Журнал обчислювальної та прикладної математики | **2016** **№ 2 (122)** | Journal of Computational & Applied Mathematics

UDC 532.595

RESONANT LIQUID SLOSHING IN AN UPRIGHT CIRCULAR TANK PERFORMING A PERIODIC MOTION

I. A. Raynovskyy, A. N. Timokha

Резюме. Використовується слабо-нелінійна модальна теорія Нарімано-ва-Моісєєва для аналізу усталених резонансних хвиль в вертикальному циліндричному резервуарі, який рухається періодично з частотою, близькою до першої власної частоти коливання рідини.

Abstract. A weakly-nonlinear Narimanov-Moiseev type modal theory is used to analyse steady-state resonant waves in an upright circular tank which moves periodically with the forcing frequency close to the lowest natural sloshing frequency.

1. Introduction

The upright circular tank is relevant for spacecraft applications, the pressure-suppression pools of Boiling Water Reactors, storage tanks, Tuned Liquid Dampers, offshore towers, and basins of the aqua-cultural engineering. Resonant sloshing due to harmonic excitations of the tank was extensively studied, theoretically and experimentally, in [1,3,4,6]. For the longitudinal tank forcing, steady-state planar (in the excitation plane), swirling and irregular (chaotic) waves were detected [1, 4, 6] when the forcing frequency is close to the lowest natural sloshing frequency. A review on sloshing due to parametric (vertical) excitations is given in [3]. However, the above-mentioned industrial applications deal, normally, with the coupled rigid tank-and-sloshing dynamics when the tank performs complex three-dimensional motions which unnecessarily occur in either meridional plane or vertical direction. This causes an interest to analytical studies on the resonant steady-state sloshing due to a three-dimensional periodic tank excitation that are done in the present paper by employing the weakly-nonlinear modal system [7].

2. Statement of the problem

An inviscid incompressible contained liquid with irrotational flows sloshes in an upright circular rigid tank with radius r_0. The tank performs small-magnitude prescribed periodic sway, surge, roll, and pitch motions which are described by the r_0-scaled generalised coordinates $\eta_1(t)$ and $\eta_2(t)$ (horizontal tank motions) and angular perturbations $\eta_4(t)$ and $\eta_5(t)$ (see, figure 1). The yaw cannot excite sloshing within the framework of the inviscid potential flow model but the heave is not considered. All geometric and physical parameters

Key words. Sloshing, multimodal method; periodic solution; response curves.

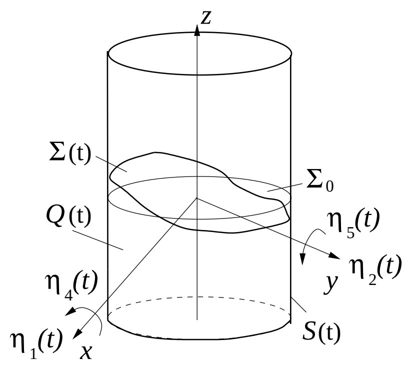

FIG. 1. The time-dependent liquid domain $Q(t)$ confined by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$. The free-surface evolution is considered in the tank-fixed coordinate system $Oxyz$ whose coordinate plane Oxy coincides with the mean (hydrostatic) free surface Σ_0 and Oz is the symmetry axis. Small-magnitude periodic tank motions are governed by the generalised coordinates $\eta_1(t)$ (surge), $\eta_4(t)$ (roll), $\eta_2(t)$ (sway), and $\eta_5(t)$ (pitch). The mean free surface Σ_0 is perpendicular to Oz

are henceforth considered *scaled* by r_0. We introduce a small parameter $0 < \epsilon \ll 1$ characterising the periodic forcing, i.e. $\eta_i(t) = O(\epsilon)$, $i = 1, 2, 4, 5$.

Figure 1 shows the time-dependent liquid domain $Q(t)$ with the free surface $\Sigma(t)$ (governed by the single-valued function $z = \zeta(r, \theta, t)$) and the wetted tank surface $S(t)$. The liquid flow is determined by the velocity potential $\Phi(r, \theta, z, t)$. The unknowns, ζ and Φ, are defined in the tank-fixed Cartesian (equivalent cylindrical) non-inertial coordinate system; they can be found from either the corresponding free-surface problem or its equivalent variational formulation. The latter formulation facilitates the multimodal method, which employs the Fourier-type representations of ζ and Φ in which the time-dependent coefficients are interpreted as *generalised coordinates and velocities*. The representations are normally based on the natural sloshing modes which are the eigenfunctions of the spectral boundary problem

$$\nabla^2\varphi = 0 \text{ in } Q_0, \quad \frac{\partial\varphi}{\partial n} = 0 \text{ on } S_0, \quad \frac{\partial\varphi}{\partial n} = \kappa\varphi \text{ on } \Sigma_0, \quad \int_{\Sigma_0} \varphi\,\mathrm{d}S = 0 \tag{1}$$

in the mean (hydrostatic) liquid domain Q_0 confined by the mean free surface Σ_0 and the wetted tank surface S_0. The r_0-scaled problem (1) has the analytical solution [4]

$$\varphi_{Mi}(r, z, \theta) = \mathcal{R}_{Mi}(r)\,\mathcal{Z}_{Mi}(z)\,\frac{\cos M\theta}{\sin M\theta}, \quad M = 0, \ldots; \; i = 1, \ldots, \tag{2a}$$

$$\mathcal{R}_{Mi}(r) = \alpha_{Mi} J_M(k_{Mi} r), \quad \mathcal{Z}_{Mi}(z) = \frac{\cosh(k_{Mi}(z + h))}{\cosh(k_{Mi} h)}, \tag{2b}$$

where $J_M(\cdot)$ is the Bessel functions of the first kind, the radial wave numbers k_{Mi} are determined by $\mathcal{R}'_{M,i}(r_1) = 0$ and the normalising multipliers α_{Mi} follow from the orthogonality condition

