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THE BEST M-TERM TRIGONOMETRIC APPROXIMATIONS OF CLASSES OF (ψ, β)-DIFFERENTIABLE PERIODIC MULTIVARIATE FUNCTIONS IN THE SPACE $L^{\psi}_{\beta,1}$

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Резюме. Встановлено порядкові оцінки найкращих M–членних тригонометричних наближень періодичних функцій D^{ψ}_{β} у просторі L_q, $1 < q \leq 2$. Використовуючи одержані результати, встановлено порядкові співвідношення цих величин для класів $L^{\psi}_{\beta,1}$.

Abstract. Obtained here are the order estimates of the best M–term trigonometric approximations of periodic functions D^{ψ}_{β} in the space $L_q, 1 < q \leq 2$. The results are applied to establish the order estimates of the same quantities for classes $L^{\psi}_{\beta,1}$.

1. Introduction

Let us introduce all necessary denotations and give a definition of the approximative characteristic to investigate.

Let $L_q(\pi_d)$, $1 \leq q \leq \infty$, — be the space of functions f, 2π–periodic by each variable, with the finite norm

$$\|f\|_{L_q(\pi_d)} = \|f\|_q = \left((2\pi)^{-d} \int\limits_{\pi_d} |f(x)|^q dx\right)^{\frac{1}{q}},\ 1 \leq q < \infty,$$

$$\|f\|_{L_\infty(\pi_d)} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \pi_d} |f(x)|,$$

where $x = (x_1, \dots, x_d)$ is the element of Euclidean space $\mathbb{R}^d$, $d \geq 1$, and $\pi_d = \prod\limits_{j=1}^{d} [-\pi, \pi]$. Suppose further that for the functions $f \in L_q(\pi_d)$ the condition

$$\int\limits_{-\pi}^{\pi} f(x) dx_j = 0,\ j = \overline{1, d},$$

holds.

Let us consider the Fourier series for the function $f \in L_1(\pi_d)$

$$\sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{i(k,x)},$$

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where

$$\widehat{f}(k) = (2\pi)^{-d} \int\limits_{\pi_d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f, $(k, x) = k_1 x_1 + \ldots + k_d x_d$.

Let $\psi_j(\cdot) \neq 0$ be arbitrary functions of the natural argument, $\beta_j \in \mathbb{R}$, $j = \overline{1, d}$. Assume that the series

$$\sum_{k \in \mathring{\mathbb{Z}}^d} \prod_{j=1}^{d} \frac{e^{i\frac{\pi\beta_j}{2}\mathrm{sgn} k_j}}{\psi_j(|k_j|)} \widehat{f}(k) e^{i(k,x)},$$

where $\mathring{\mathbb{Z}}^d = (\mathbb{Z} \setminus \{0\})^d$, are the Fourier series of some summable on π_d function. Following O. I. Stepanets [1, c. 25], (see also [2, c. 132]), let us call it (ψ, β)–derivative of the function f and denote it as f^{ψ}_{β}. A set of functions f, for which (ψ, β)–derivatives exist, is denoted as L^{ψ}_{β}.

If the condition $\left\| f^{\psi}_{\beta}(\cdot) \right\|_p \leq 1,\ 1 \leq p \leq \infty$, holds then $f \in L^{\psi}_{\beta,p}$.

The article deals with the best M–term trigonometric approximations of the functions D^{ψ}_{β} whose Fourier series are written in a form

$$\sum_{k \in \mathring{\mathbb{Z}}^d} \prod_{j=1}^{d} \psi_j\left(|k_j|\right) e^{i\frac{\pi\beta_j}{2}\mathrm{sgn} k_j} e^{i(k,x)}.$$

Note that if $\psi_j\left(|k_j|\right) = |k_j|^{-r_j}$, $r_j > 0$, $k_j \in \mathbb{Z}\backslash\{0\}$, $j = \overline{1, d}$, D^{ψ}_{β} is a multivariate analogue of the Bernoulli kernel (see, e.g., [3, c. 31]).

