Журнал обчислювальної та прикладної математики | 2016 № 2 (122) | Journal of Computational & Applied Mathematics

UDC 519.6

ON THE BOUNDARY ELEMENT METHOD FOR BOUNDARY VALUE PROBLEMS FOR CONVOLUTIONAL SYSTEMS OF ELLIPTIC EQUATIONS

Y. A. Muzychuk

Резюме. Для чисельного розв'язування крайових задач для нескінченних систем зі згортковою структурою, які складаються з еліптичних рівнянь другого порядку, запропоновано метод граничних елементів. Розв'язок подано за допомогою послідовності потенціалів простого шару. Для апроксимації невідомих густин потенціалів використано базис, який складається з кусково-сталих базисних функцій, побудованих на трикутних граничних елементах. Досліджено апріорні похибки. Наведено результати серії обчислювальних експериментів.

Abstract. For the numerical solution of boundary value problems for infinite systems with convolutional structure that consist of the second order elliptic equations, a boundary elements method is suggested. The solution is given as a sequence of single layer potentials. For the approximation of the unknown densities of the potentials a basis that consists of piece-wise constant functions built on triangular boundary elements is used. A priory error estimates are obtained. Results of a series of computational experiments are given.

1. Introduction

Boundary value problems for infinite systems that consist of elliptic partial differential equations (PDEs) can be found while investigating solutions of linear evolution problems for instance in the following works [3, 6, 10, 15, 16, 21]. Note that in [14] the well-posedness of such problems has been proven by transitioning to the corresponding variational formulations. Integral representations of the solutions of these boundary value problems that lead to equivalent boundary integral equations (BIEs) have been obtained. Properties of the BIEs method for exterior problems have been studied by the author in [17].

The main goal of the current article is such transformation of the obtained system of BIE that allows to efficiently apply the Bubnov-Galerkin method to it and prove its convergence. We also develop an algorithm for its solution by the boundary elements method (BEM) and investigate the approximation error of the obtained solution.

The paper is organized as follows. In Section 2 we formulate a Dirichlet BVP for an infinite triangular system of elliptic PDEs. We consider this problem in appropriate Sobolev spaces and introduce a notion of sequences and a new operation on them – q-convolution. In this section we also give an integral

Key words. Boundary value problems; boundary integral equations; elliptic equation; infinite system; boundary element method; convolutional system.

representation of the solution of the BVP by a combination of some surface potentials which reduces the problem to a system of BIEs.

In Section 3 we transform the system of BIEs into such sequence of BIEs all equations of which have the same boundary integral operator in the left hand side. It allows us to justify the application of the Bubnov-Galerkin method for finding the unknown functions – densities of the potentials. Afterwards, the main properties of the BEM and a priory error estimate of the numerical solution are obtained. In Section 4 some computational aspects of the systems of linear equations that appear as a result of the discretization of the BIEs are considered. Results of a series of computational experiments for the numerical solution of some model problems are given in Section 5. In this section an example of the application of the suggested approach for the solution of an initial-boundary value problem for the wave equation with homogeneous initial conditions is given. In the last section conclusions about the introduced method are given.

2. Formulation of the convolutional systems of PDE and BIE

Let $\Omega \subset \mathbb{R}^3$ be a bounded and simply connected domain with a Lipschitz boundary Γ and $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$ be an exterior domain. We consider an infinite system in Ω^+

$$\begin{cases} c_0 u_0 - \Delta u_0 = 0, \\ c_1 u_0 + c_0 u_1 - \Delta u_1 = 0, \\ c_2 u_0 + c_1 u_1 + c_0 u_2 - \Delta u_2 = 0, \\ \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \\ c_k u_0 + c_{k-1} u_1 + ... + c_0 u_k - \Delta u_k = 0, \\ \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \end{cases} \tag{1}$$

where $u_0, u_1, \; ..., \; u_k, \; ...$ are unknown functions, $c_0, c_1, \; ..., \; c_k, \; ...$ are some given constants and $c_0 > 0$. We investigate BVPs for system (1) that consist in finding its solutions that satisfy the Dirichlet condition on the boundary Γ

$$u_k|_\Gamma = \tilde{g}_k, \; k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{2}$$

where $\tilde{g}_i$ $(i \in \mathbb{N}_0)$ are given functions on Γ. In other words, we will consider the Dirichlet problem (1), (2).

Let X be an arbitrary linear space over the field of real numbers, $\mathbb{Z}$ – the set of integers. By X^∞ we denote a linear space of mappings $\mathbf{u} : \mathbb{Z} \to X$ satisfying $u(k) = 0$ when $k < 0$. For any element $\mathbf{u} \in X^\infty$ we have $u_k \equiv (\mathbf{u})_k := \mathbf{u}(k), k \in \mathbb{Z}$, and will write it as $\mathbf{u} := (u_0, u_1, ..., u_k, ...)^\top$. Henceforth we will call elements of X^∞ sequences.

Let $\widetilde{\mathbf{E}}(x, y) = \left(\widetilde{E}_0(x, y), \; \widetilde{E}_1(x, y), \; ...\right)^\top, \; x, y \in \mathbb{R}^3$, be a fundamental solution of the system (1) and sequence $\mathbf{E}(x, y) = (E_0(x, y), \; E_1(x, y), \; ...)^\top$ is calculated by the formula

$$E_i(x, y) := \widetilde{E}_i(x, y) - \widetilde{E}_{i-1}(x, y), \; i \in \mathbb{N}, \;\; E_0(x, y) = \widetilde{E}_0(x, y), \; x, y \in \mathbb{R}^3. \tag{3}$$

Note that $E_0(x,y)=\frac{e^{-\sqrt{c_0}|x-y|}}{4\pi|x-y|}$. As for the other components see, for example, [17].