$$\lambda_{(Mi)(Mj)} = \int_{r_1}^{1} r\,\mathcal{R}_{Mi}(r)\,\mathcal{R}_{Mj}(r)\,\mathrm{d}r = \delta_{ij}, \quad i,j = 1,\ldots, \tag{3}$$

where δ_{ij} is the Kronecker delta. The eigenvalues κ_{Mi} and the natural sloshing frequencies σ_{Mi} read as

$$\kappa_{Mi} = k_{Mi}\tanh(k_{Mi}h) \text{ and } \sigma_{Mi}^2 = \kappa_{Mi}\,\bar{g}/r_0 = \kappa_{Mi}\,g, \tag{4}$$

respectively, where $\bar{g}$ is the dimensional gravity acceleration.

Dealing with a small-amplitude angular tank motion requires the linearised Stokes-Joukowski potentials $\Omega_{0i}(r,z,\theta)$, $i = 1,2,3$ which are harmonic functions satisfying the Neumann boundary conditions

$$\frac{\partial\Omega_{01}}{\partial n} = -(zn_r - rn_z)\sin\theta, \ \frac{\partial\Omega_{02}}{\partial n} = (zn_r - rn_z)\cos\theta, \ \frac{\partial\Omega_{03}}{\partial n} = 0 \tag{5}$$

on Σ_0 and the wetted tank surface S_0, where n_r and n_z are the outer normal components in the r- and z- directions. This implies $\Omega_{01} = -F(r,z)\sin\theta$, $\Omega_{02} = F(r,z)\cos\theta$, $\Omega_{03} = 0$, where

$$\begin{gathered} F(r,z) = rz + \sum_{n=1}^{\infty} -\frac{2\,P_n}{k_{1n}}\,\mathcal{R}_{1n}(r)\frac{\sinh(k_{1n}(z+\frac{1}{2}h))}{\cosh(\frac{1}{2}k_{1n}h)}, \\ P_n = \int_{r_1}^{1} r^2\,\mathcal{R}_{1n}(r)\,\mathrm{d}r. \end{gathered} \tag{6}$$

When adopting (2a) and (6), the *aforementioned Fourier (modal) representation* takes the form [7]

$$\zeta(r,\theta,t) = \sum_{M,i}^{I_\theta,I_r} \mathcal{R}_{Mi}(r)\cos(M\theta)\,p_{Mi}(t) + \sum_{m,i}^{I_\theta,I_r} \mathcal{R}_{mi}(r)\sin(m\theta)\,r_{mi}(t), \tag{7a}$$

$$\begin{aligned} \Phi(r,\theta,z,t) &= \dot{\eta}_1(t)\,r\cos\theta + \dot{\eta}_2(t)\,r\sin\theta + \\ &+ F(r,z)[-\dot{\eta}_4(t)\sin\theta + \dot{\eta}_5(t)\cos\theta] + \\ &+ \sum_{M,i}^{I_\theta,I_r} \mathcal{R}_{Mi}(r)\,\mathcal{Z}_{Mi}(z)\,\cos(M\theta)\,P_{Mi}(t) + \\ &+ \sum_{m,i}^{I_\theta,I_r} \mathcal{R}_{mi}(r)\,\mathcal{Z}_{mi}(z)\,\sin(m\theta)\,R_{mi}(t), \end{aligned} \tag{7b}$$

I_θ, $I_r \to \infty$. *Here and further*, all capital summation letters imply changing from zero to I_θ but the lower case indices mean changing from one to either I_θ or I_r.

In the modal representation (7), p_{Mi} and r_{mi} play the role of the sloshing-related generalised coordinates but P_{Mi} and R_{mi} are the corresponding generalised velocities. Using the Bateman-Luke variational formulation makes it possible to derive the Euler-Lagrange equations with respect to the generalised coordinates and velocities. The procedure is described in [7] where the latter equations are explicitly written down in both fully- and weakly-nonlinear forms.

The weakly-nonlinear equations are constructed in [7] adopting the Narimanov-Moiseev asymptotic relations

$$p_{11} \sim r_{11} = O(\epsilon^{1/3}), \quad p_{0j} \sim p_{2j} \sim r_{2j} = O(\epsilon^{2/3}),$$
$$r_{1(j+1)} \sim p_{1(j+1)} \sim p_{3j} \sim r_{3j} = O(\epsilon), \quad j = 1, 2, \ldots, I_r; \quad I_r \to \infty \quad (8)$$

(see, an extensive discussion on what these relations mean for axisymmetric tanks in [5]). The equations take the form

$$\ddot{p}_{11} + \sigma_{11}^2 p_{11} + d_1 p_{11} \left(\ddot{p}_{11} p_{11} + \ddot{r}_{11} r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\right) + d_2 \left[r_{11}(\ddot{p}_{11} r_{11} - \ddot{r}_{11} p_{11}) + 2\dot{r}_{11}(\dot{p}_{11} r_{11} - \dot{r}_{11} p_{11})\right]$$
$$+ \sum_{j=1}^{I_r} \Big[d_3^{(j)}(\ddot{p}_{11} p_{2j} + \ddot{r}_{11} r_{2j} + \dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}) + d_4^{(j)}(\ddot{p}_{2j} p_{11} + \ddot{r}_{2j} r_{11})$$
$$+ d_5^{(j)}(\ddot{p}_{11} p_{0j} + \dot{p}_{11}\dot{p}_{0j}) + d_6^{(j)} \ddot{p}_{0j} p_{11}\Big] = -(\ddot{\eta}_1 - g\eta_5 - S_1\ddot{\eta}_5)\kappa_{11} P_1, \quad (9a)$$