Each of the functions $f \in L^{\psi}_{\beta,p}$ can be presented in a form of convolution

$$f(x) = \left(\varphi * D^{\psi}_{\beta}\right)(x) = (2\pi)^{-d} \int\limits_{\pi_d} \varphi(x - t) D^{\psi}_{\beta}(t) dt, \tag{1}$$

where $\|\varphi\|_p \leq 1$, and the function $\varphi(\cdot)$ almost everywhere coincides with f^{ψ}_{β}.

As an apparatus of the approximation we will use trigonometric polynomials of the form

$$P\left(\theta_M; x\right) = \sum_{k \in \theta_M} c_k e^{i(k,x)},$$

where θ_M is an arbitrary set of M different vectors $k = (k_1, \ldots k_d)$ and $c_k \in \mathbb{C}$. For $f \in L_q\left(\pi_d\right)$, $1 \leq q \leq \infty$, the quantity

$$e_M(f)_q = \inf_{\theta_M} \inf_{P(\theta_M; \cdot)} \|f(\cdot) - P\left(\theta_M; \cdot\right)\|_q. \tag{2}$$

is called the best M–term trigonometric approximation of the function f. And the quantity

$$e^{\perp}_M(f)_q = \inf_{\theta_M} \left\| f(\cdot) - \sum_{k \in \theta_M} \widehat{f}(k)\, e^{i(k,x)} \right\|_q, \tag{3}$$

is called the best orthogonal trigonometric approximation of the function f. It is obvious that the relation

$$e_M(f)_q \le e^{\perp}_M(f)_q,\ 1 \le q \le \infty, \tag{4}$$

holds. If $F \subset L_q$ is some functional class then denote

$$e_M(F)_q = \sup_{f\in F} e_M(f)_q, \tag{5}$$

and, accordingly,

$$e^{\perp}_M(F)_q = \sup_{f\in F} e^{\perp}_M(f)_q. \tag{6}$$

The quantity (2) appeared at first in the paper of S. B. Stechkin [4] in formulating an absolute convergence criterion for orthogonal series. Later the quantity (5) for classes of periodic functions of one and many variables was investigated in the papers of V. N. Temlyakov [3], [5–7], E. S. Belinskii [8–10,12], A. S. Romanyuk [13–20], A. S. Fedorenko [21–23], N. M. Konsevych [24, 25], V. V. Shkapa [26] and others.

The quantities (3) and (6) were considered by E. S. Belinskii (see, e.g., [12]), and later their exploration was further developed in the works of many authors. The detailed bibliography can be found in [20, 27].

The results of the article are formulated in order-relation terms. So, further for the quantities A and B under the notation $A \ll B$ we will understand the existance of a positive constant C_1 such that $A \le C_1 B$. If the conditions $A \ll B$ and $B \ll A$ hold then we write $A \asymp B$. All constants in order relations can depend only on the parameters that are in the definitions of class and metric in which the approximation is carried out, and on the dimension of the space $\mathbb{R}^d$.

2. Auxiliary statements

To formulate and prove the results of the article some notations and auxiliary statements will be needed.

Let D be a set of functions $\psi(\cdot)$ of natural argument that satisfy the conditions

1) $\psi(\cdot)$ are positive and nonincreasing;

2) $\exists M > 0$ such that $\forall l \in \mathbb{N}$ $\frac{\psi(l)}{\psi(2l)} \le M$.

Note that to the indicated set of functions belong, in particular, functions $\psi(|k|) = |k|^{-r}$, $\psi(|k|) = |k|^{-r} \ln^{\alpha}(|k|+1)$, $r > 0$, $k \in \mathbb{Z}\backslash\{0\}$, $\alpha \in \mathbb{R}$ and others.

Further, let us put into conformity to each vector $s = (s_1, \dots, s_d)$, $s_j \in \mathbb{N} \cup \{0\}$, $j = \overline{1,d}$, a set

$$\rho(s) = \left\{k = (k_1, \dots, k_d) :\ \left[2^{s_j - 1}\right] \le |k_j| < 2^{s_j},\ j = \overline{1,d}\right\},$$

where $[\cdot]$ is the whole part, and for $f \in L_1(\pi_d)$ put

$$\delta_s(f, x) = \sum_{k\in\rho(s)} \widehat{f}(k) e^{i(k,x)},$$

where $\widehat{f}(k)$ are the Fourier coefficients of this function. Note that the unifications of "blocks" $\rho(s)$, $(s,1) = s_1 + \ldots + s_d < n$, $n \in \mathbb{N}$, form a set Q_n that is called "step-hyperbolic cross" [3, c. 7]. The quantity of points in this set is of the order $2^n n^{d-1}$ [3, c. 70].

