tau = \tilde{u}_0, \quad 0 < T. \end{aligned}$$

Note that exponentially convergent method for approximating $v(t)$ from (5) was developed in [6] (see also [4]). So, we can consider homogeneous problem (1) ($f(t) \equiv 0$).

According to the Hille-Yosida-Phillips theorem [22] the strongly positive operator A generates a one parameter semigroup $T(t) = \mathrm{e}^{-tA}$ and solution to (1) (homogeneous case) can be represented by

$$u(t) = \mathrm{e}^{-At} u(0). \tag{6}$$

Combining the nonlocal condition from (1) and (6) we obtain

$$u(0) + B\mathrm{e}^{-AT} u(0) = u_0, \tag{7}$$

from where we have

$$u(0) = \left[I + B\mathrm{e}^{-AT}\right]^{-1} u_0,$$

in the case when $\left[I + B\mathrm{e}^{-AT}\right]^{-1}$ exists. Here I is an identity operator. So, using (6) we obtain

$$u(t) = \mathrm{e}^{-At} \left[I + B\mathrm{e}^{-AT}\right]^{-1} u_0. \tag{8}$$

Let us looking for existing conditions for $\left[I + \mathrm{e}^{-AT}B\right]^{-1}$. We have

$$\left\| \left[I + \mathrm{e}^{-AT}B\right]^{-1} \right\| \le \left(1 - \|\mathrm{e}^{-AT}B\|\right)^{-1} \le (1 - \|B\|)^{-1} \le c < \infty,$$

in the case

$$\|B\| < 1. \tag{9}$$

Remark 5. *It is possible to obtain weaker conditions than* (9) *in the case when the operator A is positive definite and selfadjoint $A = A^* \ge \lambda_0 I$, $\lambda_0 > 0$. For example if $B = A$ then we have using spectral integral representation*

$$\|B\mathrm{e}^{-AT}\| = \left\| \int_{\lambda_0}^{\infty} \mathrm{e}^{-\lambda T} \lambda dE_\lambda \right\| \le \frac{\mathrm{e}^{-1}}{T} \int_{\lambda_0}^{\infty} \|dE_\lambda\| = \frac{\mathrm{e}^{-1}}{T}.$$

Therefore, for $T > \mathrm{e}^{-1}$ we have

$$\left\| \left[I + \mathrm{e}^{-AT}B\right]^{-1} \right\| \le \left[1 - \|\mathrm{e}^{-AT}A\|\right]^{-1} < \left[1 - \frac{\mathrm{e}^{-1}}{T}\right]^{-1} = \frac{T}{T - \mathrm{e}^{-1}} < \infty.$$

3. Numerical approximation

Our aim in this section is to construct an exponentially convergent method for the solution to homogeneous problem (1) with assumption (9). Additionally we assume that the operators A and B are commutative: $AB = BA$.

Using the Dunford-Cauchy representation of $u(t)$ (see [11]) analogously to [4] we obtain

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma_I} \mathrm{e}^{-zt} \left[I + \mathrm{e}^{-zT}B\right]^{-1} (zI - A)^{-1} u_0 dz \tag{10}$$

Representation (10) makes sense only when the function $\mathrm{e}^{-zt}\left[I + \mathrm{e}^{-zT}B\right]^{-1}$ is analytic in the region enveloped by Γ_I. Let us show, that condition (9) guaranty this analyticity [8].

Actually, the analyticity of $\mathrm{e}^{-zt}\left[I + \mathrm{e}^{-zT}B\right]^{-1}$ might only be violated when $\mathrm{e}^{-zT}B = -I$, since in this case the function becomes unbounded. It is easy to see that for an arbitrary z we have

$$\left\| I + Be^{-zT} \right\| \ge |1 - \|B\|| > 0,$$

provided that (9) holds true.

We modify the representation of $u(t)$ to obtain numerical stability for small t as follows (see [4]):

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma_I} \mathrm{e}^{-zt} \left[I + \mathrm{e}^{-zT}B\right]^{-1} \left[(zI - A)^{-1} - \frac{1}{z} I\right] u_0 dz. \tag{11}$$

After discretization of the integral such modified resolvent provides better convergence speed than (10) in a neighborhood of $t = 0$ (see [4, 6] for details).