Consider a sequence of functions $\mathbf{V}\xi(x)=(V_0\xi(x),\ V_1\xi(x),\ ...)^\top$ with components

$$V_j\xi(x):=\big(V_j\xi\big)(x)=\int\limits_\Gamma \xi(y)E_j(x,y)d\Gamma_y,\ j\in\mathbb{N}_0,\ x\in\mathbb{R}^3, \tag{4}$$

where ξ is a square integrable on Γ function. It is known [17] that sequence $\mathbf{u}(x)=(u_0(x),\ u_1(x),\ ...)^\top$ built for an arbitrary sequence $\boldsymbol{\mu}=(\mu_0,\ \mu_1,\ ...)^\top$ of square integrable on Γ functions by the rule

$$u_i(x)=\sum_{j=0}^{i}V_j\mu_{i-j}(x),\ i\in\mathbb{N}_0,\ x\in\mathbb{R}^3, \tag{5}$$

will satisfy the system (1). Then in order for the sequence $\mathbf{u}$ to be a solution of the Dirichlet problem for the given sequence $\mathbf{g}=(g_0,\ g_1,\ ...)^\top$ it is enough to find such sequence $\boldsymbol{\mu}$ that would satisfy on Γ the following equalities

$$\begin{cases} V_0\mu_0=g_0,\\ V_1\mu_0+V_0\mu_1=g_1,\\ V_2\mu_0+V_1\mu_1+V_0\mu_2=g_2,\\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\ V_k\mu_0+V_{k-1}\mu_1+...+V_0\mu_k=g_k,\\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{6}$$

Lets introduce some notations. We will use the Lebesgue space $L_2(\Omega^+)$ and Sobolev spaces $H^1(\Omega^+)$ of real-valued scalar functions. Let $\gamma_0^+ : H^1(\Omega^+)\to H^{1/2}(\Gamma)$ be the trace operator, $H^{-1/2}(\Gamma):=\big(H^{-1/2}(\Gamma)\big)'$ and $\langle\cdot,\cdot\rangle_\Gamma$ denote the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Definition 4. Let $\mathbf{g}\in(H^{1/2}(\Gamma))^\infty$. Sequence $\mathbf{u}\in(H^1(\Omega^+))^\infty$ is called a generalized solution of the Dirichlet problem if it satisfies the system (1) in the sense of distributions and the boundary condition (2) in the sense of traces.

Definition 5 ([10])**.** Let X, Y and Z be arbitrary linear spaces and $q : X\times Y\to Z$ – some mapping. By a q-convolution of sequences $\mathbf{u}\in X^\infty$ and $\mathbf{v}\in Y^\infty$ we understand a sequence $\mathbf{w}\in Z^\infty$ whose components are defined by the following rule

$$w_i:=\sum_{j=0}^{i}q\left(u_{i-j},v_j\right),\ i\in\mathbb{N}_0, \tag{7}$$

and denote it $\mathbf{w}=\mathbf{u}\underset{q}{\circ}\mathbf{v}$.

In case when $X=H^{-1/2}(\Gamma),\ Y=H^{1/2}(\Gamma),\ Z=\mathbb{R}$ and $q(u,v):=<u,v>_\Gamma$, $u\in H^{-1/2}(\Gamma),\ v\in H^{1/2}(\Gamma)$, for the components of the q-convolution of arbitrary sequences $\mathbf{u}\in\big(H^{-1/2}(\Gamma)\big)^\infty$ and $\mathbf{v}\in\big(H^{1/2}(\Gamma)\big)^\infty$ we have the following formula

$$w_j = \sum_{i=0}^{j} < u_{j-i}, v_i >_\Gamma, \ \ j \in \mathbb{N}_0, \tag{8}$$

and write $\mathbf{w} := \mathbf{u} \underset{\Gamma}{\circ} \mathbf{v}$.

Another example of q-convolution is related to linear operators, when $X = \mathcal{L}(Y, Z)$ is a space of linear operators that act from Y into Z, and $q(A, v) := Av$, $A \in \mathcal{L}(Y, Z), \ v \in Y$. In this case for the components of the q-convolution of arbitrary sequences $\mathbf{A} \in \big(\mathcal{L}(Y, Z)\big)^\infty$ and $\mathbf{v} \in Y^\infty$ we obtain the formula

$$w_j = \sum_{i=0}^{j} A_{j-i} v_i, \ \ j \in \mathbb{N}_0, \tag{9}$$

and write $\mathbf{w} := \mathbf{A} \underset{Z}{\circ} \mathbf{v}$.

Definition 6 ([14]). Let $\mathbf{V} : \big(H^{-1/2}(\Gamma)\big)^\infty \to \big(H^{1/2}(\Gamma)\big)^\infty$ be a sequence of operators that act by the rule (4), where we consider the inner product in $L^2(\Gamma)$ extended to the duality on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\boldsymbol{\mu} \in \big(H^{-1/2}(\Gamma)\big)^\infty$. Sequence

$$\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu}(x) := (\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu})(x), \ \ x \in \mathbb{R}^3, \tag{10}$$

is called a single layer potential of the system (1) on the surface Γ.

Using the introduced notations, we can rewrite the system (6) as

$$\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu} = \mathbf{g} \text{ on } \Gamma. \tag{11}$$

We will call systems of type (11) that can be represented by a q-convolution systems with a convolutional structure. It is easy to see that the system of PDEs (1) also has a convolutional structure since the expressions in it's left had side (that are not related to the Laplacian) are components of the q-convolution of sequences $\mathbf{c}$ and $\mathbf{u}$.

Proposition 6 ([14]). *For an arbitrary sequence $\mathbf{g} \in l^2(H^{1/2}(\Gamma))$ there exists a unique generalized solution of the Dirichlet problem $\mathbf{u} \in l^2(H^1(\Omega))$. It can be represented as a single layer potential* (10) *whose density $\boldsymbol{\mu} \in l^2(H^{-1/2}(\Gamma))$ is a solution of the BIE* (11).

3. Boundary Elements Method for BIE System

Triangular shape of system (11) is a consequence of the convolutional structure of (1) and the application of the q-convolution in the single layer potential definition. Lets use this property to build a step-by-step process of the numerical solution of the BIE (11). This system can be represented as a sequence of Fredholm BIEs of the first kind:

$$V_0 \mu_k = \tilde{g}_k \quad \text{в} \quad H^{1/2}(\Gamma), \ \ k \in \mathbb{N}_0, \tag{12}$$

where

$$\tilde{g}_k := g_k - \sum_{i=0}^{k-1} V_{k-i} \mu_i. \tag{13}$$

As you can see, the system is reduced to a sequence of equations that have the form

$$V_0\eta = f \quad \text{в} \quad H^{1/2}(\Gamma). \tag{14}$$

They have two important properties. First, the left-hand side of the integral equation with an arbitrary index $k \in \mathbb{N}$ is defined by the same boundary operator V_0 and the right-hand side depends on the boundary condition data and on the solutions of the equations with previous indexes $i = \overline{0, k-1}$. Taking these considerations into account during the the implementation of the method makes it possible to build efficient algorithms for the numerical solution of the obtained sequence of BIEs (12) as well as for the computation of the solutions of the boundary problem.