The following propositions hold.

Proposition 7. *[27] Let $1 < q < \infty$, $\psi_j \in D$, $\beta_j \in \mathbb{R}$, $j = \overline{1,d}$, and, besides, there exists $\varepsilon > 0$ such that $\psi_j(|k_j|)\,|k_j|^{1-\frac{1}{q}+\varepsilon}$ are nonincreasing. Then for all natural M and n that satisfy the condition $M \asymp 2^n n^{d-1}$, the following relations hold*

$$\Phi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)} \ll e_M^{\perp}\left(D_\beta^\psi\right)_q \ll$$

$$\ll \Psi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)},$$

$$\Phi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)} \ll e_M^{\perp}\left(L_{\beta,1}^\psi\right)_q \ll$$

$$\ll \Psi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}.$$

where

$$\Phi(n) = \min_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right), \quad \Psi(n) = \max_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right).$$

Proposition 8. *[3, c. 28] For an arbitrary function $f \in L_q(\pi_d)$, $1 < q < p \le \infty$, holds*

$$\|f\|_q^q \gg \sum_s \|\delta_s(f,\cdot)\|_p^q \cdot 2^{(s,1)\left(\frac{1}{p}-\frac{1}{q}\right)q}.$$

To make further speculations we need one more relation which follows from a more general result of S. N. Nikolskii (see, e.g., [28, c. 25]).

Proposition 9. *For all functions $f \in L_q(\pi_d), 1 \le q < \infty$, holds*

$$e_M(f)_q = \inf_{\theta_M} \sup_{\substack{P \in L^\perp(\theta_M),\\ \|P\|_{q'} \le 1}} \left| \int_{\pi_d} f(x)\overline{P(x)}dx \right|,$$

where $L^\perp(\theta_M)$ is a set of functions that is orthogonal to the subset of trigonometric polynomials with the numbers of harmonics from the set θ_M, and $\frac{1}{q} + \frac{1}{q'} = 1$.

3. The best M-term trigonometric approximations

The following statement holds.

Theorem 1. *Let $1 < q \le 2$, $\psi_j \in D$, $\beta_j \in \mathbb{R}$, $j = \overline{1,d}$, and, besides, there exists $\varepsilon > 0$ such that $\psi_j(|k_j|)\,|k_j|^{1-\frac{1}{q}+\varepsilon}$ are nonincreasing. Then for arbitrary natural M and n that satisfy condition $M \asymp 2^n n^{d-1}$, we have the estimate*

$$\Phi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)} \ll e_M\left(D_\beta^\psi\right)_q \ll$$

$$\ll \Psi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, \tag{7}$$

where

$$\Phi(n) = \min_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right), \quad \Psi(n) = \max_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right).$$

Proof. The upper estimate follows from (4) and proposition 7, that is

$$e_M\left(D_\beta^\psi\right)_q \le e_M^\perp\left(D_\beta^\psi\right)_q \ll$$

$$\ll \Psi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, \ 1 < q \le 2. \tag{8}$$

Let us go to the establishment of the lower estimate in (7). For the given M let us choose n so that the relation $M \asymp 2^n n^{d-1}$ holds. Note that the consideration of the case $\beta = 0$ is sufficient to receive a corresponding estimate.

