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INTERPOLATION FORMULAS FOR FUNCTIONS, DEFINED ON THE SETS OF MATRICES WITH DIFFERENT MULTIPLICATION RULES

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Резюме. Розглядається задача інтерполяції функції від матриці у випадку множення за правилами Йордана, Адамара, Фробеніуса, Кронекера і Лапласа. Отримано новий клас інтерполяційних многочленів Лагранжа і Ньютона фіксованого степеня для функцій, визначених на множинах скінченних і нескінченних матриць. Вказано вигляд операторних поліномів, для яких ці формули інваріантні.

Abstract. We consider the problem of matrix functions interpolation in the case of Jordan, Hadamard, Frobenius, Kronecker and Laplace multiplication rules. We give a new class of Lagrange and Newton interpolation polynomials of fixed degree for functions, defined on the sets of finite and infinite matrices. The type of operator polynomials, for which these formulas are invariant, is indicated.

1. Introduction

Let X be a set of square or rectangular matrices of the fixed size. The operator $F : X \to Y$, where Y is a given set, is called a function of the matrix. In particular, Y may coincide with X, may be some other set of matrices, a numerical set, a function space and others.

Approximation of functions of the matrix variables is a part of a more general problem – interpolation of operators [1–4].

General form of the interpolation formulas is determined by the structure and properties of elements of the set X, on which the interpolated function $F(A)$ is given, as well as the interpolation nodes. A number of interpolation formulas on the sets of square and rectangular matrices was obtained in the works [1, 2; 5–8].

Along with the commonly accepted operation of matrix multiplication, the other matrix multiplication rules are also used and can be applied in mathematics and its applications. Such an approach is also effective at constructing of interpolation methods for functions of matrices. In this paper the interpolation formulas, using both the ordinary matrix multiplication and the matrix multiplication by Jordan, Hadamard, Frobenius and others, are obtained.

Key words. Interpolation; matrix functions; interpolation matrix polynomial; interpolation formula of Lagrange and Newton type; matrix multiplication by Jordan, Hadamard, Frobenius and Kronecker, Laplace discrete convolution.

2. Interpolation formulas with multiplication of square matrix by Jordan

Let X be a set of square matrices of the fixed size, the operator $F : X \to X$. The Jordan product $A \circ B$ of two matrices A and B from X is defined by the following rule: $A \circ B = \frac{1}{2}(AB + BA)$. It is commutative, but not associative. So, if the Jordan product contains more than two matrices, then in some cases it is required to indicate the execution order of the multiplication in the given product for uniqueness.

Let us first consider interpolation formulas of Lagrange type of the arbitrary order, which are constructed on the basis of such rules of multiplication of square matrices. Here are three variants of the formulas for constant matrices. We denote by $l_{nk}(A)$ the product

$$l_{nk}(A) = B_{k0} \circ (A - A_0) \circ B_{k1} \circ ... \circ B_{k,k-1} \circ$$

$$\circ (A - A_{k-1}) \circ B_{kk} \circ (A - A_{k+1}) \circ B_{k,k+1} \circ ... \circ B_{k,n-1} \circ (A - A_n) \circ B_{nn},$$

where A_k $(k = 0, 1, ..., n)$ are interpolation nodes, $B_{k\nu} \equiv B_{k,\nu}$ $(k, \nu = 0, 1, ..., n)$ are arbitrarily given matrices. Let the order of execution of multiplication operation in $l_{nk}(A)$ be determined in advance. We introduce the matrix polynomials of the form

$$L_{0n}(A) = \sum_{k=0}^{n} F(A_k) \circ \left\{ l_{nk}^{-1}(A_k) \circ l_{nk}(A) \right\} \tag{1}$$

$$L_{n0}(A) = \sum_{k=0}^{n} \left\{ F(A_k) \circ l_{nk}^{-1}(A_k) \right\} \circ l_{nk}(A), \tag{2}$$

in which first the multiplication operation in the curly brackets is performed. Since $l_{nk}^{-1}(A_k) \circ l_{nk}(A_\nu) = \delta_{k\nu} I$ $(k, \nu = 0, 1, ..., n)$, where $\delta_{k\nu}$ is the Kronecker symbol, than for the formula (1) in the nodes A_k the interpolation conditions $L_{0n}(A_k) = F(A_k)$ are met.

These conditions are satisfied for the formula (2), if the associator

$$\left\{ F(A_\nu),\ l_{n\nu}^{-1}(A_\nu),\ l_{n\nu}(A_\nu) \right\} = 0.$$

It takes place in virtue of the equality

$$\left\{ F(A_\nu),\ l_{n\nu}^{-1}(A_\nu),\ l_{n\nu}(A_\nu) \right\} =$$

$$= \left(F(A_\nu) \circ l_{n\nu}^{-1}(A_\nu) \right) \circ l_{n\nu}(A_\nu) - F(A_\nu) \circ \left(l_{n\nu}^{-1}(A_\nu) \circ l_{n\nu}(A_\nu) \right) =$$

$$= \left(F(A_\nu) \circ l_{n\nu}^{-1}(A_\nu) \right) \circ l_{n\nu}(A_\nu) - F(A_\nu) = 0.$$

It follows that (2) is the interpolation formula.

It is easy to check that the matrix polynomial of the n-th degree

$$L_n(A) = \sum_{k=0}^{n} F(A_k) \circ l_{kk}(A), \tag{3}$$

where

$$l_{kk}(A) = \prod_{\nu=0, \nu \neq k}^{n} B_\nu \left\{ (A - A_\nu) \circ (A_k - A_\nu)^{-1} \right\} B_\nu^{-1},$$

B_ν $(\nu = 0, 1, ..., n)$ are arbitrary invertible matrices, also satisfies the conditions $L_n(A_k) = F(A_k)$ $(k = 0, 1, ..., n)$, at that the product of matrices, indicated in curly brackets, on B_ν and B_ν^{-1} can be understood as the ordinary or in the sense of Jordan. In the both cases $l_{kk}(A_\nu) = \delta_{k\nu} I$ $(k, \nu = 0, 1, ..., n)$.

The interpolation polynomials (1)–(3) are exact for the matrix polynomials

$$P_{0n}(A) = \sum_{\nu=0}^{n} D_\nu \circ \left\{ l_{n\nu}^{-1}(A_\nu) \circ l_{n\nu}(A) \right\},$$

$$P_{n0}(A) = \sum_{\nu=0}^{n} \left\{ D_\nu \circ l_{n\nu}^{-1}(A_\nu) \right\} \circ l_{n\nu}(A), \quad P_n(A) = \sum_{\nu=0}^{n} D_\nu \circ l_{\nu\nu}(A),$$

respectively, where D_ν are arbitrary square matrices. As already mentioned, the interpolation conditions for the formula (2) are satisfied, if and only if associator

$$\left\{ F(A_k),\ l_{nk}^{-1}(A_k),\ l_{nk}(A_k) \right\} = 0 \quad (k = 0, 1, ..., n).$$

This imposes additional conditions on the operator F and the interpolation nodes.

If $n = 1$, and $B_{k\nu}$ $(k, \nu = 0, 1)$ are the identity matrices, then the formula (1) with the nodes A_0 and A_1 is reduced to the equality

$$L_{01}(A) = F(A_0) + [F(A_1) - F(A_0)] \circ \left\{ (A_1 - A_0)^{-1} \circ (A - A_0) \right\}. \qquad (4)$$

It is exact (invariant) for the polynomials $P_{01}(A) = D \circ \left\{ (A_1 - A_0)^{-1} \circ A \right\} + C$, where D and C are arbitrary matrices.