Another feature of the obtained system is that the boundary integral operator on left-hand side of the equations corresponds to the elliptic operator $c_0 I - \Delta$, where I is the identity operator, and is well studied in the literature (see, e.g., [2, 4, 5, 13]). In our case, it gives us the opportunity not only to prove the existence and the uniqueness of the solutions of the obtained sequence of BIEs, but also to get the corresponding numerical solutions using BEM, which is considered as a representative of the Bubnov-Galerkin method family [8]. A large number of publications (see, e.g., the literature review in [9, 20]) confirms the effectiveness and the versatility of this method regarding the numerical solution of boundary value problems for different types of elliptic equations and systems of elliptic equations of smaller dimension.

Investigation of the solutions of BIE (14) and the approximation by the Bubnov-Galerkin scheme is based on the ellipticity and the boundedness of the operator V_0:

$$\langle V_0\eta, \eta\rangle_\Gamma \geq c_1 ||\eta||^2_{H^{-1/2}(\Gamma))}, \quad ||V_0\eta||_{H^{1/2}(\Gamma))} \leq c_2 ||\eta||_{H^{-1/2}(\Gamma))}, \quad \forall \eta \in H^{-1/2}(\Gamma)),$$

where $c_1 > 0$ and $c_2 > 0$ are constants.

Consider a sequence of finite-dimensional subspaces $X_M \subset H^{-1/2}(\Gamma)$, $M \in \mathbb{N}$, that are linear spans of functions $\{\phi_i\}_{i=1}^M$ that form a basis in X_M. According to the Bubnov-Galerkin method, we seek a numerical solution of the equation (14) in the form of a linear combination

$$\eta^M := \sum_{i=1}^{M} \eta_i \phi_i \in X_M \tag{15}$$

as a solution of such variational problem

$$\langle V_0 \eta^M, \eta\rangle_\Gamma = \langle f, \eta\rangle_\Gamma, \quad \forall \eta \in X_M. \tag{16}$$

In order to find the vector of the unknown coefficients $\boldsymbol{\eta}^{[M]} := \{\eta_i\}_{i=1}^M \in \mathbb{R}^M$ lets take the basis functions ϕ_j as the test ones. Then from the variational equations we obtain a system of linear algebraic equations (SLAE) regarding the unknown coefficients η_i:

$$V_0^{[M]} \boldsymbol{\eta}^{[M]} = \mathbf{f}^{[M]}, \tag{17}$$

where $V_0^{[M]}[j,i] := \langle V_0\phi_i, \phi_j\rangle_\Gamma$, $f_j^{[M]} := \langle f, \phi_j\rangle_\Gamma$, $\ i, j = \overline{1, M}$.

Note that the matrix of the obtained system if symmetric. Moreover, as a result of the $H^{-1/2}(\Gamma)$-ellipticity of the operator V_0, it is positive definite. Therefore, with an arbitrary right-hand side the system (17) will have a unique solution i.e. $\forall M \in \mathbb{N}$ by using the Bubnov-Galerkin method we will get an approximate solution of the equation (14). By the Cea lemma (see, e.g., [20, Theorem 8.1]) such approximate solution satisfies the inequality

$$||\eta_M||_{H^{-1/2}(\Gamma)} \leq c_1||f||_{H^{1/2}(\Gamma))}, \tag{18}$$

and there exists an estimate for its error

$$||\eta - \eta_M||_{H^{-1/2}(\Gamma)} \leq \frac{c_2}{c_1} \inf_{\xi \in X^M} ||\eta - \xi||_{H^{-1/2}(\Gamma)}. \tag{19}$$

Hence the convergence in $H^{-1/2}(\Gamma)$ of the approximate solution $\eta_M \to \eta \in H^{-1/2}(\Gamma)$ when $M \to \infty$, where η is the solution of the corresponding BIE in the sequence (12). Note that convergence of the numerical solution follows from the approximation property of the trial space X_M.

Lets specificate the numerical scheme (17) using the boundary elements method [8, 19, 20]. Let $\Gamma_{\widetilde{M}} = \bigcup_{l=1}^{\widetilde{M}} \overline{\tau}_l$ be some approximation of the surface Γ built by triangular boundary elements $\{\tau_l\}_{l=1}^{\widetilde{M}}$ with vertices $\{x^{[l_1]}, x^{[l_2]}, x^{[l_3]}\}$ and $h := \max\limits_{l=\overline{1,\widetilde{M}}} \left(\int_{\tau_l} \mathrm{d}s \right)^{1/2}$ – parameter of the approximation. We assume that vertices of all triangles have global numeration $\{x_k\}_{k=1}^{M^*}$.

Lets build a set of linearly-independent on $\Gamma_{\widetilde{M}}$ piece-wise constant functions $\{\varphi_l^0\}_{l=1}^{M}$, $M = \widetilde{M}$:

$$\varphi_l^0(x) = \begin{cases} 1, & x \in \tau_l, \\ 0, & x \notin \tau_l. \end{cases} \tag{20}$$

We will consider finite-dimensional spaces of functions $S_h^0(\Gamma) := X^M = \operatorname{span}\{\varphi_l^s\}_{l=1}^{M}$, $\dim S_h^0(\Gamma) = M$ as approximating spaces for the numerical scheme (17).