Let

$$D^\psi(x) = D_0^\psi(x) = 2^d \sum_s \sum_{k\in\rho^+(s)} \prod_{j=1}^{d} \psi_j(k_j) \cos k_j x_j,$$

where $\rho^+(s) = \{k = (k_1, \ldots, k_d): \ \left[2^{s_j-1}\right] \le k_j < 2^{s_j}, \ j = \overline{1,d}\}$. By S we denote a set of vectors $s \in \mathbb{N}^d$, such that $(s,1) = n$ and $|\theta_M \cap \rho^+(s)| \le \frac{1}{2}|\rho^+(s)|$ hold. Then, using proposition 8 (if $p = 2$), we get

$$I_1 = \left\|D_\beta^\psi(\cdot) - P(\theta_M;\cdot)\right\|_q \gg$$

$$\gg \left(\sum_s \left\|\delta_s\left(D^\psi(\cdot) - P(\theta_M;\cdot)\right)\right\|_2^q \cdot 2^{(s,1)\left(\frac{1}{2}-\frac{1}{q}\right)q}\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{(s,1)=n} \left\|\delta_s\left(D^\psi(\cdot) - P(\theta_M;\cdot)\right)\right\|_2^q\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{s\in S} \left\|\sum_{k\in\rho^+(s)} \left(D^\psi(\cdot) - P(\theta_M;\cdot)\right)\right\|_2^q\right)^{\frac{1}{q}}.$$

Further, according to the Parseval equality, we can write

$$I_1 \gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{s\in S} \left(\sum_{k\in\rho^+(s)} \left(\prod_{j=1}^{d} \psi_j(k_j)\right)^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{s\in S} \left(\min_{(s,1)=n} \prod_{j=1}^{d} \psi_j\left(2^{s_j}\right)\right)^q 2^{(s,1)\frac{q}{2}}\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)} \Phi(n) \left(2^{\frac{nq}{2}} |S|\right)^{\frac{1}{q}} \gg$$

$$\gg 2^{n\left(\frac{1}{2}-\frac{1}{q}\right)}\Phi(n)2^{\frac{n}{2}}n^{\frac{d-1}{q}} = \Phi(n)2^{n\left(1-\frac{1}{q}\right)}n^{\frac{d-1}{q}} \tag{9}$$

So, taking into account that $M \asymp 2^n n^{d-1}$, from (9) we receive

$$e_M\left(D_\beta^\psi\right)_q \gg \Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)},\ 1<q\le 2. \tag{10}$$

The lower estimate is proven. The relation (7) follows from (8) and (10).

The theorem is proven.

Remark 2. *In the case* $\psi_j\left(|k_j|\right) = |k_j|^{-r_j}$, $r_j > 1-\frac{1}{q}$, $1\le q\le 2$, $k_j\in\mathbb{Z}\backslash\{0\}$, $j=\overline{1,d}$, *corresponding results were obtained by E. S. Belinskii [8, 9].*

Further, by using the lower estimate established in theorem 1, we get estimates of the best M–term trigonometric approximations for the classes of functions $L_{\beta,1}^\psi$.

The theorem holds.

Theorem 2. *Let* $1<q\le 2$, $\psi_j\in D$, $\beta_j\in\mathbb{R}$, $j=\overline{1,d}$, *and, besides, there exists* $\varepsilon>0$ *such that* $\psi_j\left(|k_j|\right)|k_j|^{1-\frac{1}{q}+\varepsilon}$ *are nonincreasing. Then for arbitrary natural* M *and* n *that satisfy condition* $M\asymp 2^n n^{d-1}$, *the relation holds*

$$\begin{gathered}\Phi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)} \ll e_M\left(L_{\beta,1}^\psi\right)_q \ll \\ \ll \Psi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}.\end{gathered} \tag{11}$$

Proof. The upper estimate follows from the relation (4) and the already known result for the best orthogonal trigonometric approximations. Given proposition 7 we get

$$\begin{gathered}e_M\left(L_{\beta,1}^\psi\right)_q \ll e_M^\perp\left(L_{\beta,1}^\psi\right)_q \ll \\ \ll \Psi(n)M^{1-\frac{1}{q}}(\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, 1<q\le 2.\end{gathered}$$

Let us obtain the lower estimate. By virtue of proposition 9 and (1) we can write

$$\begin{gathered}e_M\left(L_{\beta,1}^\psi\right)_q = \sup_{f\in L_{\beta,1}^\psi}\inf_{\theta_M}\sup_{\substack{P_\beta^\psi\in L^\perp(\theta_M),\\ \|P_\beta^\psi\|_{q'}\le 1}}\left|\int_{\pi_d} f(x)\overline{P_\beta^\psi(x)}dx\right| = \\ = \sup_{\|\varphi\|_1\le 1}\inf_{\theta_M}\sup_{\substack{P_\beta^\psi\in L^\perp(\theta_M),\\ \|P_\beta^\psi\|_{q'}\le 1}}\left|\int_{\pi_d}\left((2\pi)^{-d}\int_{\pi_d}\varphi(t)D_\beta^\psi(x-t)dt\right)\overline{P_\beta^\psi(x)}dx\right|.\end{gathered} \tag{12}$$