In the particular case, when $A_1 - A_0 = I$, the linear interpolation formula (4) takes the form

$$L_{01}(A) = F(A_0) + \\ + \frac{1}{2} \left[(F(A_1) - F(A_0))(A - A_0) + (A - A_0)(F(A_1) - F(A_0)) \right]$$

and it will be invariant for the matrix polynomials $P_1(A) = DA + AD + C$, where D and C are arbitrary fixed matrices.

Here is another formula of the linear interpolation with the multiplication by Jordan:

$$L_1(A) = F(A_0) + (A - A_0) \circ B + \\ + [F(A_1) - F(A_0) - (A_1 - A_0) \circ B] \circ \left\{ (A_1 - A_0)^{-1} \circ (A - A_0) \right\},$$

where B is an arbitrary given matrix. This interpolation formula is exact for polynomials of the form

$$P_1(A) = D \circ \left\{ (A_1 - A_0)^{-1} \circ A \right\} + B \circ A + C.$$

One of the quadratic interpolation formulas of the kind (3) has the form

$$L_{21}(A) = L_{01}(A) + \left\{ (A - A_1) \circ (A_2 - A_1)^{-1} \right\} \circ \left[\left\{ (A - A_0) \circ (A_2 - A_0)^{-1} \right\} \circ \\ \circ (F(A_2) - F(A_1)) - \left\{ (A - A_0) \circ (A_1 - A_0)^{-1} \right\} \circ (F(A_1) - F(A_0)) \right],$$

where, as before, at the beginning matrices in the curly brackets are found, and then in the usual order – in square brackets; $L_{01}(A)$ is the matrix polynomial of the first degree (4). For it the following equalities $L_{21}(A_i) = F(A_i)$, $(i = 0, 1, 2)$ are valid.

Example 2.1. It is not difficult to show that the interpolation polynomial

$$L_{10}(A) = F(A_0) + \left\{(A - A_0) \circ (A_1 - A_0)^{-1}\right\} \circ [F(A_1) - F(A_0)],$$

where the function $F(A) = A^2$, and the nodes

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix},$$

has the form

$$L_{10}(A) = \frac{1}{2} A \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} A - \begin{bmatrix} 6 & 8 \\ 12 & 18 \end{bmatrix}.$$

Next, we consider the formulas of the linear and quadratic interpolation on the set of square functional matrices, which are determined by the matrix Stieltjes integrals. Let $X = C(T)$ be the set of continuous on $T = [a, b]$ square matrices; $F : X \to X$, $A_0(t)$, $A_1(t)$ be interpolation nodes from X.

On the set of matrices with the Jordan multiplication, the interpolation polynomial of the first degree with respect to the nodes $A_0(t)$ and $A_1(t)$ has the form

$$\begin{aligned} \tilde{L}_{10}(A) = F(A_0) + \int_T \left\{[A(\tau) - A_0(\tau)] \circ [A_1(\tau) - A_0(\tau)]^{-1}\right\} \circ \\ \circ d_\tau F[A_0(\cdot) + \chi(\tau, \cdot)(A_1(\cdot) - A_0(\cdot))]. \end{aligned} \tag{5}$$

In the formula (5), as before, first the multiplication operation in the curly brackets is carried out. This formula is invariant with respect to the polynomials

$$P_1(A) = K_0 + \int_T \left\{A(t) \circ [A_1(t) - A_0(t)]^{-1}\right\} \circ K(t) \circ [A_1(t) - A_0(t)]\, dt,$$

where K_0, $K(t)$ are some given matrices.

Example 2.2. The interpolation matrix polynomial of the form (5) with respect to the nodes $A_0(t)$ and $A_1(t)$ for the function $F(A) = \int_a^b A^2(t)\, dt$ takes the form

$$\tilde{L}_{10}(A) = F(A_0) + \int_a^b G[A(\tau), A_0(\tau), A_1(\tau)]\, d\tau,$$

where

$$\begin{aligned} G[A, A_0, A_1] &= \\ &= \frac{1}{4}\left\{(A - A_0)(A_1 - A_0)^{-1} + (A_1 - A_0)^{-1}(A - A_0)\right\}\left(A_1^2 - A_0^2\right) + \\ &+ \frac{1}{4}\left(A_1^2 - A_0^2\right)\left\{(A - A_0)(A_1 - A_0)^{-1} + (A_1 - A_0)^{-1}(A - A_0)\right\}. \end{aligned}$$

Next, we consider the interpolation polynomials of the arbitrary degree for functions of two matrix variables. Let $F(A, B)$ be a function of two variable square matrices A and B, the interpolation nodes $\{A_\nu, B_\nu\}$ $(\nu = 0, 1, ..., n)$ are given, where A_ν, B_ν are some square matrices. We introduce the following notations: $r_l \equiv r_l(A,B)$ is the vector with matrix coordinates $\{A-A_l, B-B_l\}$;

$$r_{l,k} \equiv r_{lk} \equiv r_{lk}(A_k,B_k) \equiv r_l(A_k,B_k)\,(l,\ k=0,1,...,n)\,.$$

The vector r_{lk} has coordinates $\{A_k-A_l, B_k-B_l\}$. It's obvious that $r_{ll} = r_l(A_l,B_l) = 0$. Assume that

$$(r_l, r_{lk}) = (A-A_l)\circ(A_k-A_l) + (B-B_l)\circ(B_k-B_l)\,(l,\ k=0,1,...,n)\,,$$

$$(r_{lk}, r_{lk}) = (A_k-A_l)^2 + (B_k-B_l)^2$$

and, accordingly, we denote

$$l_k(A,B) = (r_0, r_{0k})\,...\,(r_{k-1}, r_{k-1,k})\,(r_{k+1}, r_{k+1,k})\,...\,(r_n, r_{nk}) \times$$

$$\times\left[(r_{0k}, r_{0k})\,...\,(r_{k-1,k}, r_{k-1,k})\,(r_{k+1,k}, r_{k+1,k})\,...\,(r_{nk}, r_{nk})\right]^{-1}.$$

Since $l_k(A_\nu,B_\nu) = \delta_{k\nu} I$, then the matrix polynomial

$$L_{1n}(A,B) = \sum_{k=0}^{n} l_k(A,B)\,F(A_k,B_k)\,, \tag{6}$$

where the product of the matrices $l_k(A,B)$ and $F(A_k,B_k)$ on the right side of (6) may be usual or in the sense of Jordan, is also the interpolation polynomial for the function $F(A,B)$ with respect to the nodes (A_k,B_k) $(k=0,1,...,n)$.

We give a slightly modified version of the interpolation formula of the form (6). We introduce the notations

$$\tilde{l}_k(A,B) = \prod_{\nu=0,\nu\neq k}^{n} \tilde{l}_{\nu k}(A,B)\,,\tilde{l}_{\nu k}(A,B) = (r_\nu, r_{\nu k})\circ(r_{\nu k}, r_{\nu k})^{-1}.$$

Since $\tilde{l}_{\nu k}(A_k,B_k) = I$, $\tilde{l}_{\nu k}(A_\nu,B_\nu) = 0$, then $\tilde{l}_k(A_\nu,B_\nu) = \delta_{k\nu} I$. Thus, the formula

$$L_n(A,B) = \sum_{k=0}^{n} \tilde{l}_k(A,B)\,F(A_k,B_k)$$

is the interpolation polynomial of the degree not higher than n, for which the equalities $L_n(A_\nu,B_\nu) = F(A_\nu,B_\nu)$ $(\nu=0,1,...,n)$ are true.