Let the operator equation (14) correspond to some k-th equation of the sequence (12). Its approximate (numerical) solution μ_k^h can be represented as a linear combination of piece-wise constant functions:

$$\mu_k^h = \sum_{l=1}^{M} \mu_{k,l}^h \varphi_l^0 \ \in S_h^0(\Gamma), \ \ k \in \mathbb{N}_0. \tag{21}$$

Here $\left\{\mu_{k,l}^h\right\}_{l=1}^{M} =: \boldsymbol{\mu}_k^h \in \mathbb{R}^M$ is a vector of unknown coefficients that can be found from the following system of algebraic equations:

$$\mathbf{V}_0^h \boldsymbol{\mu}_k^h = \tilde{\mathbf{g}}_k^h, \ \ k \in \mathbb{N}_0. \tag{22}$$

Matrix $\mathbf{V}_0^h$ is a concrete representation of the matrix of the system (17). Its elements can be given as

$$V_0^h[i,l] = \int_{\tau_i} \int_{\tau_l} E_0(x-y) \mathrm{d}s_y \mathrm{d}s_x, \ \ i,l = \overline{1,M}, \tag{23}$$

and the components of the right-hand side vector in (22) have the following form

$$\tilde{g}_k^h[i] = \int_{\tau_i} \Big\{ g_k(x) - \sum_{j=0}^{k-1} \big(V_{k-j}\mu_j^h\big)(x) \Big\} \mathrm{d}s_x, \;\; j = \overline{1, M}. \tag{24}$$

Sequence $\boldsymbol{\mu}^h := \big(\mu_0^h, \mu_1^h, ...\big)^\top$ can be treated as a numerical solution of the system of BIEs (12). After finding the consequent solution $\boldsymbol{\mu}_k^h$ of the algebraic system (22), we can approximate the corresponding density element using the formula (21) and calculate the k-th component of the numerical solution of the Dirichlet problem at an arbitrary point $x \in \Omega^+$:

$$u_k^h(x) = \sum_{j=0}^{k} \big(V_{k-j}\mu_j^h\big)(x), \; x \in \Omega^+. \tag{25}$$

The sequence $\mathbf{u}^h := \big(u_0^h, u_1^h, ...\big)^\top$ can be treated as a numerical solution of the Dirichlet problem.

Lets find an apriory estimate for the error of its components after introducing some Sobolev spaces [9]. Let the boundary Γ be given as a union $\Gamma = \bigcup_{i=1}^{\widetilde{N}} \overline{\Gamma}_i$ of surfaces Γ_i ($\Gamma_i \cap \Gamma_j = \oslash$ when $i \neq j$) each of which has a sufficiently smooth parameterization

$$\Gamma_i := \big\{ x \in \mathbb{R}^3 \, : \, x = \widetilde{\chi}_i(\xi), \; \xi \in \widetilde{\tau}_i \subset \mathbb{R}^2 \big\}.$$

By using a set of non-negative functions $\phi_i \in C_0^\infty(\mathbb{R}^3)$ such that

$$\sum_{i=1}^{\widetilde{N}} \phi_i(x) = 1 \;\; \forall x \in \Gamma, \quad \phi_i(x) = 0 \;\; \forall x \in \Gamma \setminus \Gamma_i,$$

each function v given on the boundary Γ can be written in a form

$$v(x) = \sum_{i=1}^{\widetilde{N}} \phi_i(x) v(x) = \sum_{i=1}^{\widetilde{N}} v_i(x) \;\; \forall x \in \Gamma, \tag{26}$$

where $v_i(x) := \phi_i(x) v(x) \;\; \forall x \in \Gamma_i$. We consider the Sobolev spaces $H^m(\widetilde{\tau}_i)$ when $m \in \mathbb{N}_0$, elements of which are functions $\widetilde{v}_i(\xi) := v_i(\widetilde{\chi}_i(\xi))$ when $\xi \in \widetilde{\tau}_i$, with a norm and a half-norm

$$||\widetilde{v}_i||_{H^m(\widetilde{\tau}_i)} := \Bigg(\sum_{|\alpha| \le m} ||\partial^\alpha \widetilde{v}_i||^2_{L^2(\widetilde{\tau}_i)} \Bigg)^{1/2}, \;\; |\widetilde{v}_i|_{H^m(\widetilde{\tau}_i)} := \Bigg(\sum_{|\alpha| = m} |\partial^\alpha \widetilde{v}_i|^2_{L^2(\widetilde{\tau}_i)} \Bigg)^{1/2}, \tag{27}$$

correspondingly. Here ∂^α is a notation of the partial derivative with a multi-index $\alpha = (\alpha_1, \alpha_2)$. Then for the functions, given on the whole surface Γ, we will use the Sobolev spaces $H^m(\Gamma)$ with a norm and a half-norm

$$||v||_{H^m(\Gamma)} := \Bigg(\sum_{i=1}^{\widetilde{N}} ||\widetilde{v}_i||^2_{H^m(\widetilde{\tau}_i)} \Bigg)^{1/2}, \;\; |v|_{H^m(\Gamma)} := \Bigg(\sum_{i=1}^{\widetilde{N}} |\widetilde{v}_i|^2_{H^m(\widetilde{\tau}_i)} \Bigg)^{1/2}, \tag{28}$$

correspondingly.

For non-integer values of the indexes $s = m + \sigma$, $m \in \mathbb{N}_0$, $\sigma \in (0,1)$, we will use Sobolev-Slobodetski spaces $H^s(\widetilde{\tau}_i)$ and $H^s(\Gamma)$ with corresponding half-norms and norms

$$|\widetilde{v}_i|_{H^s(\widetilde{\tau}_i)} := \left(\sum_{|\alpha|=m} \int_{\widetilde{\tau}_i} \int_{\widetilde{\tau}_i} \frac{|\partial^\alpha \widetilde{v}_i(\xi) - \partial^\alpha \widetilde{v}_i(\eta)|^2}{|\xi - \eta|^{2+2\sigma}} ds_\xi ds_\eta \right)^{1/2},$$

$$||\widetilde{v}_i||_{H^s(\widetilde{\tau}_i)} := \left(||\widetilde{v}_i||^2_{H^m(\widetilde{\tau}_i)} + |\widetilde{v}_i|^2_{H^s(\widetilde{\tau}_i)} \right)^{1/2}, \tag{29}$$

$$|v|_{H^s(\Gamma)} := \left(\sum_{i=1}^{\widetilde{N}} |\widetilde{v}_i|^2_{H^s(\widetilde{\tau}_i)} \right)^{1/2}, \quad ||v||_{H^s(\Gamma)} := \left(||v||^2_{H^m(\Gamma)} + |v|^2_{H^s(\Gamma)} \right)^{1/2},$$

and also spaces of piece-wise smooth functions

$$H^s_{pw}(\Gamma) := \left\{ v \in L^2(\Gamma) : v|_{\Gamma_i} \in H^s(\Gamma_i) \right\}, \tag{30}$$

for which

$$||v||_{H^s_{pw}(\Gamma)} := \left(\sum_{i=1}^{\widetilde{N}} ||v_{|\Gamma_i}||^2_{H^s(\Gamma_i)} \right)^{1/2}, \quad |v|_{H^s_{pw}(\Gamma)} := \left(\sum_{i=1}^{\widetilde{N}} |v_{|\Gamma_i}|^2_{H^s(\Gamma_i)} \right)^{1/2}. \tag{31}$$