$$\ddot{r}_{11} + \sigma_{11}^2 r_{11} + d_1 r_{11} \left(\ddot{p}_{11} p_{11} + \ddot{r}_{11} r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\right) + d_2 \left[p_{11}(\ddot{r}_{11} p_{11} - \ddot{p}_{11} r_{11}) + 2\dot{p}_{11}(\dot{r}_{11} p_{11} - \dot{p}_{11} r_{11})\right]$$
$$+ \sum_{j=1}^{I_r} \Big[d_3^{(j)}(\ddot{p}_{11} r_{2j} - \ddot{r}_{11} p_{2j} + \dot{p}_{11}\dot{r}_{2j} - \dot{p}_{2j}\dot{r}_{11}) + d_4^{(j)}(\ddot{r}_{2j} p_{11} - \ddot{p}_{2j} r_{11})$$
$$+ d_5^{(j)}(\ddot{r}_{11} p_{0j} + \dot{r}_{11}\dot{p}_{0j}) + d_6^{(j)} \ddot{p}_{0j} r_{11}\Big] = -(\ddot{\eta}_2 + g\eta_4 + S_1\ddot{\eta}_4)\kappa_{11} P_1; \quad (9b)$$

$$\ddot{p}_{2k} + \sigma_{2k}^2 p_{2k} + d_{7,k}(\dot{p}_{11}^2 - \dot{r}_{11}^2) + d_{9,k}(\ddot{p}_{11} p_{11} - \ddot{r}_{11} r_{11}) = 0, \quad (10a)$$
$$\ddot{r}_{2k} + \sigma_{2k}^2 r_{2k} + 2 d_{7,k} \dot{p}_{11}\dot{r}_{11} + d_{9,k}(\ddot{p}_{11} r_{11} + \ddot{r}_{11} p_{11}) = 0, \quad (10b)$$
$$\ddot{p}_{0k} + \sigma_{0k}^2 p_{0k} + d_{8,k}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{10,k}(\ddot{p}_{11} p_{11} + \ddot{r}_{11} r_{11}) = 0; \quad (10c)$$

$$\ddot{p}_{3k} + \sigma_{3k}^2 p_{3k} + d_{11,k}\left[\ddot{p}_{11}(p_{11}^2 - r_{11}^2) - 2 p_{11} r_{11} \ddot{r}_{11}\right] + d_{12,k}\left[p_{11}(\dot{p}_{11}^2 - \dot{r}_{11}^2) - 2 r_{11}\dot{p}_{11}\dot{r}_{11}\right]$$
$$+ \sum_{j=1}^{I_r} \Big[d_{13,k}^{(j)}(\ddot{p}_{11} p_{2j} - \ddot{r}_{11} r_{2j}) + d_{14,k}^{(j)}(\ddot{p}_{2j} p_{11} - \ddot{r}_{2j} r_{11})$$
$$+ d_{15,k}^{(j)}(\dot{p}_{2j}\dot{p}_{11} - \dot{r}_{2j}\dot{r}_{11})\Big] = 0, \quad (11a)$$

$$\ddot{r}_{3k} + \sigma_{3k}^2 r_{3k} + d_{11,k}\left[\ddot{r}_{11}(p_{11}^2 - r_{11}^2) + 2 p_{11} r_{11} \ddot{p}_{11}\right] + d_{12,k}\left[r_{11}(\dot{p}_{11}^2 - \dot{r}_{11}^2) + 2 p_{11}\dot{p}_{11}\dot{r}_{11}\right]$$
$$+ \sum_{j=1}^{I_r} \Big[d_{13,k}^{(j)}(\ddot{p}_{11} r_{2j} + \ddot{r}_{11} p_{2j}) + d_{14,k}^{(j)}(\ddot{p}_{2j} r_{11} + \ddot{r}_{2j} p_{11})$$
$$+ d_{15,k}^{(j)}(\dot{p}_{2j}\dot{r}_{11} + \dot{r}_{2j}\dot{p}_{11})\Big] = 0, \quad k = 1, ..., I_r; \quad (11b)$$

$$
\begin{aligned}
&\ddot{p}_{1n} + \sigma_{1n}^2 p_{1n} + d_{16,n}(\ddot{p}_{11}p_{11}^2 + r_{11}p_{11}\ddot{r}_{11}) + d_{17,n}(\ddot{p}_{11}r_{11}^2 - r_{11}p_{11}\ddot{r}_{11}) \\
&\qquad + d_{18,n}p_{11}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{19,n}(r_{11}\dot{p}_{11}\dot{r}_{11} - p_{11}\dot{r}_{11}^2) \\
&\qquad + \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{p}_{2j} + r_{11}\ddot{r}_{2j}) \\
&\qquad + d_{22,n}^{(j)}(\dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}) + d_{23,n}^{(j)}\ddot{p}_{11}p_{0j} + d_{24,n}^{(j)}p_{11}\ddot{p}_{0j} + d_{25,n}^{(j)}\dot{p}_{11}\dot{p}_{0j}\Big] \\
&\qquad = -(\ddot{\eta}_1 - g\eta_5 - S_n\ddot{\eta}_5)\kappa_{1n} P_n, \qquad (12a)
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{r}_{1n} + \sigma_{1n}^2 r_{1n} + d_{16,n}(\ddot{r}_{11}r_{11}^2 + r_{11}p_{11}\ddot{p}_{11}) + d_{17,n}(\ddot{r}_{11}p_{11}^2 - r_{11}p_{11}\ddot{p}_{11}) \\
&\qquad + d_{18,n}r_{11}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{19,n}(p_{11}\dot{p}_{11}\dot{r}_{11} - r_{11}\dot{p}_{11}^2) \\
&\qquad + \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}r_{2j} - \ddot{r}_{11}p_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{r}_{2j} - r_{11}\ddot{p}_{2j}) \\
&\qquad + d_{22,n}^{(j)}(\dot{p}_{11}\dot{r}_{2j} - \dot{r}_{11}\dot{p}_{2j}) + d_{23,n}^{(j)}\ddot{r}_{11}p_{0j} + d_{24,n}^{(j)}r_{11}\ddot{p}_{0j} + d_{25,n}^{(j)}\dot{r}_{11}\dot{p}_{0j}\Big] \\
&\qquad = -(\ddot{\eta}_2 + g\eta_4 + S_n\ddot{\eta}_4)\kappa_{1n} P_n, \quad n = 2, ..., I_r. \qquad (12b)
\end{aligned}
$$