Parameterizing the integral (11) by (4) we get

$$u(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(t, \xi) d\xi, \tag{12}$$

with

$$\mathcal{F}(t,\xi) = F_A(t,\xi)u_0,$$

$$F_A(t,\xi) = \mathrm{e}^{-z(\xi)t} z'(\xi)[I + B\mathrm{e}^{-z(\xi)T}]^{-1} \left[(z(\xi)I - A)^{-1} - \frac{1}{z(\xi)} I \right],$$

$$z'(\xi) = a_I \sinh \xi - i b_I \cosh \xi.$$

Supposing $u_0 \in D(A^\alpha),\ 0 < \alpha < 1$ it was shown in [4,6] that

$$\|\mathrm{e}^{-z(\xi)t} z'(\xi) \left[(z(\xi)I - A)^{-1} - \frac{1}{z(\xi)} I \right] u_0\|$$

$$\le (1+M)K \frac{b_I}{a_I} \left(\frac{2}{a_I} \right)^\alpha \mathrm{e}^{-a_I t \cosh \xi - \alpha|\xi|} \|A^\alpha u_0\|,\ \xi \in \mathbb{R},\ t \ge 0.$$

The part responsible for the nonlocal condition in (12), can be estimated in the following way

$$\left\| \left(I + B\mathrm{e}^{-z(\xi)T} \right)^{-1} \right\| \le (1 - \|B\|)^{-1} = Q.$$

Thus, we obtain the following estimate for $\mathcal{F}(t,\xi)$ using commutative property of operators A and B:

$$\|\mathcal{F}(t,\xi)\| \le Q(1+M)K \frac{b_I}{a_I} \left(\frac{2}{a_I} \right)^\alpha \mathrm{e}^{-a_I t \cosh \xi - \alpha|\xi|} \|A^\alpha u_0\|,\ \xi \in \mathbb{R},\ t \ge 0. \quad (13)$$

Further, we have to estimate a strip around the real axis where the function $\mathcal{F}(t,\xi)$ permit analytical extension (with respect to ξ). The analyticity of function $\mathcal{F}(t,\xi + i\nu)$, in the strip

$$D_{d_1} = \{(\xi,\nu) : \xi \in (-\infty,\infty), |\nu| < d_1/2\},$$

with some d_1 could be violated if the resolvent or the part related to the nonlocal condition become unbounded. To avoid this we have to choose d_1 so that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola set $\Gamma(\nu)$ remains in the right half-plane of the complex plane. For $\nu = -d_1/2$ the corresponding hyperbola is going through the origin $(0,0)$. For $\nu = d_1/2$ it coincides with the spectral hyperbola and therefore for all $\nu \in (-d_1/2, d_1/2)$ the set $\Gamma(\nu)$ does not intersect the spectral sector.

The above requirements are fulfilled when (see [4])

$$d_1 = \arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}} \right) - \varphi, \quad (14)$$

where $\cos\varphi = \frac{\rho_0}{\sqrt{\rho_0^2+b_0^2}}$, $\sin\varphi = \frac{b_0}{\sqrt{\rho_0^2+b_0^2}}$. And for a_I, b_I

$$\begin{aligned}
a_I &= \sqrt{\rho_0^2+b_0^2}\cos\left(\frac{d_1}{2}+\varphi\right) \\
&= \rho_0\frac{\cos\left(\frac{d_1}{2}+\varphi\right)}{\cos\varphi} = \rho_0\frac{\cos\left(\arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2+b_0^2}}\right)/2+\varphi/2\right)}{\cos\varphi}, \\
b_I &= \sqrt{\rho_0^2+b_0^2}\sin\left(\frac{d_1}{2}+\varphi\right) \\
&= \rho_0\frac{\cos\left(\frac{d_1}{2}+\varphi\right)}{\cos\varphi} = \rho_0\frac{\cos\left(\arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2+b_0^2}}\right)/2+\varphi/2\right)}{\cos\varphi}.
\end{aligned} \tag{15}$$

For a_I and b_I defined as above the vector valued function $\mathcal{F}(t,w)$ is analytic in the strip D_{d_1} with respect to $w=\xi+i\nu$ for any $t\geq 0$.