Lemma 1. *Let $\boldsymbol{\mu} \in \left(H^s_{pw}(\Gamma)\right)^\infty$ be a solution of the system* (12) *for some $s \in (0,1]$, that satisfies the inequality*

$$\sum_{j=0}^{\infty} |\mu_j|_{H^s_{pw}(\Gamma)} < +\infty. \tag{32}$$

Then for the components of the numerical solutions of the system of BIEs (12) *and the Dirichlet problem* (1), (2) *obtained by BEM the following asymptotic estimates hold*

$$\left\| \mu_k - \mu_k^h \right\|_{H^{-1/2}(\Gamma)} \le c_k h^{s+1/2} |\mu_k|_{H^s_{pw}(\Gamma)}, \ k \in \mathbb{N}_0, \tag{33}$$

$$|u_k(x) - u_k^h(x)| \le \tilde{c}_k h^{s+1/2} \sum_{j=0}^{k} |\mu_j|_{H^s_{pw}(\Gamma)}, \quad x \in \Omega^+, \quad k \in \mathbb{N}_0, \tag{34}$$

where c_k and $\tilde{c}_k$ are some values that do not depend on the parameter h.

Proof. Validity of the statement regarding (33) directly follows from a known theorem ([7], [20, Theorem 12.3]).

A priory error of the k-th component of the numerical solution of the Dirichlet problem at an arbitrary point $x \in \Omega^+$ can be given as

$$|u_k(x) - u_k^h(x)| = \Big| \sum_{i=0}^{k} V_{k-i}\big(\mu_i - \mu_i^h\big)(x) \Big| = \Big| \sum_{i=0}^{k} \langle \big(\mu_i - \mu_i^h\big), \ E_{k-i}(x - \cdot) \ \rangle_\Gamma \Big|.$$

Note, that for an arbitrary fixed point $x \in \Omega^+$ all the functions $E_j(x - \cdot)$ are infinitely-differentiable and bounded together with all their derivatives on Γ, i.e. $\left\|E_j(x-\cdot)\right\|_{H^{1/2}(\Gamma)} \le c_j^* = const$. Using the generalized Cauchy-Schwarz inequality, we get

$$|u_k(x) - u_k^h(x)| \le \sum_{i=0}^{k} \left|\langle (\mu_i - \mu_i^h),\ E_{k-i}(x-\cdot)\,\rangle_\Gamma\right| \le$$
$$\le \sum_{i=0}^{k} \left\|\mu_i - \mu_i^h\right\|_{H^{-1/2}(\Gamma)} \left\|E_{k-i}(x-\cdot)\right\|_{H^{1/2}(\Gamma)}.$$

Then, taking into account the inequality (33), we obtain

$$|u_k(x) - u_k^h(x)| \le h^{s+1/2} \sum_{i=0}^{k} c_{k-i}^* c_i |\mu_i|_{H^s_{pw}(\Gamma)} \le \tilde{c}_k h^{s+1/2} \sum_{i=0}^{k} |\mu_i|_{H^s_{pw}(\Gamma)},$$

where $\tilde{c}_k = \max\limits_{0\le i\le k} \{c_{k-i}^* c_i\}$ does not depend on the parameter h. □

4. Computational aspects of the method

Effectiveness of the numerical solution of the Dirichlet problem depends in great length on the approaches for the calculation of the surface potential in the domain and the trace on the boundary. In practice, it means a combination of algorithms for numerical integration and analytic calculation of some singular integrals over the boundary elements.

If the point, at which the trace of the potentials mentioned above is calculated, is not located on the boundary element over which the integration is performed, then the kernels of these potentials are infinitely-differentiable functions on the corresponding boundary element. Hence, the calculation of the majority of the elements in corresponding SLAE and also the components of the numerical solution of the problem at the observational points can be performed using numerical integration and the Gauss quadrature in particular.

Lets consider the calculation of integrals over singular functions that can be obtained during the construction of the matrix of the SLAE and correspond to the boundary operator $\mathbf{V}_0$ (23):

$$V_0^h[k,l] = \frac{1}{4\pi} \int_{\tau_k} \int_{\tau_l} \frac{e^{-\sqrt{c_0}|x-y|}}{|x-y|} ds_y ds_x, \quad k,l = \overline{1,M}. \tag{35}$$

If the boundary elements τ_k and τ_l coincide or are adjacent then the integrand of the internal integral has a weak singularity when the points $x \in \tau_k$ and $y \in \tau_l$ coincide. It can be explicitly eliminated if the element of the matrix is given as

$$V_0^h[k,l] = \frac{1}{4\pi} \int_{\tau_k} \int_{\tau_l} \frac{e^{-\sqrt{c_0}|x-y|} - 1}{|x-y|} ds_y ds_x + \frac{1}{4\pi} \int_{\tau_k} I_l(x) ds_x, \tag{36}$$

where

$$I_l(x) = \int_{\tau_l} \frac{1}{|x-y|} ds_y. \tag{37}$$

Integrand of the first integral in (36) allows continuous definition at $x = y$ (it can be verified if the exponential function is expanded in a Maclaurin series over the variable $r = |x-y|$), so the value of this integral can be found numerically using the Gauss quadrature rules. The integral (37) can be found analytically as a function [11,18–20], parameterized by the geometric data of the boundary element τ_l and the coordinates of the point x.

In the integrals

$$V_j^h[k,l] = \int_{\tau_k}\int_{\tau_l} E_j(x,y)ds_y ds_x, \quad k,l = \overline{1,M}, \; j \in \mathbb{N}, \tag{38}$$

that correspond to the boundary operator $\mathbf{V}_j$, $j \in \mathbb{N}$, and are found during the construction of the right-hand side, the integrands are continuous for any location of the boundary elements τ_k and τ_l. Hence these integrals can also be found numerically using the Gauss quadratures.

Note, that all relations of the suggested approach can be applied to interior BVP without any changes.