Next, we consider formulas of the other form for the linear interpolation of functions of two matrix variables on the set of constant matrix with the multiplication by Jordan. Let $F(A,B)$ be a function of matrix variables A and B; (A_i,B_i) be interpolation nodes $(i=0,1,2)$.

We introduce the following notations:

$$\tilde{l}_0(A,B) = [(A-A_1)\circ(B_1-B_2) - (A_1-A_2)\circ(B-B_1)]\circ D^{-1},$$

$$\tilde{l}_1(A,B) = [(A-A_0)\circ(B_2-B_0) - (A_2-A_0)\circ(B-B_0)]\circ D^{-1},$$

$$\tilde{l}_2(A,B) = [(A-A_0)\circ(B_0-B_1) - (A_0-A_1)\circ(B-B_0)]\circ D^{-1},$$

where

$$D = (A_0-A_1)\circ(B_1-B_2) - (A_1-A_2)\circ(B_0-B_1)\,.$$

Note that the relations $\tilde{l}_i(A_j, B_j) = \delta_{ij} I$; $\tilde{l}_0(A,B) + \tilde{l}_1(A,B) + \tilde{l}_2(A,B) = I$ take place. It is not difficult to verify that for matrix polynomial of the variables A and B of the first degree of the form

$$\tilde{L}_1(A,B) = \tilde{l}_0(A,B) \circ F(A_0, B_0) + \tilde{l}_1(A,B) \circ F(A_1, B_1) + \tilde{l}_2(A,B) \circ F(A_2, B_2) \quad (7)$$

the interpolation conditions $\tilde{L}_1(A_i, B_i) = F(A_i, B_i)$ $(i = 0, 1, 2)$ are carried out.

3. Interpolation formulas with matrix multiplication by Hadamard

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be some matrices of the same dimension. The matrix $C = A \cdot B$ of the same size with elements $c_{ij} = a_{ij} b_{ij}$ is called the Hadamard product of the matrices A and B. It is commutative, associative and distributive with respect to the addition of matrices. The role of the identity matrix for such rule of multiplication carries the matrix J, all elements of which are equal to one. By $A^{\overset{\bullet}{-1}} = \left[\frac{1}{a_{ij}}\right]$ we denote the matrix that is inverse in the sense of Hadamard for the matrix $A = [a_{ij}]$ with the elements $a_{ij} \neq 0$.

By the definition, the n-th degree of matrix $A = [a_{ij}]$ in the sense of Hadamard, which is denoted as $A^{\overset{\bullet}{n}}$, is the matrix $A^{\overset{\bullet}{n}} = \left[a_{ij}^n\right]$, where $A^{\overset{\bullet}{n}} = J$ for $n = 0$. The function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of the matrix $A = [a_{ij}]$, analytical in a neighborhood of each element of this matrix, is defined on the set of matrices with Hadamard multiplication by the formula $f(A) = \sum_{k=0}^{\infty} a_k A^{\overset{\bullet}{k}}$ and, accordingly, it is the matrix $f(A) = [f(a_{ij})]$.

Here are the special cases of interpolation formula [8] of the form

$$L_{0n}(A) = \sum_{k=0}^{n} F(A_k) \cdot l_{nk}^{\overset{\bullet}{-1}}(A_k) \cdot l_{nk}(A) =$$
$$= \sum_{k=0}^{n} \left[\frac{f_{ij}^k \left(a_{ij} - a_{ij}^0\right) \ldots \left(a_{ij} - a_{ij}^{k-1}\right)\left(a_{ij} - a_{ij}^{k+1}\right) \ldots \left(a_{ij} - a_{ij}^n\right)}{\left(a_{ij}^k - a_{ij}^0\right) \ldots \left(a_{ij}^k - a_{ij}^{k-1}\right)\left(a_{ij}^k - a_{ij}^{k+1}\right) \ldots \left(a_{ij}^k - a_{ij}^n\right)} \right], \quad (8)$$

where

$$l_{nk}(A) = (A - A_0) \cdot \ldots \cdot (A - A_{k-1}) \cdot (A - A_{k+1}) \cdot \ldots \cdot (A - A_n),$$

matrices $l_{nk}(A_k)$ do not have zero elements, matrix A and nodes A_k of the same dimension, f_{ij}^k are elements of the matrix $F(A_k)$ $(k = 0, 1, \ldots, n)$. It is obvious that the equalities $L_{0n}(A_i) = F(A_i)$ $(i = 0, 1, \ldots, n)$ hold.

Consider the linear case of the interpolation formula (8). Let the interpolation nodes $A_0 = \left[a_{ij}^0\right]$, $A_1 = \left[a_{ij}^1\right]$ be such that all elements of the matrix $A_0 - A_1 = \left[a_{ij}^0 - a_{ij}^1\right]$ are different from zero. Then for the formula

$$L_{01}(A) = F(A_0) \cdot (A_0 - A_1)^{\overset{\bullet}{-1}} \cdot (A - A_1) + F(A_1) \cdot (A_1 - A_0)^{\overset{\bullet}{-1}} \cdot (A - A_0)$$

or, what is the same thing, for the formula

$$L_{01}(A) = F(A_0) \cdot \left[\frac{a_{ij} - a_{ij}^1}{a_{ij}^0 - a_{ij}^1}\right] + F(A_1) \cdot \left[\frac{a_{ij} - a_{ij}^0}{a_{ij}^1 - a_{ij}^0}\right],$$

where $A = [a_{ij}]$ is current matrix variable, the interpolation conditions $L_{01}(A_i) = F(A_i)$ $(i = 0, 1)$ are fulfilled.

During the construction of interpolation formulas, based on the Hadamard multiplication of square matrices, it is useful to introduce yet another analogue of the inverse matrix. Let $A = [a_{ij}]$ be a square matrix and $a_{ii} \neq 0$. By $A^{(-1)}$ we denote the matrix, for which $A \cdot A^{(-1)} = A^{(-1)} \cdot A = I$, where I is the identity matrix in the ordinary sense of the same dimension as the matrix A. This matrix will be $A^{(-1)} = \operatorname{diag}\left[\frac{1}{a_{ii}}\right]$.

We give formulas of the linear interpolation with the ordinary and the Hadamard multiplication. Let $A = [a_{ij}]$ be some square matrix that has nonzero diagonal elements. Then for the linear interpolation formula

$$L_{01}(A) = F(A_0)\left\{(A_0 - A_1)^{(-1)} \cdot (A - A_1)\right\} + \\ + F(A_1)\left\{(A_1 - A_0)^{(-1)} \cdot (A - A_0)\right\},$$

or for the same formula in another form

$$L_{01}(A) = F(A_0)\operatorname{diag}\left[\frac{a_{ii} - a_{ii}^1}{a_{ii}^0 - a_{ii}^1}\right] + F(A_1)\operatorname{diag}\left[\frac{a_{ii} - a_{ii}^0}{a_{ii}^1 - a_{ii}^0}\right],$$

equalities $L_{10}(A_0) = F(A_0)$, $L_{10}(A_1) = F(A_1)$ hold.