They couple all generalised coordinates up to the $O(\epsilon)$-order as $I_r \to \infty$; $r_{kl} \sim p_{kl} = o(\epsilon)$, $k \geq 4$ and, therefore, are neglected. The hydrodynamic coefficients of (9)–(12) are functions of the nondimensional liquid depth h. The system needs either initial or periodicity condition that determines transient and steady-state solutions, respectively.

3. Steady-state (periodic) resonant solutions

Applicability of (9)–(12) for studying the steady-state (periodic) waves requires that

– the generalised coordinates $\eta_i(t), i = 1, 2, 4, 5$, are the given $2\pi/\sigma$-periodic functions,

$$
\eta_i(t) = \eta_{ia}^{(0)} + \sum_{k=1}^{\infty}\Big[\eta_{ia}^{(k)}\cos(k\sigma t) + \mu_{ia}^{(k)}\sin(k\sigma t)\Big], \quad \eta_{ia}^{(k)} \sim \mu_{ia}^{(k)} = O(\epsilon), \qquad (13)
$$

where σ is the circular forcing frequency; the lowest-order harmonic component should not be zero, i.e.

$$
\sum_{i=1,2,4,5} |\eta_{ia}^{(1)}| + |\mu_{ia}^{(1)}| \neq 0; \qquad (14)
$$

– the forcing frequency σ is close to the lowest natural sloshing frequency σ_{11} so that the so-called Moiseev detuning condition

$$
\bar{\sigma}_{11}^2 - 1 = O(\epsilon^{2/3}), \quad \bar{\sigma}_{11} = \sigma_{11}/\sigma \qquad (15)
$$

is satisfied;

– there are no resonance amplifications of $p_{mj}, r_{mj},\ m\,j \neq 1$ that implies

$$\begin{aligned} & m - \bar\sigma_{1k} \geq O(1), \quad \bar\sigma_{mi} = \sigma_{mi}/\sigma, \quad m,k \geq 2; \\ & \bar\sigma_{0i}^2 - 4 \sim \bar\sigma_{2i}^2 - 4 \sim \bar\sigma_{3i}^2 - 9 \sim \bar\sigma_{1(i+1)}^2 - 9 \geq O(1), \quad i \geq 1; \end{aligned} \tag{16}$$

the second row means that there is no the so-called secondary resonance [2].

Our goal consists of constructing an asymptotic periodic solution of (9)–(12) and (13). The right-hand sides of (9) are

$$\sigma^2 P_1 \kappa_{11} \sum_{k=1}^{\infty} \Big[(k\eta_{1a}^{(k)} - (kS_1 - g/\sigma^2)\eta_{5a}^{(k)}) \cos(k\sigma t) + (k\mu_{1a}^{(k)} - (kS_1 - g/\sigma^2)\mu_{5a}^{(k)}) \sin(k\sigma t) \Big],$$

$$\sigma^2 P_1 \kappa_{11} \sum_{k=1}^{\infty} \Big[(k\eta_{2a}^{(k)} + (kS_1 - g/\sigma^2)\eta_{4a}^{(k)}) \cos(k\sigma t) + (k\mu_{2a}^{(k)} + (kS_1 - g/\sigma^2)\mu_{4a}^{(k)}) \sin(k\sigma t) \Big].$$

Because of (15), neglecting the higher-order terms, $o(\epsilon)$, allows for replacing $g/\sigma^2 \to g/\sigma_{11}^2$ and, therefore, amplitudes of the first Fourier harmonics are

$$\begin{aligned} \epsilon_x &= P_1 \kappa_{11} (\eta_{1a}^{(1)} - (S_1 - g/\sigma_{11}^2)\eta_{5a}^{(1)}), \\ \bar\epsilon_x &= P_1 \kappa_{11} (\mu_{1a}^{(1)} - (S_1 - g/\sigma_{11}^2)\mu_{5a}^{(1)}), \\ \bar\epsilon_y &= P_1 \kappa_{11} (\eta_{2a}^{(1)} + (S_1 - g/\sigma_{11}^2)\eta_{4a}^{(1)}), \\ \epsilon_y &= P_1 \kappa_{11} (\mu_{2a}^{(1)} + (S_1 - g/\sigma_{11}^2)\mu_{4a}^{(1)}). \end{aligned} \tag{17}$$

Here, ϵ_x and $\bar\epsilon_x$ appear in the front of $\cos\sigma t$ and $\sin\sigma t$ and imply the forcing components in the Ox direction, but $\bar\epsilon_y$ and ϵ_y correspond to the $\cos\sigma t$ and $\sin\sigma t$ harmonics along the Oy axis. Because of (14), rotating the Oxy frame around Oz can always help getting the non-zero first-harmonic forcing component along Ox, i.e. $\epsilon_x^2 + \bar\epsilon_x^2 \neq 0$. Furthermore, the periodicity condition is defined within to an arbitrary phase shift and one can assume, without loss of generality, that

$$\epsilon_x > 0, \quad \bar\epsilon_x = 0. \tag{18}$$

Henceforth, we follow the Bubnov-Galerking procedure [2] by posing the lowest-order asymptotic solution component