5. Results of the computational experiment

Lets demonstrate the usage of the suggested method to find numerical solutions of some model Dirichlet problems. We assume that in (1) and (2) components of the sequences $\mathbf{c}$ and $\mathbf{g}$ have the form $c_k = (k+1)\kappa$ and $g_k = v_k$, $k \in \mathbb{N}_0$, correspondingly, where κ is some parameter and the sequence $\mathbf{v}$ consists of functions

$$\begin{aligned} v_k(x) =& \frac{e^{-\kappa(|x-x^*|-1)}\big(L_k(\kappa(|x-x^*|) - L_{k-1}(\kappa(|x-x^*|)\big)}{|x-x^*|}, \quad k \in \mathbb{N}, \\ v_0(x) =& \frac{e^{-\kappa(|x-x^*|-1)}}{|x-x^*|}, \end{aligned} \tag{39}$$

parameterized by some point x^*, L_k, $k \in \mathbb{N}_0$, are the Laguerre polynomials [1]. Up to a factor the sequence $\mathbf{v}$ coincides with the fundamental solution of the system (1), so it will be used to build the analytical solution of the Dirichlet problem. Note, that the variable x will denote points on the boundary Γ and in the domain where the numerical solution is sought, and the parameter x^* is located in the complement of this domain to the whole space $\mathbb{R}^3$.

We consider the following domains in the model problem: a unit sphere, its exterior in $\mathbb{R}^3$, a cube $\Omega := (-1,1)\times(-1,1)\times(-1,1)$ and its exterior $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$.

Lets consider first the model boundary value problems for the first equation of the system (1).

Example 1. Find a numerical solution u_0^h of the exterior ($x^* = (0,0,0)$) and interior ($x^* = (2,0,0)$) Dirichlet problems in case of the cubic boundary when $g_0 = v_0$.

Table 1 contains corresponding numerical solutions of the exterior problem using the decomposition of the cube's boundary into $\overline{M} = 1200$ boundary elements. As we can see, with increasing value of κ the solutions are decreasing rapidly when moving further from the boundary. Next, we examine the errors of the numerical solutions of this problem with a fixed value of the parameter κ, for example, take $\kappa = 2$.

TABL. 1. Numerical solutions $u_0^h(x)$ of the problem 1 for different values of κ

x_1	Value of the parameter κ				
	0.5	1.0	2.0	4.0	8.0
1.2	$7.48314 \cdot 10^{-1}$	$6.75199 \cdot 10^{-1}$	$5.49666 \cdot 10^{-1}$	$3.64214 \cdot 10^{-1}$	$1.59908 \cdot 10^{-1}$
2.0	$3.02031 \cdot 10^{-1}$	$1.82901 \cdot 10^{-1}$	$6.70731 \cdot 10^{-2}$	$9.01743 \cdot 10^{-3}$	$1.62947 \cdot 10^{-4}$
3.0	$1.22230 \cdot 10^{-1}$	$4.49190 \cdot 10^{-2}$	$6.06765 \cdot 10^{-3}$	$1.10698 \cdot 10^{-4}$	$3.68426 \cdot 10^{-8}$
4.0	$5.56144 \cdot 10^{-2}$	$1.23988 \cdot 10^{-2}$	$6.16492 \cdot 10^{-4}$	$1.52428 \cdot 10^{-6}$	$9.29979 \cdot 10^{-12}$

TABL. 2. Errors of the numerical solution $u_0^h(x)$ of the problem 1

	Exterior problem			Interior problem		
$\overline{M}$	δ^h	eoc	$\epsilon^h(\%)$	δ^h	eoc	$\epsilon^h(\%)$
300	0.01384		3.10	0.01324		2.99
588	0.00702	2.018	1.55	0.00673	2.012	1.50
768	0.00537	2.010	1.18	0.00515	2.005	1.14
972	0.00421	2.061	0.93	0.00404	2.058	0.90
1200	0.00340	2.030	0.75	0.00326	2.027	0.72
1728	0.00234	2.039	0.51	0.00225	2.037	0.50
2700	0.00149	2.033	0.33	0.00143	2.031	0.32

In order to find the dependency between the error of the numerical solution and the parameter h that defines the triangulation of the boundary surface we will consider the values $\delta^h := ||u_0^h - u_0||_{L^2(a,b)}$ and $\epsilon^h := \frac{\delta^h}{||u_0||_{L^2(a,b)}} \cdot 100\%$, where (a, b) is an inverval in space from which the points of observation x are taken. We will also calculate the value of the estimated order of convergence [19]

$$eoc := \frac{\ln \delta^{h_j} - \ln \delta^{h_{j+1}}}{\ln h_j - \ln h_{j+1}}, \tag{40}$$

where h_j and h_{j+1} are the parameters of the two consequent triangulations of the boundary surface into boundary elements. Results of the calculations given in table 2 highlight the equal orders of errors of the numerical solutions of the interior and exterior problems. Moreover, the obtained result has $eoc \approx 2.0$.

Now lets demonstrate that the developed method gives us ability to find components of the numerical solutions with other values of the indexes.

Example 2. Find N components of the numerical solution u_i^h, $i = \overline{0, N}$, of the exterior Dirichlet problem (1), (2) if $\tilde{h}_i = v_i$, $\kappa = 2$ and $x^* = (0, 0, 0)$.

Charts of the obtained numerical solutions are given on figure 1. They demonstrate rapid decrease of the functions $u_i^h(x)$, $i = 0, 10, 20$, with the increase of their index. Numerical solutions obtained on $\overline{M} = 1200$ boundary elements are given in table 3 and indicate the commensurability of the errors of components of the numerical solutions $u_i^h(x)$ when $i = 10$ and $i = 20$ with the corresponding error of $u_0^h(x)$.