We consider the case $n = 2$ of the interpolation formula (8). The quadratic interpolation formula with respect to the nodes $A_0 = \left[a_{ij}^0\right]$, $A_1 = \left[a_{ij}^1\right]$ and $A_2 = \left[a_{ij}^2\right]$, such that all elements of the matrices

$$A_0 - A_1 = \left[a_{ij}^0 - a_{ij}^1\right], A_0 - A_2 = \left[a_{ij}^0 - a_{ij}^2\right], A_1 - A_2 = \left[a_{ij}^1 - a_{ij}^2\right]$$

are different from zero, is a matrix polynomial of the form

$$L_{02}(A) = F(A_0) \cdot \left[\frac{\left(a_{ij} - a_{ij}^1\right)\left(a_{ij} - a_{ij}^2\right)}{\left(a_{ij}^0 - a_{ij}^1\right)\left(a_{ij}^0 - a_{ij}^2\right)}\right] +$$

$$+F(A_1) \cdot \left[\frac{\left(a_{ij} - a_{ij}^0\right)\left(a_{ij} - a_{ij}^2\right)}{\left(a_{ij}^1 - a_{ij}^0\right)\left(a_{ij}^1 - a_{ij}^2\right)}\right] + F(A_2) \cdot \left[\frac{\left(a_{ij} - a_{ij}^0\right)\left(a_{ij} - a_{ij}^1\right)}{\left(a_{ij}^2 - a_{ij}^0\right)\left(a_{ij}^2 - a_{ij}^1\right)}\right],$$

for which the conditions $L_{02}(A_i) = F(A_i)$ $(i = 0, 1, 2)$ are fulfilled.

Next, we give formulas of the quadratic interpolation with the ordinary and the Hadamard multiplication. Let $A = [a_{ij}]$ be some square matrix that has different from zero diagonal elements. For quadratic interpolation with the same restrictions on the nodes A_0, A_1 and A_2, as in the previous case, we have the formula

$$L_{02}(A) = F(A_0)\operatorname{diag}\left[\frac{(a_{ii}-a_{ii}^1)(a_{ii}-a_{ii}^2)}{(a_{ii}^0-a_{ii}^1)(a_{ii}^0-a_{ii}^2)}\right]+$$

$$+F(A_1)\operatorname{diag}\left[\frac{(a_{ii}-a_{ii}^0)(a_{ii}-a_{ii}^2)}{(a_{ii}^1-a_{ii}^0)(a_{ii}^1-a_{ii}^2)}\right]+F(A_2)\operatorname{diag}\left[\frac{(a_{ii}-a_{ii}^0)(a_{ii}-a_{ii}^1)}{(a_{ii}^2-a_{ii}^0)(a_{ii}^2-a_{ii}^1)}\right],$$

which satisfies the conditions $L_{02}(A_i) = F(A_i)$ $(i = 0, 1, 2)$.

Example 3.1. On the set of matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with matrix multiplication only in the sense of Hadamard for the function $F(A) = A^2$ with respect to the nodes

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix},$$

the interpolation polynomial

$$L_{01}(A) = F(A_0) + [F(A_1) - F(A_0)] \cdot (A_1 - A_0)^{\overset{\bullet}{-1}} \cdot (A - A_0)$$

takes the form

$$L_{01}(A) = \begin{bmatrix} 7 & 5 \\ 9 & 13 \end{bmatrix} \cdot A - \begin{bmatrix} 0 & 0 \\ 12 & 30 \end{bmatrix} = \begin{bmatrix} 7a_{11} & 5a_{12} \\ 9a_{21} & 13a_{22} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 12 & 30 \end{bmatrix}.$$

For the constructed polynomial the interpolation conditions

$$L_{01}(A_0) = F(A_0) = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}, L_{01}(A_1) = F(A_1) = \begin{bmatrix} 0 & 0 \\ 6 & 9 \end{bmatrix}.$$

are also true. In the case if the interpolation nodes $A_k = \alpha_k J$, where α_k $(k = 0, 1, ..., n)$ are different in pairs numbers, then the formula (8) takes the form

$$L_n(A) =$$

$$= \sum_{k=0}^{n} \frac{(A-\alpha_0 J)\cdot ... \cdot(A-\alpha_{k-1}J)\cdot(A-\alpha_{k+1}J)\cdot ... \cdot(A-\alpha_n J)}{(\alpha_k-\alpha_0)...(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})...(\alpha_k-\alpha_n)} \cdot F(\alpha_k J).$$

Next, we consider interpolation formulas for operators, defined on the set of functional matrices. Let $X = C(T)$ be the set of continuous on $T = [a, b]$ square matrices; an operator $F : X \to X$ and $A_0(t)$, $A_1(t)$ be interpolation nodes from X. Suppose also that $A = A(t)$, interpolation nodes $A_0(t)$, $A_1(t)$ are matrices of the same order, defined on the segment $[a, b]$, and operator $F(A)$ is defined at the nodes $A_0(t)$, $A_1(t)$ and on the matrix curve $A_0(t)+\chi(\tau, t)(A_1(t) - A_0(t))$, where the function

$$\chi(\tau, t) = \begin{cases} 1, & \tau \geq t; \\ 0, & \tau < t, \end{cases} \quad \chi(a, t) \equiv 0, \quad \chi(b, t) \equiv 1 \quad (a \leq \tau, t \leq b).$$

One of the linear interpolation formulas on the set of continuous on the segment $[a, b]$ matrices can be written using the Stieltjes integral in the form

$$L_{10}(A) = F(A_0) + \int_a^b [A(\tau) - A_0(\tau)] \cdot [A_1(\tau) - A_0(\tau)]^{\stackrel{\bullet}{-1}} \cdot d_\tau \times$$
$$\times F[A_0(t) + \chi(\tau, t)(A_1(t) - A_0(t))],$$

on condition that all elements of the matrix $A_1(t) - A_0(t)$ are different from zero on $[a, b]$ and in this formula integral exists. The equalities $L_{10}(A_i) = F(A_i)$ $(i = 0, 1)$ are true.

In the space $C^m[a, b]$ of rectangular matrices $A(t) = [a_{ij}(t)]$ of the dimension $p \times q$, for which the derivative $A^{(m)}(t) = \left[a_{ij}^{(m)}(t)\right]$ of order m is continuous on the $[a, b]$, we consider the matrix polynomial of the first degree

$$P_1(A) = B + \sum_{j=0}^{n} A(t_j) \cdot C_j + \sum_{k=0}^{m} \int_a^b D_k(t, s) \cdot A^{(k)}(s) ds \tag{9}$$

where $B = B(t)$, $C_j = C_j(t)$, $D_k(t, s)$ $(j = 0, 1, ...n;\ k = 0, 1, ..., m)$ are fixed $(p \times q)$-matrices.

We denote by $\sigma_{1i}(t)$ and $H_i(t)$ the matrices

$$\sigma_{1i}(t) = A_0(t) + A_1(t_i) - A_0(t_i), \quad H_i(t) = A(t) - A_0(t) - A(t_i) + A_0(t_i),$$

where t_i $(i = 0, 1, ...n)$ are given points of the segment $[a, b]$; $A_0(t)$ and $A_1(t)$ are interpolation nodes such that the matrices $A_1(t_i) - A_0(t_i)$ are reversible in the sense of Hadamard.