$$p_{11}(t) = a\cos(\sigma t) + \bar a \sin(\sigma t) + O(\epsilon), \; r_{11}(t) = \bar b \cos(\sigma t) + b \sin(\sigma t) + O(\epsilon), \tag{19}$$

where $a, \bar a, \bar b$, and b are of $O(\epsilon^{1/3})$. The second- and third-order generalised coordinates can be found from (10) and (11), (12), respectively. This gives, in particular,

$$\begin{aligned} p_{0k}(t) &= s_{0k}(a^2 + \bar a^2 + b^2 + \bar b^2) \\ &+ s_{1k} \left[(a^2 - \bar a^2 - b^2 + \bar b^2)\cos(2\sigma t) + 2(a\bar a + b\bar b)\sin(2\sigma t) \right] + o(\epsilon), \end{aligned} \tag{20a}$$

$$p_{2k}(t) = c_{0k}(a^2+\bar{a}^2-b^2-\bar{b}^2) \\ + c_{1k}\left[(a^2-\bar{a}^2+b^2-\bar{b}^2)\cos(2\sigma t)+2(a\bar{a}-b\bar{b})\sin(2\sigma t)\right]+o(\epsilon), \quad (20b)$$

$$r_{2k}(t) = 2c_{0k}(a\bar{b}+b\bar{a}) \\ + 2c_{1k}\left[(a\bar{b}-b\bar{a})\cos(2\sigma t)+(ab+\bar{a}\bar{b})\sin(2\sigma t)\right]+o(\epsilon), \quad (20c)$$

where

$$s_{0k}=\tfrac{1}{2}\left(\frac{d_{10,k}-d_{8,k}}{\bar{\sigma}_{0k}^2}\right),\quad s_{1k}=\frac{d_{10,k}+d_{8,k}}{2(\bar{\sigma}_{0k}^2-4)},\quad \bar{\sigma}_{0k}=\frac{\sigma_{0k}}{\sigma},$$
$$c_{0k}=\tfrac{1}{2}\left(\frac{d_{9,k}-d_{7,k}}{\bar{\sigma}_{2k}^2}\right),\quad c_{1k}=\frac{d_{9,k}+d_{7,k}}{2(\bar{\sigma}_{2k}^2-4)},\quad \bar{\sigma}_{2k}=\frac{\sigma_{2k}}{\sigma}. \quad (21)$$

Substituting (19) and (20) into (9) and gathering the first harmonic terms, $\cos\sigma t$ and $\sin\sigma t$, lead to the solvability (secular) equations

$$\begin{cases} ① : \ a\left[(\bar{\sigma}_{11}^2-1)+m_1(a^2+\bar{a}^2+\bar{b}^2)+m_3b^2\right]+(m_1-m_3)\bar{a}\bar{b}b=\epsilon_x, \\ ② : \ b\left[(\bar{\sigma}_{11}^2-1)+m_1(b^2+\bar{b}^2+\bar{a}^2)+m_3a^2\right]+(m_1-m_3)\bar{a}a\bar{b}=\epsilon_y, \\ ③ : \ \bar{a}\left[(\bar{\sigma}_{11}^2-1)+m_1(a^2+\bar{a}^2+b^2)+m_3\bar{b}^2\right]+(m_1-m_3)a\bar{b}b=0, \\ ④ : \ \bar{b}\left[(\bar{\sigma}_{11}^2-1)+m_1(b^2+\bar{b}^2+a^2)+m_3\bar{a}^2\right]+(m_1-m_3)\bar{a}ab=\bar{\epsilon}_y \end{cases} \quad (22)$$

with respect to $a, \bar{a}, \bar{b}$ and b. The coefficients m_1 and m_3 are computed by the formulas

$$m_1=-\tfrac{1}{2}d_1+\sum_{j=1}^{I_r}\Big[c_{1j}\left(\tfrac{1}{2}d_3^{(j)}-2d_4^{(j)}\right) \\ + s_{1j}\left(\tfrac{1}{2}d_5^{(j)}-2d_6^{(j)}\right)-s_{0j}d_5^{(j)}-c_{0j}d_3^{(j)}\Big], \quad (23a)$$

$$m_3=\tfrac{1}{2}d_1-2d_2+\sum_{j=1}^{I_r}\Big[c_{1j}\left(\tfrac{3}{2}d_3^{(j)}-6d_4^{(j)}\right) \\ + s_{1j}\left(-\tfrac{1}{2}d_5^{(j)}+2d_6^{(j)}\right)-s_{0j}d_5^{(j)}+c_{0j}d_3^{(j)}\Big]. \quad (23b)$$

After finding $a, \bar{a}, \bar{b}$ and b from (22), the second- and third-order components of the asymptotic solution are fully determined. Coefficients in this solution as well as m_1 and m_3 in (22) are functions of h, r_1 and the forcing frequency σ. Utilising (15) shows that the latter dependence can be neglected by substituting $\sigma=\sigma_{11}$ into the corresponding expressions. Dependence on σ *remains only* in the $(\bar{\sigma}_{11}^2-1)$-quantity of (22).

Calculations show that (16) is fulfilled for fairy deep liquid depths, $1.2 \lesssim h$, and the conditions

$$O(1)=m_1<0 \text{ and } O(1)=m_1+m_3>0 \quad (24)$$

are satisfied. This means, in particular, that $m_3>0$ and $m_1\neq m_3$.