TABL. 3. Solutions $u_i^h(x)$, $i = 10, 20$ of the problem 2 when $\overline{M} = 1200$.

x_1	$u_{10}(x)$	$u_{10}^h(x)$	$u_{20}(x)$	$u_{20}^h(x)$
1.5	$8.8570 \cdot 10^{-2}$	$9.0932 \cdot 10^{-2}$	$-7.6672 \cdot 10^{-2}$	$-7.5956 \cdot 10^{-2}$
2.0	$5.6502 \cdot 10^{-2}$	$5.6254 \cdot 10^{-2}$	$4.0784 \cdot 10^{-2}$	$4.1496 \cdot 10^{-2}$
3.0	$-1.9676 \cdot 10^{-2}$	$-1.9664 \cdot 10^{-2}$	$-1.0549 \cdot 10^{-2}$	$-1.0619 \cdot 10^{-2}$
4.0	$4.3413 \cdot 10^{-3}$	$4.3359 \cdot 10^{-3}$	$2.9939 \cdot 10^{-3}$	$3.0045 \cdot 10^{-3}$

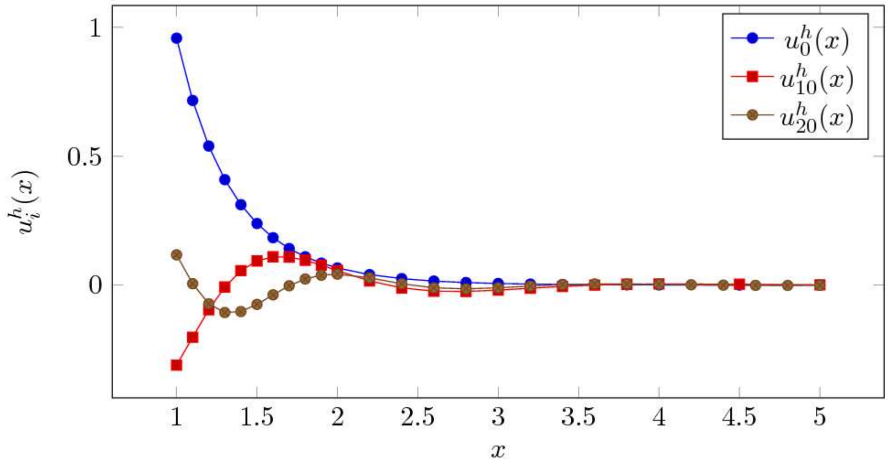

FIG. 1. Charts of the components $u_0^h(x)$, $u_{10}^h(x)$, $u_{20}^h(x)$ of the numerical solution of the problem 2 when $\overline{M} = 768$

As it has been mentioned above, the Dirichlet problem (1), (2) can be obtained by means of the application of the Laguerre transform by the time variable to a certain class of linear evolutionary problems. For instance, the system (1), that is mentioned in problems 1 and 2, can be obtained from a homogeneous wave equation with homogeneous boundary conditions. After finding for some N the components u_i^h, $i = \overline{0, N}$, the numerical solution of the mixed problem can be given as a partial sum of the Laguerre-Fourier expansion

$$u^{h,N}(x,t) = \frac{1}{\kappa} \sum_{i=0}^{N} u_i^h(x) L_i(\kappa t), \quad (x,t) \in \Omega^+ \times (0, \infty). \tag{41}$$

To generate the data for the boundary conditions (2) we use a "spherical impulse" with a center at x^*

$$v(x,t) = \frac{f(t - |x - x^*|)}{4\pi |x - x^*|}, \quad (x,t) \in \overline{\Omega^+} \times [0, \infty), \tag{42}$$

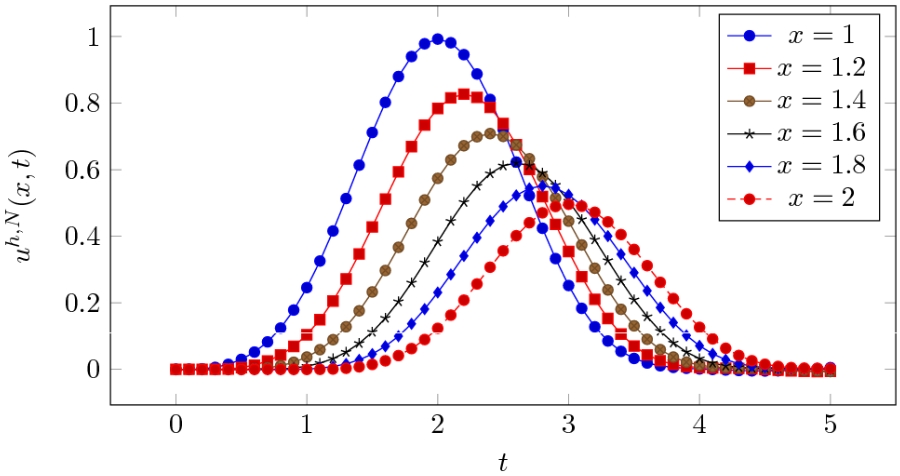

FIG. 2. Chart of the solution of the problem 3 in the exterior of the sphere with $N = 40$, $\overline{M} = 720$

where f is a cubical β-spline [12], and apply to it the Laguerre transform

$$v_k(x) = \int\limits_{\mathbb{R}_+} v(x,t)\, L_k(\kappa t)\, e^{-\kappa t} dt, \quad x \in \Gamma, \; k \in \mathbb{N}_0. \tag{43}$$

Example 3. In the exterior Ω^+ of the unit sphere calculate the numerical solution of the Dirichlet problem for the wave equation with homogeneous initial conditions and the boundary condition defined by (42) at $x^* = (0,0,0)$.

Let the problem (1), (2) correspond to the initial-boundary value problem 3 when $\kappa = 2$. After finding $N = 40$ components of the numerical solution u_i^h, $i = \overline{0,N}$, with the use of $\overline{M} = 720$ boundary elements, the numerical solution of the problem 3 at the points along the axis Ox_1 is calculated by the formula (41). As it can be seen from the charts of the numerical solution, given on the figure 2, the obtained results are well representing the physics of the wave propagation from the boundary surface, especially, passing through the observation points of the front and rear disturbance fronts.

Note that the formulation of the problem 3 gives us ability to find the coefficients u_i, $i \in \mathbb{N}_0$, of the expansion of the precise solution $u(x,t)$ into series (41) analytically. So it can be compared how the partial sums of the series (41) with analytical coefficients and coefficients found by the suggested approach approximate the precise solution of the evolution problem. As it can be see from the table 4, values of such partial sums are pointwise (regarding the time variable) close.