For the formula

$$\begin{aligned} L_1(A) = {} & F(A_0) + \\ & + \frac{1}{n+1} \sum_{i=0}^{n} [A(t_i) - A_0(t_i)] \cdot [A_1(t_i) - A_0(t_i)]^{\stackrel{\bullet}{-1}} \cdot [F(\sigma_{1i}) - F(A_0)] + \\ & + \frac{1}{n+1} \sum_{i=0}^{n} \int_0^1 \delta F[\sigma_{1i}(\cdot) + \tau(A_1(\cdot) - \sigma_{1i}(\cdot));\, H_i(\cdot)] d\tau \end{aligned} \tag{10}$$

the conditions $L_1(A_i) = F(A_i)$ $(i = 0,\ 1)$ hold, and it is exact for matrix polynomials of the form (9).

Really, the equation $L_1(A_0) = F(A_0)$ is satisfied, since the second and third terms in (10) become zero. Execution of interpolation condition at the second node is also easy to verify, taking into account that in this case the integral in (10) can be calculated exactly.

Let $F(A, B)$ be also a function of two matrix variables A and B; (A_i, B_i) be interpolation nodes $(i = 0, 1, 2)$. We introduce the following notations:

$$l_{10}(A, B) = [(A - A_1) \cdot (B_1 - B_2) - (A_1 - A_2) \cdot (B - B_1)] \cdot D^{\stackrel{\bullet}{-1}},$$

$$l_{11}(A, B) = [(A - A_0) \cdot (B_2 - B_0) - (A_2 - A_0) \cdot (B - B_0)] \cdot D^{\stackrel{\bullet}{-1}},$$

$$l_{12}(A, B) = [(A - A_0) \cdot (B_0 - B_1) - (A_0 - A_1) \cdot (B - B_0)] \cdot D^{\stackrel{\bullet}{-1}}.$$

Here the matrix $\overset{\bullet}{D}^{-1}$ is reversible in the sense of Hadamard for

$$D = (A_0 - A_1) \cdot (B_1 - B_2) - (A_1 - A_2) \cdot (B_0 - B_1);$$

A, B are independent variables, interpolation nodes (A_i, B_i) and values $F(A_i, B_i)$ $(i = 0, 1, 2)$ are rectangular matrices of the same dimension.

For the interpolation formula

$$\begin{aligned} L_{11}(A, B) = l_{10}(A, B) \cdot F(A_0, B_0) + \\ + l_{11}(A, B) \cdot F(A_1, B_1) + l_{12}(A, B) \cdot F(A_2, B_2) \end{aligned} \tag{11}$$

the conditions $L_{11}(A_i, B_i) = F(A_i, B_i)$ $(i = 0, 1, 2)$ are satisfied. The formula (11) is invariant with respect to matrix polynomials of the form

$$P_1(A, B) = l_{10}(A, B) \cdot C_0 + l_{11}(A, B) \cdot C_1 + l_{12}(A, B) \cdot C_2. \tag{12}$$

At that in the equation (12) arbitrary rectangular matrices C_i are of the same dimension as the matrices $F(A_i, B_i)$ $(i = 0, 1, 2)$.

Example 3.2. Let $A = [a_{ij}]$, $B = [b_{ij}]$ $(i, j = 1, 2)$ be square matrices of the second order. The interpolation formula (11) for the function $F(A, B) = (AB)^2$ with respect to the nodes

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

takes the form

$$L_{11}[A, B] = \begin{bmatrix} 8 - 8a_{11} + 4b_{11} & 1 + 4b_{12} \\ -11 + 11a_{21} + 10b_{21} & 7 + 4a_{22} - 2b_{22} \end{bmatrix}.$$

For $L_{11}[A, B]$ the interpolation conditions

$$L_{11}[A_0, B_0] = F(A_0, B_0) = \begin{bmatrix} 0 & 9 \\ 0 & 9 \end{bmatrix},$$

$$L_{11}[A_1, B_1] = F(A_1, B_1) = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix},$$

$$L_{11}[A_2, B_2] = F(A_2, B_2) = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$

are true.

Note that in [1, 46 p.] the matrix Γ is constructed as a sum of the powers of the Hadamard matrices, which plays an important role in the construction of the set of interpolating polynomials in the Hilbert space and in the justification of a number of the results obtained on this set.

4. Interpolation formulas with matrix multiplication by Frobenius

Suppose that the matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ have the same dimension. Their product in the sense of Frobenius is defined as

$$A\diamondsuit B=\sum_{i,j}a_{ij}b_{ij}.$$

This operation is commutative, and its result is a scalar. Interpolation formulas for functions of matrices may be also constructed on the basis of such multiplication rule.

Let interpolation nodes A_k $(k=0,1,...,n)$ be different stationary or functional matrices, and $F(A_k)$ be given fixed matrices, which dimension may differ from the dimension of A_k, or some other mathematical objects over the field of real or complex numbers. Then in the case of rectangular matrices of the same dimension (including square matrices) for the formula

$$L_n(F;A)=\sum_{k=0}^{n}\frac{l_{nk}(A)}{l_{nk}(A_k)}F(A_k), \qquad (13)$$

where

$$l_{nk}(A)=[(A-A_0)\diamondsuit(A_k-A_0)]\,...\,[(A-A_{k-1})\diamondsuit(A_k-A_{k-1})]\times$$
$$\times[(A-A_{k+1})\diamondsuit(A_k-A_{k+1})]\,...\,[(A-A_n)\diamondsuit(A_k-A_n)],$$

the equalities $L_n(F;A_\nu)=F(A_\nu)$ $(\nu=0,\ 1,\ ...,\ n)$ take place.

If the interpolation nodes A_k such that $\operatorname{tr}(A_k-A_\nu)\neq 0$ $(k,\nu=0,1,...,n)$, then on the set of square matrices for the similar formula

$$L_n(F;A)=\sum_{k=0}^{n}\frac{\tilde{l}_{nk}(A)}{\tilde{l}_{nk}(A_k)}F(A_k),$$

where

$$\tilde{l}_{nk}(A)=\operatorname{tr}(A-A_0)\operatorname{tr}(A_k-A_0)\cdots\operatorname{tr}(A-A_{k-1})\operatorname{tr}(A_k-A_{k-1})\times$$
$$\times\operatorname{tr}(A-A_{k+1})\operatorname{tr}(A_k-A_{k+1})\cdots\operatorname{tr}(A-A_n)\operatorname{tr}(A_k-A_n),$$

the same interpolation conditions are fulfilled.

Obviously, the equation (13) remains an interpolation, if $l_{nk}(A)$ is replaced by any number function $\phi_{nk}(A)$ of matrix function arguments such that $\phi_{nk}(A_k)\neq 0$ for $k=0,1,...,n$.

In particular, if $n=2$ and $n=1$, then the formula (13) takes the form

$$L_2(F;A)=\frac{[(A-A_1)\diamondsuit(A_0-A_1)]\,[(A-A_2)\diamondsuit(A_0-A_2)]}{[(A_0-A_1)\diamondsuit(A_0-A_1)]\,[(A_0-A_2)\diamondsuit(A_0-A_2)]}F(A_0)+$$
$$+\frac{[(A-A_0)\diamondsuit(A_1-A_0)]}{[(A_1-A_0)\diamondsuit(A_1-A_0)]}\frac{[(A-A_2)\diamondsuit(A_1-A_2)]}{[(A_1-A_2)\diamondsuit(A_1-A_2)]}F(A_1)+$$
$$+\frac{[(A-A_0)\diamondsuit(A_2-A_0)]\,[(A-A_1)\diamondsuit(A_2-A_1)]}{[(A_2-A_0)\diamondsuit(A_2-A_0)]\,[(A_2-A_1)\diamondsuit(A_2-A_1)]}F(A_2)$$

and

$$L_1(F;A) = \frac{(A-A_1)\diamondsuit(A_0-A_1)}{(A_0-A_1)\diamondsuit(A_0-A_1)}F(A_0) + \\ + \frac{(A-A_0)\diamondsuit(A_1-A_0)}{(A_1-A_0)\diamondsuit(A_1-A_0)}F(A_1), \tag{14}$$

respectively.