One can follow [2] to study the stability of the asymptotic solution by using the linear stability analysis and the multi-timing technique. For this purpose, we introduce the slowly varying time $\tau=\epsilon^{2/3}\sigma t/2$ (the order $\epsilon^{2/3}$ is chosen to match the lowest asymptotic terms in the multi-timing technique), the Moiseev detuning (15), and express the infinitesimally perturbed solution

$$\begin{aligned} p_{11} &= (a+\alpha(\tau))\cos\sigma t + (\bar{a}+\bar{\alpha}(\tau))\sin\sigma t + o(\epsilon^{1/3}),\\ r_{11} &= (\bar{b}+\bar{\beta}(\tau))\cos\sigma t + (b+\beta(\tau))\sin\sigma t + o(\epsilon^{1/3}),\end{aligned} \tag{25}$$

where $a, \bar{a}, b$ and $\bar{b}$ are known and $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ are their relative perturbations depending on τ. Inserting (25) into the Narimanov-Moiseev modal equations, gathering terms of the lowest asymptotic order and keeping linear terms in $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ lead to the following linear system of ordinary differential equations

$$\frac{d\mathbf{c}}{d\tau} + \mathcal{C}\mathbf{c} = 0, \tag{26}$$

where $\mathbf{c} = (\alpha, \bar{\alpha}, \beta, \bar{\beta})^T$ and the matrix $\mathcal{C}$ has the elements

$$\begin{aligned}
&c_{11} = -[2a\bar{a}\, m_1 + (m_1 - m_3)\, b\bar{b}];\\
&c_{12} = -[(\bar{\sigma}_{11}^2 - 1) + m_1(a^2 + 3\bar{a}^2 + b^2) + m_3\, \bar{b}^2],\\
&c_{13} = -[2\bar{a}b\, m_1 + (m_1 - m_3)\, a\bar{b}];\ c_{14} = -[2\bar{a}\bar{b}\, m_3 + (m_1 - m_3)\, ab],\\
&c_{21} = (\bar{\sigma}_{11}^2 - 1) + m_1(3a^2 + \bar{a}^2 + \bar{b}^2) + m_3\, b^2;\ c_{22} = 2a\bar{a}\, m_1 + (m_1 - m_3)\, b\bar{b},\\
&c_{23} = 2ab\, m_3 + (m_1 - m_3)\, \bar{a}\bar{b};\ c_{24} = 2a\bar{b}\, m_1 + (m_1 - m_3)\, \bar{a}b,\\
&c_{31} = 2m_1\, a\bar{b} + (m_1 - m_3)\, b\bar{a};\ c_{32} = 2m_3\, \bar{a}\bar{b} + (m_1 - m_3)\, ab,\\
&c_{33} = 2m_1\, b\bar{b} + (m_1 - m_3)\, a\bar{a};\ c_{34} = (\bar{\sigma}_{11}^2 - 1) + m_1(b^2 + 3\bar{b}^2 + a^2) + m_3\, \bar{a}^2,\\
&c_{41} = -[2m_3\, ab + (m_1 - m_3)\, \bar{a}\bar{b}];\ c_{42} = -[2\bar{a}b\, m_1 + (m_1 - m_3)\, a\bar{b}],\\
&c_{43} = -[(\bar{\sigma}_{11}^2 - 1) + m_1(3b^2 + \bar{b}^2 + \bar{a}^2) + m_3\, a^2];\\
&c_{44} = -[2b\bar{b}\, m_1 + (m_1 - m_3)\, a\bar{a}].
\end{aligned}$$

The instability of the asymptotic solution can be evaluated by studying (26). Its fundamental solution depends on the eigenvalue problem $\det[\lambda E + C] = 0$, where E is the identity matrix. Computations give the following characteristic polynomial

$$\lambda^4 + c_1\lambda^2 + c_0 = 0, \tag{27}$$

where c_0 is the determinant of $\mathcal{C}$ and c_1 is a complicated function of the elements of $\mathcal{C}$. As discussed in [2], the stability requires $c_0 > 0, c_1 > 0$ and $c_1^2 - 4c_0 > 0$. When at least one of the inequalities is not fulfilled, the steady-state wave regime associated with the dominant amplitudes $a, \bar{a}, b$ and $\bar{b}$ is not stable.

4. Classification of steady-state (periodic) solutions

The steady-state (periodic) sloshing can be classified by analysing the lowest-order component (19) which gives the dominant wave contribution. The lowest-order amplitudes $a, \bar{a}, b$ and $\bar{b}$ follow from the secular system (22) which does not involve the super-harmonic components from (13). This means that the resonant sloshing regimes are, within to the higher-order terms, the same as if the tank performs the *artificial horizontal harmonic* motions

$$\begin{aligned}(\kappa_{11}P_1)\,\eta_1(t) &= \epsilon_x\cos\sigma t,\\ (\kappa_{11}P_1)\,\eta_2(t) &= \bar\epsilon_y\cos\sigma t+\epsilon_y\sin\sigma t,\quad \eta_4(t)=\eta_5(t)=0\end{aligned} \tag{28}$$

that define, by accounting for (18), either longitudinal ($\epsilon_y = 0$) or *elliptic (rotary)* ($\epsilon_y \neq 0$) harmonic tank motion. The latter occurs along the trajectory

$$\frac{\epsilon_y^2+\bar\epsilon_y^2}{\epsilon_x^2}x^2+y^2-2\frac{\bar\epsilon_y}{\epsilon_x}xy=\epsilon_y^2. \tag{29}$$

For the longitudinal tank motions ($\epsilon_y = 0$), one can rotate the Oxy frame around Oz to get the artificial tank vibrations by (28) occurring along the Ox axis. The forcing amplitudes become then $\epsilon_x > 0$ and $\bar\epsilon_y = \epsilon_y = 0$ and the secular system (22) has only two analytical solutions well known from, for example, [4]. The first solution implies the so-called *planar* steady-state wave ($\bar a = \bar b = b = 0$). The nonzero lowest-order amplitude parameter a is governed by

$$a\left[(\bar\sigma_{11}^2-1)+m_1a^2\right]=\epsilon_x. \tag{30}$$

This solution is characterised by the zero transverse wave component, namely, $r_{mi}(t) \equiv 0$. The second solution corresponds to *swirling* whose longitudinal ($a \neq 0$) and transverse ($b \neq 0$) amplitude parameters come from the system

$$\begin{cases} a\left[(\bar\sigma_{11}^2-1)+(m_1+m_3)a^2\right]=\dfrac{m_1}{m_1-m_3}\epsilon_x,\\ b^2=-\dfrac{(\bar\sigma_{11}^2-1)+m_3a^2}{m_1}>0.\end{cases} \tag{31}$$

Why the solution $\bar a = \bar b = 0$, $ab \neq 0$ is called swirling is discussed in [4].