6. Conclusions

Application of the surface potentials built using the q-convolution operation is an effective way to obtain the integral representation of the solutions of

TABL. 4. Comparison of the numerical solution of the problem 3 $u^{h,N}(x,t)$ (the row above) with the values of the partial sum (41)(the row below), in which the coefficients are calculated analytically

t	$x_1 = 1.0$	$x_1 = 1.2$	$x_1 = 1.4$	$x_1 = 1.6$	$x_1 = 1.8$	$x_1 = 2.0$
0.0	0.00037	0.00043	-0.00011	0.00013	-0.00010	-0.00013
	0.00012	-0.00034	0.00012	0.00000	0.00008	-0.00004
0.4	0.01521	0.00136	-0.00007	-0.00006	-0.00003	-0.00002
	0.01595	0.00178	-0.00006	-0.00004	0.00004	0.00001
1.2	0.41588	0.20541	0.08926	0.03234	0.00852	0.00100
	0.42386	0.20880	0.09113	0.03376	0.00898	0.00100
2.0	0.99249	0.78406	0.57393	0.38364	0.23108	0.12327
	0.99860	0.78846	0.57774	0.38853	0.23510	0.12553

boundary value problems for infinite systems of PDE with convolutional structure. Such approach makes it possible to reduce the boundary value problem to an equivalent BIE system, develop efficient projection methods for its numerical solution and justify their usage. The results of a series of numerical experiments that confirm the theoretical statements and demonstrate the applicability of the proposed methods for modeling of evolutionary processes are given.

BIBLIOGRAPHY

1. Abramowitz M. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables / M. Abramowitz, I. Stegun. — Dover Publications, 1965. — 1046 p.
2. Bamberger A. Formulation variationnelle espace-temps pour le calcul par potentiel retarde de la diffraction d'une onde acoustique (I) / A. Bamberger, A. Ha-Duong // Math. Methods Appl. Sci. – 1986. – Vol. 8, № 3. – P. 405-435.
3. Chapko R. On the numerical solution of initial boundary value problems by the Laguerre transformation and boundary integral equations / R. Chapko, R, Kress // Integral and Integrodifferential Equations: Theory, Methods and Applications. Series in Mathematical Analysis and Applications. – 2000. – № 2. – P. 55-69.
4. Costabel M. Boundary integral operators on Lipschitz domains: elementary results / M. Costabel // SIAM J. Math. Anal. – 1988. – № 19. – P. 613-626.
5. Dautray R. Mathematical analysis and numerical methods for science and technology, Vol. 2. Functional and Variational Methods / R. Dautray, Jacques-Louis Lions. – Berlin: Springer, 2000. – 592 p.
6. Galazyuk V. A. An integral equation method for time-depended diffraction problems // V. A. Galazyuk, J. V. Lyudkevych, A. O. Muzychuk. – Lviv. univ. – 1984. – Dep. v UkrNIINTI, № 601Uk-85 Dep. – 16 p. (in Russian).
7. Hsiao G. C. The Aubin–Nitsche lemma for integral equations / G. C. Hsiao, W. L. Wendland // J. Int. Equat. – 1981. – 3. – P. 299–315.
8. Hsiao G. C. Boundary element methods: foundation and error analysis / G .C. Hsiao, W. L. Wendland // Encyclopaedia of Computational Mechanics. – Chichester: John Wiley and Sons Publ., 2004. – Vol. 1. – P. 339–373.
9. Hsiao G. C. Boundary Integral Equations / G. C. Hsiao, W. L. Wendland. – Springer-Verlag: Berlin, 2008. – 640 p.

10. Litynskyy S. On weak solutions of boundary problems for an infinite triangular system of elliptic equations / S. Litynskyy, Yu. Muzychuk, A. Muzychuk // Visnyk of the Lviv university. Series of Applied mathematics and informatics. – 2009. Vol. 15. – P. 52–70. (in Ukrainian).
11. Litynskyy S. Boundary elements method for some triangular system of boundary integral equations / S. Litynskyy, Yu. Muzychuk // Proceedings of XIV International Seminar/Workshop on Direct and Inverse problem of Electromagnetic and Acoustic Wave Theory (DIPED-2009). – 2009. – P. 208-211.
12. Marchuk G. Introduction into projectional methods / G. Marchuk, V. Agoshkov. – M.: Nauka, 1981. – 416 p. (in Russian).
13. McLean W. Strongly Elliptic Systems and Boundary Integral Equations / William McLean. – Cambridge: Cambridge University Press, 2000. – 372 p.
14. Muzychuk Yu. On variational formulations of inner boundary value problems for infinite systems of elliptic equations of special kind / Yu. A. Muzychuk, R. S. Chapko // Matematychni Studii. – 2012. – Vol. 38, № 1. – P. 12-34.
15. Muzychuk Yu. A. On the boundary integral equation method for boundary-value problems for a system of elliptic equations of the special type in partially semi-infinite domains / Yu. Muzychuk, R. Chapko // Reports of the National Academy of Sciences of Ukraine. – 2012. – № 11. – P. 20-27. (in Ukrainian).
16. Muzychuk Yu. On numerical solution of interior boundary value problems for infinite systems of elliptic equations / Yu. Muzychuk // Visn. Lviv. univ. Ser. prykl. matem. inform. – 2013. – Vol. 20. – P. 49-56. (in Ukrainian).
17. Muzychuk Yu. On the boundary integral equation method for exterior boundary value problems for infinite systems of elliptic equations of special kind / Yu. Muzychuk // Journal of Computational and Applied Mathematics. – 2014. – Vol. 116, № 2. – P. 96-116.
18. Ostudin B. A. On numerical appoach to solve some three-dimensional boundary value problems in potential theory based on integral equation method / B. A. Ostudin, Y. S. Garasym // Journal of Computational and Applied Mathematics. – 2003. – Vol. 88, № 1. – P. 17-28.
19. Rjasanow S. The Fast Solution of Boundary Integral Equations / S. Rjasanow, O. Steinbach. – New-York: Springer Science+Business Media, LLC, 2007. – 284 p.
20. Steinbach O. Numerical Approximation Methods for Elliptic Boundary Value Problems / O. Steinbach. – Springer Science, 2008. – 396 p.
21. Vavrychuk V. H. Numerical solution of mixed non-stationary problem of thermal conductivity in partially unbounded domain / V. H. Vavrychuk // Visnyk of the Lviv university. Series of Applied mathematics and informatics. – 2011. – Vol. 17. – P. 62-72. (in Ukrainian).

Y. A. Muzychuk,
Faculty of Applied Mathematics and Informatics,
Ivan Franko National University of Lviv,
1, Universytets'ka Str., Lviv, 79000, Ukraine.

Received 25.05.2016