Example 4.1. The interpolation formula (14), based on the nodes

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$$

for the function $F(A) = A^2$, has the form

$$L_1(F;A) = \frac{1}{2}\mathrm{tr}\, A \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

Example 4.2. Let $A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$ be a functional matrix and

$$A_0 = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 0 \end{bmatrix}$$

be the interpolation nodes. Then

$$(A_0-A_1)\diamondsuit(A_0-A_1) = (A_1-A_0)\diamondsuit(A_1-A_0) = 2,$$

and the interpolation formula (14) takes the form

$$L_1(F;A) = \frac{1}{2}(x_{13}+x_{21}-3)F(A_0) - \frac{1}{2}(x_{13}+x_{21}-5)F(A_1),$$

and, therefore, we get that $L_1(F;A_0) = F(A_0)$, $L_1(F;A_1) = F(A_1)$.

Next, we consider a formula of the linear interpolation, similar to (7) and (11), with the multiplication in the case of Frobenius. We introduce the following notation:

$$\tilde{l}_{00}(A,B) = \frac{1}{D}\left[(A-A_1)\diamondsuit(B_1-B_2) - (A_1-A_2)\diamondsuit(B-B_1)\right],$$

$$\tilde{l}_{11}(A,B) = \frac{1}{D}\left[(A-A_0)\diamondsuit(B_2-B_0) - (A_2-A_0)\diamondsuit(B-B_0)\right],$$

$$\tilde{l}_{22}(A,B) = \frac{1}{D}\left[(A-A_0)\diamondsuit(B_0-B_1) - (A_0-A_1)\diamondsuit(B-B_0)\right],$$

where D is the numeric value, which is calculated by the formula

$$D = (A_0-A_1)\diamondsuit(B_1-B_2) - (A_1-A_2)\diamondsuit\diamondsuit(B_0-B_1).$$

The interpolation formula

$$\tilde{L}_{11}(A,B) = \tilde{l}_{00}(A,B)F(A_0,B_0) + \\ + \tilde{l}_{11}(A,B)F(A_1,B_1) + \tilde{l}_{22}(A,B)F(A_2,B_2) \tag{15}$$

satisfies the interpolation conditions $\tilde{L}_{11}(A_i,B_i) = F(A_i,B_i)$ $(i=0,1,2)$.

Example 4.3. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be square matrices of the second order. We construct interpolation formulas of the form (7), (11) and (15) for the function $F(A, B) = (AB)^2$ on the nodes

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix};$$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

In the case of the formula (7) we have

$$L_{11}[A, B] = \frac{1}{26} \times$$

$$\times \begin{bmatrix} -38 + 91a_{11} - 156a_{12} + 22a_{21} + 123a_{22} + 122b_{11} - 144b_{12} + 16b_{21} + 246b_{22} \\ 1440 + 32a_{11} - 388a_{12} + 206a_{21} - 192a_{22} + 58b_{11} - 328b_{12} + 176b_{21} - 378b_{22} \end{bmatrix.}$$

$$\begin{matrix} -102 + 94a_{11} - 16a_{12} - 52a_{21} + 126a_{22} + 16b_{11} + 8b_{12} - 64b_{21} + 168b_{22} \\ 128 + 211a_{11} - 364a_{12} + 138a_{21} + 131a_{22} + 82b_{11} - 136b_{12} + 24b_{21} + 262b_{22} \end{matrix} \Bigg].$$

Using the rule (11), we get that

$$L_{11}[A, B] = \begin{bmatrix} 8 - 8a_{11} + 4b_{11} & 1 + 4b_{12} \\ -11 + 11a_{21} + 10b_{21} & 7 + 4a_{22} - 2b_{22} \end{bmatrix}.$$

Finally, for the formula (15) the value $D = -3$, and the required polynomial has the form

$$L_{11}[A, B] =$$

$$= \frac{1}{3} \Bigg[\begin{matrix} 4(-4 + 2a_{11} - a_{12} + 2a_{22} - b_{11} - 2b_{12} + 2b_{21} + 5b_{22}) \\ -40 + 20a_{11} - 21a_{12} + 22a_{21} + 20a_{22} + b_{11} - 20b_{12} + 20b_{21} + 39b_{22} \end{matrix}$$

$$\begin{matrix} 35 - 4a_{11} + 4a_{21} - 4a_{22} + 4b_{11} + 4b_{12} - 4b_{21} - 12b_{22} \\ 19 + 4a_{11} - 14a_{12} + 24a_{21} + 4a_{22} + 10b_{11} - 4b_{12} + 4b_{21} - 2b_{22} \end{matrix} \Bigg].$$

We note that all formulas, obtained in this example, have a different form, but for them the same interpolation conditions

$$L_{11}[A_0, B_0] = F(A_0, B_0) = \begin{bmatrix} 0 & 9 \\ 0 & 9 \end{bmatrix},$$

$$L_{11}[A_1, B_1] = F(A_1, B_1) = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix},$$

$$L_{11}[A_2, B_2] = F(A_2, B_2) = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$

are fulfilled.

5. Kronecker matrix multiplication and corresponding matrix polynomials

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are some matrices of the dimensions $m \times n$ and $p \times q$, respectively, then the Kronecker product of these matrices $A \otimes B$ is a matrix of dimension $mp \times nq$, which is defined by the formula

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ ... & ... & ... & ... \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{bmatrix}.$$

In general, the Kronecker product of matrices, in contrast to the Jordan multiplication, non-commutative, but has the property of associativity. The Kronecker multiplication is distributive with respect to the addition of matrices.

Let X be a set of square matrices, an operator $F : X \to Y$, where Y is also a set of square matrices of the fixed dimension, interpolation nodes $A_k \in X$ $(k = 0, 1, ..., n)$ and there are inverse matrices $(A_i - A_j)^{-1}$ $(i \neq j)$. In addition, the dimension of matrices of the set Y coincides with the dimension of square matrices of the form $(A - A_\nu) \otimes I$.

We introduce the notation

$$l_k(A) = [(A - A_0) \otimes I] ... [(A - A_{k-1}) \otimes I] [(A - A_{k+1}) \otimes I] ... [(A - A_n) \otimes I].$$

Then for the polynomials

$$L_{0n}(A) = \sum_{k=0}^{n} F(A_k) l_k^{-1}(A_k) l_k(A), \tag{16}$$

$$L_{n0}(A) = \sum_{k=0}^{n} l_k(A) l_k^{-1}(A_k) F(A_k) \tag{17}$$

the equalities $L_{0n}(A_k) = L_{n0}(A_k) = F(A_k)$ are true, because

$$l_k^{-1}(A_k) l_k(A_\nu) = l_k(A_\nu) l_k^{-1}(A_k) = \delta_{k\nu} I.$$

Here and further the orders of matrices $F(A_k)$ are consistent with the order of the interpolation fundamental square matrices $l_k(A)$. If we select the expression

$$l_k(A) = [I \otimes (A - A_0)] ... [I \otimes (A - A_{k-1})] [I \otimes (A - A_{k+1})] ... [I \otimes (A - A_n)]$$

for the function $l_k(A)$ in (16) and (17), we come to some other kind of these formulas.