When the artificial horizontal harmonic motions occur along an elliptic trajectory ($\epsilon_y \neq 0$), rotating the Oxy frame around Oz helps superposing Ox with the major axis of the ellipse. This new position of the Oxy frame implies that

$$\bar\epsilon_y=0,\quad 0<\epsilon_y\le\epsilon_x\neq 0 \tag{32}$$

in (22). The following equalities

$$\begin{aligned}\bar a\cdot①-a\cdot③ &= \bar b\cdot②-b\cdot④\\ &=(m_1-m_3)[a\bar a(\bar b^2-b^2)+\bar b b(\bar a^2-a^2)]=\bar a\epsilon_x=\bar b\epsilon_y,\end{aligned} \tag{33a}$$

$$\begin{aligned}\bar b\cdot①-a\cdot④ &= \bar a\cdot②-b\cdot③\\ &=(m_1-m_3)[b\bar a(\bar b^2-a^2)+\bar b a(\bar a^2-b^2)]=\bar b\epsilon_x=\bar a\epsilon_y,\end{aligned} \tag{33b}$$

$$b\cdot①-a\cdot②=(m_1-m_3)(a^2-b^2)(ab-\bar a\bar b)=b\epsilon_x-a\epsilon_y \tag{33c}$$

can then be treated as solvability conditions of (22).

When $0 < \epsilon_y < \epsilon_x$, the homogeneous linear system (33a)–(33b) with respect to $\bar a$ and $\bar b$ has only trivial solution $\bar a = \bar b = 0$. Equation (33c) shows then that $a\,b \neq 0$ (and $a \neq b$) and, therefore, the only nonzero amplitudes a and b

always determine *swirling*. The amplitudes are governed by (22) which can be rewritten in the equivalent form

$$\begin{cases} b\left[(m_1-m_3)b^2+\left(\dfrac{\epsilon_x}{a}-(m_1-m_3)a^2\right)\right]=\epsilon_y, \\ (\bar\sigma_{11}^2-1)=\dfrac{\epsilon_x}{a}-m_1a^2-m_3b^2, \quad a\neq 0. \end{cases} \tag{34}$$

The first equality is a depressed cubic with respect to b whose coefficient at the linear term is a function of a. The second equality computes the forcing frequency, σ/σ_{11} $(\bar\sigma_{11}^2-1)$, as a function of a and b. A numerical procedure may suggest varying a in an admissible range, solving the depressed cubic (finding $b=b(a)$), and computing σ/σ_{11} as a function a and $b=b(a)$. When solving the depressed cubic, one should check for the discriminant

$$\Delta=-4\left(\frac{\epsilon_x}{a\,(m_1-m_3)}-a^2\right)^3-27\left(\frac{\epsilon_y}{m_1-m_3}\right)^2, \quad 0<\epsilon_y<\epsilon_x. \tag{35}$$

Cartano's theorem says that, (i) if $\Delta>0$, then there are three distinct real roots for b, (ii) if $\Delta=0$, then the equation has at least one multiple root and all its roots are real, and (iii) if $\Delta<0$, then the equation has one real root and two nonreal complex conjugate roots.

When considering Δ as a function of a, a simple analysis shows that, if $m_1-m_3<0$, there exists only a negative real root $a_*<0$ of $\Delta(a_*)=0$ so that $\Delta(a)>0$ for $a<a_*$ and $0<a$ (three real solutions) but $\Delta(a)<0$ for $a_*<a<0$ (a unique real solution). Analogously, if $m_1-m_3>0$, there exists only a positive real root $a_*>0$ of $\Delta(a_*)=0$ so that $\Delta(a)>0$ for $a<0$ and $a_*<a$ but $\Delta(a)<0$ for $0<a<a_*$.

When $\bar\epsilon_y=0$, $\epsilon_y=\epsilon_x\neq 0$ (artificial rotary harmonic motions of the tank), equations (33a) and (33b) are unable to derive that $\bar a$ and $\bar b$ are zeros but deduce, instead, $\bar a=\bar b=c$. The latter makes ③ ≡ ④ in (22). By taking the sum ① + ② and the difference ① − ②, we transform (22) to the form

$$\begin{cases} (a+b)\big\{(\bar\sigma_{11}^2-1)+m_1(a^2+b^2)+ \\ \qquad\qquad +(3m_1-m_3)c^2-(m_1-m_3)ab\big\}=2\epsilon, \\ (a-b)\big[(\bar\sigma_{11}^2-1)+m_1(a^2+b^2)+ \\ \qquad\qquad +(m_1+m_3)c^2+(m_1-m_3)ab\big]=0, \\ c\left[(\bar\sigma_{11}^2-1)+m_1(a^2+b^2)+(m_1+m_3)c^2+(m_1-m_3)ab\right]=0, \end{cases} \tag{36}$$

in which the two homogeneous equations contain identical expressions in the square bracket. These expressions are multiplied by $(a-b)$ and c, respectively.