Similarly to [15] (see [4]), we introduce the space $\mathbf{H}^p(D_d)$, $1\leq p\leq\infty$ of all vector-valued functions $\mathcal{F}$ analytic in the strip

$$D_d = \{z\in\mathbb{C} : -\infty < \Re z < \infty, |\Im z| < d\},$$

equipped by the norm

$$\|\mathcal{F}\|_{\mathbf{H}^p(D_d)} = \begin{cases} \lim_{\epsilon\to 0}(\int_{\partial D_d(\epsilon)}\|\mathcal{F}(z)\|^p|dz|)^{1/p} & \text{if } 1\leq p<\infty, \\ \lim_{\epsilon\to 0}\sup_{z\in\partial D_d(\epsilon)}\|\mathcal{F}(z)\| & \text{if } p=\infty, \end{cases}$$

where

$$D_d(\epsilon) = \{z\in\mathbb{C} : |\mathrm{Re}(z)| < 1/\epsilon, |\mathrm{Im}(z)| < d(1-\epsilon)\}$$

and $\partial D_d(\epsilon)$ is the boundary of $D_d(\epsilon)$.

Similarly to [4] we have estimate for $\|\mathcal{F}(t,w)\|$

$$\begin{aligned}
&\|\mathcal{F}(t,\cdot)\|_{\mathbf{H}^1(D_{d_1})} \leq \|A^\alpha u_0\|[C_-(\varphi,\alpha) \\
&+ C_+(\varphi,\alpha)]\int_{-\infty}^{\infty} e^{-\alpha|\xi|}d\xi = C(\varphi,\alpha)\|A^\alpha u_0\|
\end{aligned} \tag{16}$$

with

$$\begin{aligned}
C(\varphi,\alpha) &= \frac{2}{\alpha}[C_+(\varphi,\alpha)+C_-(\varphi,\alpha)], \\
C_\pm(\varphi,\alpha) &= (1+M)QK\tan\left(\frac{d_1}{2}+\varphi\pm\frac{d_1}{2}\right)\left(\frac{2\cos\varphi}{\rho_0\cos\left(\frac{d_1}{2}+\varphi\pm\frac{d_1}{2}\right)}\right)^\alpha.
\end{aligned}$$

Note that the influence of both the smoothness parameter of u_0 given by α and of the spectral characteristics of the operator A given by φ and ρ_0 is accounted by that fact, that the constant $C(\varphi,\alpha)$ from (15) tends to ∞ if $\alpha\to 0$, $\varphi\to\pi/2$ or $\rho_1\to 0$ (in this case due to (14) $d_1\to\frac{\pi}{2}-\varphi$).

We approximate integral (12) by the following Sinc-quadrature [4,6,15]:

$$u_N(t) = \frac{h}{2\pi i}\sum_{k=-N}^{N}\mathcal{F}(t,z(kh)), \tag{17}$$

with an error

$$\|\eta_N(\mathcal{F},h)\| = \|u(t) - u_{h,N}(t)\|$$

$$\leq \|u(t) - \frac{h}{2\pi i}\sum_{k=-\infty}^{\infty}\mathcal{F}(t,z(kh))\| + \|\frac{h}{2\pi i}\sum_{|k|>N}\mathcal{F}(t,z(kh))\|$$

$$\leq \frac{1}{2\pi}\frac{e^{-\pi d_1/h}}{2\sinh(\pi d_1/h)}\|\mathcal{F}\|_{\mathbf{H}^1(D_{d_1})}$$

$$+\frac{C(\varphi,\alpha)h\|A^\alpha u_0\|}{2\pi}\sum_{k=N+1}^{\infty}\exp[-a_I t\cosh(kh) - \alpha kh]$$

$$\leq \frac{c\|A^\alpha u_0\|}{\alpha}\left\{\frac{e^{-\pi d_1/h}}{\sinh(\pi d_1/h)} + \exp[-a_I t\cosh((N+1)h) - \alpha(N+1)h]\right\},$$

where the constant c does not depend on h, N, t. Equalizing the both exponentials for $t = 0$ implies

$$\frac{2\pi d_1}{h} = \alpha(N+1)h,$$

or after the transformation

$$h = \sqrt{\frac{2\pi d_1}{\alpha(N+1)}}. \tag{18}$$

With this step-size the following error estimate holds true

$$\|\eta_N(\mathcal{F},h)\| \leq \frac{c}{\alpha}\exp\left(-\sqrt{\frac{\pi d_1\alpha}{2}(N+1)}\right)\|A^\alpha u_0\|, \tag{19}$$