The formulas $L_{0n}(A)$ and $L_{n0}(A)$ are exact for the matrix polynomials

$$P_{0n}(A) = \sum_{k=0}^{n} B_k l_k^{-1}(A_k) l_k(A), P_{n0}(A) = \sum_{k=0}^{n} l_k(A) l_k^{-1}(A_k) B_k,$$

where B_ν $(\nu = 0, 1, ..., n)$ are arbitrary matrices from the set Y, respectively.

We consider formulas of the linear interpolation

$$L_{01}(A) = F(A_0) + [F(A_1) - F(A_0)] \left[I \otimes (A_1 - A_0)^{-1}\right] [I \otimes (A - A_0)],$$

$$L_{10}(A) = F(A_0) + [(A - A_0) \otimes I] \left[(A_1 - A_0)^{-1} \otimes I\right] [F(A_1) - F(A_0)].$$

The formula $L_{10}(A)$ is exact for matrix polynomials of the form $P_{10}(A) = A \otimes B + D$. Really,

$$L_{10}[P_{10}; A] = A_0 \otimes B + D + [(A - A_0) \otimes I] \left[(A_1 - A_0)^{-1} \otimes I\right] [(A_1 - A_0) \otimes B] =$$

$$= A_0 \otimes B + D + (A - A_0)(A_1 - A_0)^{-1}(A_1 - A_0) \otimes B =$$
$$= A_0 \otimes B + D + (A - A_0) \otimes B = P_{10}(A).$$

Similarly, the formula $L_{01}(A)$ is exact for matrix polynomials of the form $P_{01}(A) = B \otimes A + D$.

We consider the application of the Lagrange–Sylvester formula to construct the corresponding interpolation formulas, using several properties of the Kronecker multiplication for this. One of the important properties of this multiplication for the given problem is that the spectrum of the Cartesian product of matrices is clearly expressed through the spectrum of its factors.

Suppose that the matrix C has the form $C = A \otimes B$, and square matrices A and B of the orders p and q have the eigenvalues λ_i $(i = 1, 2, ..., p)$ and μ_j $(j = 1, 2, ..., q)$, respectively. Then [9] the matrix C has pq eigenvalues $\lambda_i \mu_j$ $(i = 1, 2, ..., p;\ j = 1, 2, ..., q)$.

If the eigenvalues $\lambda_i \mu_j$ are different, then for the matrix C the Lagrange–Sylvester formula takes the form

$$F(C) = \sum_{k=1}^{p} \sum_{\nu=1}^{q} \frac{l_{k\nu}(C)}{l_{k\nu}(\lambda_k \mu_\nu)} F(\lambda_k \mu_\nu),$$

where

$$l_{k\nu}(C) = \prod_{i=1, i\neq k}^{p} \prod_{j=1, j\neq \nu}^{q} (C - \lambda_i \mu_j I_{pq}),$$

$$l_{k\nu}(\lambda_k \mu_\nu) = \prod_{i=1, i\neq k}^{p} \prod_{j=1, j\neq \nu}^{q} (\lambda_k \mu_\nu - \lambda_i \mu_j),$$

I_{pq} is the identity matrix of the pq-dimension.

We give the trigonometric variant of the Lagrange-Sylvester formula for the Kronecker product of matrices $C = A \otimes B$:

$$F(C) = \sum_{k=1}^{p} \sum_{\nu=1}^{q} \frac{\tilde{l}_{k\nu}(C)}{\tilde{l}_{k\nu}(\lambda_k \mu_\nu)} \times$$
$$\times \left(\frac{F(\lambda_k \mu_\nu) + F(-\lambda_k \mu_\nu)}{2} I_{pq} + \frac{F(\lambda_k \mu_\nu) - F(-\lambda_k \mu_\nu)}{2 \sin(\lambda_k \mu_\nu)} \sin C \right),$$

where

$$\tilde{l}_{k\nu}(C) = \prod_{i=1, i\neq k}^{p} \prod_{j=1, j\neq \nu}^{q} (\cos C - \cos(\lambda_i \mu_j) I_{pq}),$$

$$\tilde{l}_{k\nu}(\lambda_k \mu_\nu) = \prod_{i=1, i\neq k}^{p} \prod_{j=1, j\neq \nu}^{q} (\cos(\lambda_k \mu_\nu) - \cos(\lambda_i \mu_j)),$$

and I_{pq} is the identity matrix of the pq-dimension as before.

6. Infinite matrix and some interpolation formulas

Operators of the discrete convolution, as well as continuous, are widely used in the solution of many mathematical and applied problems [10–12]. Discrete convolutions can be applied to the interpolation problem of functions with many variables and infinite matrix variables.

Matrix $A = [a_{ij}]$ with real or complex elements a_{ij} is called infinite, if $i, j = 1, 2, ...$ or at least one of the indices i or j has infinite number of the values. Addition and multiplication of the infinite matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined the same way as in the finite-dimensional case. In contrast to the finite matrices, the product $AB = [c_{ij}]$ may not exist, since the series $c_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{kj}$ $(i, j = 1, 2, ...)$ may be divergent or nonsummable for all or only for the several i and j values. Moreover, if there is the existing product BA, the product AB may not exist. In general, the multiplication of infinite matrices is not associative: $(AB)\,C \neq A\,(BC)$.

On the set of infinite matrix A, on condition that the matrices A^k $(k \geq 2)$ exist, for entire functions $f\,(z)$ $(z \in \mathbb{C})$ the matrices $f\,(A)$ may be determined by the usual rules.

The theory of infinite matrices, as one of the sections of mathematical analysis, and its applications are interconnected with the theory of separable Hilbert spaces, including the coordinate Hilbert space l_2.

We consider some formulas for the interpolation of functions, given on the set of infinite sequences, which we denote by l. Each element x (infinite-dimensional vector) from l is defined by its coordinates: $x = \{x_k\}_{k=0}^{\infty} = \{x_0, x_1, x_2, ...\}$, where x_k $(k = 0, 1, ...)$ are complex numbers or complex random values with given distribution laws. Here the addition of elements of the set and its multiplication by a number are determined by the usual rules, and the product $x * y$ is given by the discrete convolution of the Laplace according to the rule

$$x * y = \left\{ \sum_{\nu=0}^{k} x_{k-\nu} y_\nu \right\}_{k=0}^{\infty};$$

the product $x * y$ also belongs to l. For this multiplication rule the sequence $I = \{1, 0, 0, ...\}$ is the unit, and in this case the set l is a commutative algebra.

Let F be operator, mapping the set l into l, and the elements $x_0 = \alpha_0 I$, $x_1 = \alpha_1 I$ and $x_2 = \alpha_2 I$, where I is the unit element in l, $\alpha_i \in \mathbb{C}$, $\alpha_j \neq \alpha_i$ for $j \neq i$ $(i, j = 0, 1, 2)$, are taken as the interpolation nodes. Then simplest on l formulas are formulas of the linear and quadratic interpolation

$$L_1\,(F; x) = F\,(x_0) + \frac{1}{\alpha_1 - \alpha_0}\,[F\,(x_1) - F\,(x_0)] * (x - x_0)\,,$$

$$\begin{aligned} L_2\,(F; x) = {} & \frac{1}{(\alpha_0 - \alpha_1)\,(\alpha_0 - \alpha_2)} F\,(x_0) * (x - x_1) * (x - x_2) + \\ & + \frac{1}{(\alpha_1 - \alpha_0)\,(\alpha_1 - \alpha_2)} F\,(x_1)\,(x - x_0) * (x - x_2) + \\ & + \frac{1}{(\alpha_2 - \alpha_0)\,(\alpha_2 - \alpha_1)} F\,(x_2) * (x - x_0) * (x - x_1)\,, \end{aligned}$$

respectively, for which $L_1(F;x_0) = F(x_0)$, $L_1(F;x_1) = F(x_1)$ and $L_2(F;x_i) = F(x_i)$ $(i = 0,1,2)$.