We adopt $a_+=\frac{1}{2}(a+b), a_-=\frac{1}{2}(a-b)$ instead of a and b. When both a_- and c are zeros, we arrive at

$$\bar a=\bar b=0, \quad a_+=a=b, \quad a_+\left[(\bar\sigma_{11}^2-1)+(m_1+m_3)a_+^2\right]=\epsilon \tag{37}$$

which imply *rotary* (circular swirling) waves characterised by equal longitudinal (along Ox) and transverse (along Oy) amplitude components, $p_{11}(t)=a_+\cos(\sigma t)+O(\epsilon)$, $r_{11}(t)=a_+\sin(\sigma t)+O(\epsilon)$. The rotary waves are co-directed with the rotary tank motion.

When either $a_- \neq 0$ or $c \neq 0$, the square bracket expression of (36) must be zero. This makes the second and third equalities of (36) automatically satisfied and, therefore, three amplitude parameters a_+, a_- and c should be found from the two equalities

$$\begin{cases} a_+ \left[(\bar{\sigma}_{11}^2 - 1) + 4m_1 a_+^2\right] = -\dfrac{m_1 + m_3}{2(m_1 - m_3)}\,\epsilon, \\ a_-^2 + c^2 = -\dfrac{(\bar{\sigma}_{11}^2 - 1) + (3m_1 - m_3)a_+^2}{(m_1 + m_3)} > 0, \end{cases} \tag{38}$$

which define the following lowest-order steady-state solution component

$$\begin{aligned} p_{11}(t) &= (a_+ + a_-)\cos(\sigma t) + c\sin(\sigma t) + O(\epsilon), \\ r_{11}(t) &= (a_+ - a_-)\sin(\sigma t) + c\cos(\sigma t) + O(\epsilon). \end{aligned} \tag{39}$$

The amplitude a_+ can be found from the first equation of (38) but the amplitudes a_- and c are not uniquely defined. Only the sum $a_-^2 + c^2$ can be found for any fixed pair $(\bar{\sigma}_{11}^2, a_+)$ from the first cubic equation. This defines a manifold $a_+ = a_+(\sigma/\sigma_{11}), a_-^2 + c^2 = F(\sigma/\sigma_{11}, a_+)$ in the four-dimensional space $(\sigma/\sigma_{11}, a_+, a_-, c)$. Numerical analysis of the solution (39) shows that it is *unstable on the aforementioned manifold* due to $c_0 = 0$ in the characteristic equation (27).

When $c = 0$, system (38) defines the three-dimensional response curves $a_+ = a_+(\sigma/\sigma_{11})$, $a_- = a_-(\sigma/\sigma_{11})$ which implies the solution

$$p_{11}(t) = (a_+ + a_-)\cos(\sigma t) + O(\epsilon), \quad r_{11}(t) = (a_+ - a_-)\sin(\sigma t) + O(\epsilon) \tag{40}$$

which has the same form as for the elliptically-excited swirling with $\epsilon_y < \epsilon_x$.

5. Conclusions

By using the Narimanov-Moiseev type modal theory [7], the steady-state (periodic) resonant waves in an upright circular cylindrical tank with a fairly deep liquid depth are analysed when the tank performs an arbitrary small-magnitude sway-surge-pitch-roll periodic motion. The forcing frequency is close to the lowest natural sloshing frequency. The analysis shows that, within to the higher-order terms, the resonant sloshing is the same as that due to either longitudinal or elliptic/rotary horizontal harmonic tank motions. The longitudinal case is well known from the literature. Planar (in the excitation plane) and swirling waves were established and described. In the present paper, the cases of elliptic and rotary excitations are studied to show that they always lead to swirling, which can be either co- or counter-directed with respect to the forcing direction. The co-directed wave converts then to the rotary wave regime when the elliptic forcing tends to the rotary one. The effective frequency range of the stable counter-directed swirling becomes unstable in this limit case.

The second author acknowledges the financial support of the Centre of Autonomous Marine Operations and Systems (AMOS) whose main sponsor is the Norwegian Research Council (Project number 223254–AMOS).

Bibliography

1. Abramson H. N. Some studies of nonlinear lateral sloshing in rigid containers / H. N. Abramson, W. H. Chu, D. D. Kana // Journal of Applied Mechanics. – 1966. – Vol. 33. – P. 66-74.
2. Faltinsen O. M. Resonant three-dimensional nonlinear sloshing in a square base basin / O. M. Faltinsen, O. F. Rognebakke, A. N. Timokha // Journal of Fluid Mechanics. – 2003. – Vol. 487. – P. 1-42.
3. Ibrahim R. A. Recent advances in physics of fluid parametric sloshing and related problems / R. A. Ibrahim // Journal of Fluids Engineering. – 2015. – Vol. 137. – ID 090801-1.
4. Lukovsky I. A. Nonlinear Dynamics: Mathematical Models for Rigid Bodies with a Liquid / I. A. Lukovsky. – De Gruyter, 2015.
5. Lukovsky I. Combining Narimanov–Moiseev' and Lukovsky–Miles' schemes for nonlinear liquid sloshing / I. Lukovsky, A. Timokha // Journal of Numerical and Applied Mathematics. – 2011. – Vol. 105. – P. 69-82.
6. Royon-Lebeaud A. Liquid sloshing and wave breaking in circular and square-base cylindrical containers / A. Royon-Lebeaud, E. Hopfinger, A. Cartellier // Journal of Fluid Mechanics. – 2007. – Vol. 577. – P. 467-494.
7. Timokha A. N. The Narimanov–Moiseev modal equations for sloshing in an annular tank / A. N. Timokha // Proceedings of the Institute of Mathematics of the NASU. – 2015. – Vol. 12, № 5. – P. 122-147.

I. A. Raynovskyy,
Institute of Mathematics,
National Academy of Sciences,
3, Tereschenkivs'ka Str., Kyiv, 01004, Ukraine;
A. N. Timokha,
Centre for Autonomous Marine Operations and Systems,
Department of Marine Technology,
Norwegian University of Science and Technology,
NO-7491, Trondheim, Norway.

Received 11.05.2016