where the constant c independent of t, N. In the case $t > 0$ the first summand in the argument of $\exp[-a_I t\cosh((N+1)h) - \alpha(N+1)h]$ from the estimate for $\|\eta_N(\mathcal{F},h)\|$ contributes mainly to the error order. Setting in this case $h = c_1\ln N/N$ with some positive constant c_1 we remain, asymptotically for a fixed t, with an error

$$\|\eta_N(\mathcal{F},h)\| \leq c\left[e^{-\pi d_1 N/(c_1\ln N)} + e^{-c_1 a_I tN/2 - c_1\alpha\ln N}\right]\|A^\alpha u_0\|, \tag{20}$$

where c is a positive constant. Thus, we have proven the following result.

Theorem 1. *Let A be a densely defined strongly positive operator and $u_0 \in D(A^\alpha)$, $\alpha \in (0,1)$, then the Sinc-quadrature (17) represents an approximate solution of the homogeneous nonlocal value problem (1) (i.e. the case when $f(t) \equiv 0$) and possesses an exponential convergence rate which is uniform with respect to $t \geq 0$ and is of the order $\mathcal{O}(\mathrm{e}^{-c\sqrt{N}})$ uniformly in $t \geq 0$ for $h = \mathcal{O}(1/\sqrt{N})$ (estimate (19)) and of the order $\mathcal{O}\left(max\left\{\mathrm{e}^{-\pi dN/(c_1\ln N)},\ \mathrm{e}^{-c_1 a_I tN/2 - c_1\alpha\ln N}\right\}\right)$ for each fixed $t > 0$ when $h = c_1\ln N/N$ (estimate (20)).*

TABL. 1. The error for $x = 0.5,\ t = 0.5$.

N	$\varepsilon_{1,N}$	$\varepsilon_{2,N}$
8	0.4686576088595737062e-1	0.1900886270925846e-2
16	0.934021577137014178e-2	0.852946984325721275711e-4
32	0.1546349721567053042e-3	0.810358320985172283872e-5
64	0.0159641801061596051e-3	0.01035505780238307696e-5
128	0.735484912605954949e-5	0.91841759148488051333e-6
256	0.146908016254907436e-7	0.24806555113840622551e-7
512	0.8577765610e-8	0.1165963141e-8
1024	0.7339799837e-11	0.1591565422e-11

TABL. 2. The estimate of c

N	c
4	2.372652515388745588587496
8	1.120148732795449515627946
16	1.458741976765153165445005
32	1.527648924601130131250452
64	1.476794596387591759032900
128	1.499935011373075736075927
256	1.506597339081609844717370

4. NUMERICAL EXAMPLE

We consider the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) + Bu(x,1) = u_0, \end{aligned}$$

with

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} \tag{21}$$

$$u_0(x,t) = \begin{pmatrix} (1+0.2\mathrm{e}^{-\pi^2})\sin(\pi x) + 0.1\mathrm{e}^{-4\pi^2}\sin(2\pi x) \\ 0.1\mathrm{e}^{-\pi^2}\sin(\pi x) + (1+0.4\mathrm{e}^{-4\pi^2})\sin(2\pi x) \end{pmatrix} \tag{22}$$

It is easy to check that exact solution is

$$u(x,t) = \begin{pmatrix} \sin(\pi x) \\ \sin(2\pi x) \end{pmatrix}, \tag{23}$$

The error of computation is presented in Tabl. 1.

Due to Theorem 1 the error should not be greater then $\varepsilon_N = \mathcal{O}\left(\mathrm{e}^{-c\sqrt{N}}\right)$. The constant c in the exponent can be estimated using the following a-posteriori relation:

$$c = \ln\left(\frac{\varepsilon_N}{\varepsilon_{2N}}\right)(\sqrt{2}-1)^{-1}N^{-1/2} = \ln(\mu_N)(\sqrt{2}-1)^{-1}N^{-1/2}.$$

The numerical results are presented in Tabl. 2. Note that the constant can be estimated as $c \approx 1.5$ when $N \to \infty$.

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Received 11.05.2016