For the same system of interpolation nodes $x_i = \alpha_i I$ on condition that $\alpha_j \neq \alpha_i$, $j \neq i$ $(i,j = 0,1,2,...,n)$, the Lagrange formula of the n-th order is written in the analogous form

$$L_n(F;x) = \sum_{k=0}^{n} \omega_{nk}(x) * F(\alpha_k I), \tag{18}$$

where

$$\omega_{nk}(x) = \frac{(x-\alpha_0 I)\,(x-\alpha_1 I)\cdots(x-\alpha_{k-1}I)\,(x-\alpha_{k+1}I)\cdots(x-\alpha_n I)}{(\alpha_k-\alpha_0)\,(\alpha_k-\alpha_1)\cdots(\alpha_k-\alpha_{k-1})\,(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n)},$$

I is the unit element of the algebra l. It's obvious that $L_n(F;x_k) = F(x_k)$ $(k = 0,1,...,n)$.

Let us consider a slightly different variant of (18). By $l^{m\times m}$ we denote the set of $m\times m$-matrices of the form $X = [x^{ij}]$, where x^{ij} are elements from l, i.e. $x^{ij} = \left\{x_k^{ij}\right\}_{k=0}^{\infty}$ $(i,j = 1,2,...,m)$. Here the operations of addition and multiplication of matrices by a number are ordinary, and the multiplication of matrices $X = [x^{ij}]$ and $Y = [y^{ij}]$ from $l^{m\times m}$ is carried out according to the rule:

$$C = X * Y = \left[c^{ij}\right],$$

where $c^{ij} = \sum_{\nu=1}^{m} x^{i\nu} * y^{\nu j}$, i.e. $x^{i\nu} * y^{\nu j}$ means the product of sequences $x^{i\nu}$ and $y^{\nu j}$ also in the sense of the Laplace convolution given above. This set of matrices with indicated rules of multiplication also form an algebra.

We consider the formula of the form (18), in which the interpolation nodes x_ν are $m\times m$-matrices

$$x_\nu = \begin{bmatrix} x_\nu^{11} & x_\nu^{12} & \cdots & x_\nu^{1m} \\ x_\nu^{21} & x_\nu^{22} & \cdots & x_\nu^{2m} \\ \cdots & \cdots & \cdots & \cdots \\ x_\nu^{m1} & x_\nu^{m2} & \cdots & x_\nu^{mm} \end{bmatrix} \quad (\nu = 0,1,...,n)$$

with the elements x_ν^{ij} from l. It is required of nodes x_ν that the matrices $x_\nu - x_k$ are reversible in the ordinary sense.

Let the interpolation nodes be matrices of the form

$$\tilde{x}_\nu = x_\nu I = \left[x_\nu^{ij}, 0, 0, ...\right] \quad (i,j = 1,2,...,m;\ \nu = 0,1,...,n).$$

Then for an operator $F : l^{m\times m} \to l^{m\times m}$ and the formula

$$\tilde{L}_n(F;x) = \sum_{k=0}^{n} \tilde{\omega}_{nk}(x) * F(\tilde{x}_k),$$

where

$$\tilde{\omega}_{nk}(x) = l_{k,0}(x) * l_{k,1}(x) * \cdots * l_{k,k-1}(x) * l_{k,k+1}(x) * \cdots * l_{k,n}(x),$$

$$l_{k,\nu}(x) = (x-\tilde{x}_\nu) * (\tilde{x}_k - \tilde{x}_\nu)^{-1} \equiv (x-\tilde{x}_\nu) * (x_k - x_\nu)^{-1} I \quad (k,\nu = 0,1,...,n)$$

the interpolation conditions $\tilde{L}_n(F;\tilde{x}_k) = F(\tilde{x}_k)$ $(k = 0, 1, ..., n)$ are fulfilled. These conditions take place by virtue of the equalities $\tilde{\omega}_{nk}(\tilde{x}_\nu) = \delta_{k\nu} I$, where, as before, $\delta_{k\nu}$ is the Kronecker symbol.

Example 6.1. Let A and B be infinite rectangular matrices of the dimensions $2 \times \infty$ and $\infty \times 2$, respectively:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \\ \vdots & \vdots \end{bmatrix}.$$

Their product is a (2×2)-matrix $AB = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, where the elements S_{ij} $(1 \le i, j \le 2)$ are given by series

$$S_{11} = \sum_{i=1}^{\infty} a_{1i} b_{i1},\ S_{12} = \sum_{i=1}^{\infty} a_{1i} b_{i2},\ S_{21} = \sum_{i=1}^{\infty} a_{2i} b_{i1},\ S_{22} = \sum_{i=1}^{\infty} a_{2i} b_{i2}.$$

For the existence of the product AB it is required that these series are converging in some sense. For example, if the elements of matrix A and B are random values or processes, then one of the variants of the convergence may be the convergence of mathematical expectations of the summands of these series. We consider an example with this type of convergence.

Suppose that

$$a_{1i} = \frac{1}{(2i-1)!} W^{4i-2}(t),\ a_{2i} = \frac{1}{(2i-2)!} W^{4i+2}(t);$$

$$b_{i1} = \frac{(-1)^{1+i}}{[(4i-3)!!]^2} \xi^{4i-2}(t),\ b_{i2} = \frac{(-1)^{1-i}}{[(4i+1)!!]^2} \xi^{4i+2}(t),$$

where $W(t)$ is standard Wiener process, $\xi(t)$ is a random Gaussian process with zero mean value and variance $\sigma = \sigma(t)$. We assume that these processes are stochastically independent. We remind that the k-th moments of the processes $W(t)$ and $\xi(t)$ are given [13] by the equalities

$$E\left\{W^k(t)\right\} = \begin{cases} (2\nu-1)!!t^\nu, & k = 2\nu; \\ 0, & k = 2\nu+1, \end{cases}$$

$$E\left\{\xi^k(t)\right\} = \begin{cases} (2\nu-1)!!\sigma^\nu, & k = 2\nu; \\ 0, & k = 2\nu+1 \end{cases}$$

$(\nu = 0, 1, ...)$. In this case, the series $E\{S_{j\nu}\}$ $(j = 1, 2;\ \nu = 1, 2)$ converge. Since

$$E\{S_{11}\} = \sum_{i=1}^{\infty} E\{a_{1i} b_{i1}\} = \sin(t\sigma(t)),\ E\{S_{22}\} = \sum_{i=1}^{\infty} E\{a_{2i} b_{i2}\} = t\sigma(t)\cos(t\sigma(t)),$$

then the mathematical expectation of the trace of matrix AB has the simple form

$$E\{\mathrm{tr}(AB)\} = \sin(t\sigma(t)) + t^3\sigma^3(t)\cos(t\sigma(t)).$$

Construction and research of interpolation operator polynomials in the Hilbert spaces, which theory in some cases is interconnected with the infinite matrix theory, are considered in the articles [14–15].

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