Now we are going to verify the conditions of the Fubini theorem (see, e.g., [30, c. 336]) for an integral on the right side of (12). Let us consider the integral

$$\int_{\pi_d}\varphi(t)\left(\int_{\pi_d}D_\beta^\psi(x-t)\overline{P_\beta^\psi(x)}dx\right)dt. \tag{13}$$

Since $D^\psi_\beta \in L_q$, $1<q<\infty$, and $P^\psi_\beta \in L_{q'}$, then using the Holder's inequality we get

$$\int\limits_{\pi_d} D^\psi_\beta(x-t)\overline{P^\psi_\beta(x)}dx \le \left\|D^\psi_\beta\right\|_q \left\|P^\psi_\beta\right\|_{q'},$$

and then for an arbitrary function $\varphi \in L_1$ the integral (13) is convergent.

After changing the order of integration in (12) we receive

$$e_M\left(L^\psi_{\beta,1}\right)_q = \sup_{\|\varphi\|_1\le 1} \inf_{\theta_M} \sup_{\substack{P^\psi_\beta \in L^\perp(\theta_M),\\ \|P^\psi_\beta\|_{q'}\le 1}} \int\limits_{\pi_d} \varphi(t)\times$$

$$\times\left((2\pi)^{-d}\int\limits_{\pi_d} D^\psi_\beta(x-t)\overline{P^\psi_\beta(x)}dx\right)dt.$$

Using first the Holder's inequality (if $p=1$, $p'=\infty$) and then proposition 9 we get

$$e_M\left(L^\psi_{\beta,1}\right)_q = \inf_{\theta_M} \sup_{\substack{P^\psi_\beta \in L^\perp(\theta_M),\\ \|P^\psi_\beta\|_{q'}\le 1}} \left\|(2\pi)^{-d}\int\limits_{\pi_d} D^\psi_\beta(x-t)\overline{P^\psi_\beta(x)}dx\right\|_\infty \ge$$

$$\ge \inf_{\theta_M} \sup_{\substack{P^\psi_\beta \in L^\perp(\theta_M),\\ \|P^\psi_\beta\|_{q'}\le 1}} \left|(2\pi)^{-d}\int\limits_{\pi_d} D^\psi_\beta(x-t)\overline{P^\psi_\beta(x)}dx\right| =$$

$$= (2\pi)^{-d} e_M\left(D^\psi_\beta\right)_q.$$

By virtue of theorem 7 we can write

$$e_M\left(L^\psi_{\beta,1}\right)_q \gg \Phi(n) M^{1-\frac{1}{q}} (\log M)^{2(d-1)\left(\frac{1}{q}-\frac{1}{2}\right)}, \quad 1<q\le 2.$$

The lower estimate and consequently theorem 2 is proven.

Remark 3. *The corresponding statement if* $\psi_j(|k_j|) = |k_j|^{-r_j}$, $r_j > 1-\frac{1}{q}$, $1<q\le 2$, $k_j \in \mathbb{Z}\backslash\{0\}$, $j=\overline{1,d}$, *was formulated by A.S. Romanyuk [17].*

4. Conclusions

The paper continues investigation of the approximative characteristics that where considered earlier by Temlyakov V. N., Stepanets A. I., Romanyuk A. S. and other mathematicians. Many results for the best M-term and orthogonal trigonometric approximations of classes of functions $B^r_{p,\theta}$, $W^r_{\beta,p}$, H^r_p are already obtained. Note that the great attention was paid to classes of functions of one variable. Nevertheless the problem of estimation of the best M-term approximations of classes $L^\psi_{\beta,1}$ of multivariate (ψ,β)-differentiable functions remained unsolved until now. We have obtained order relations of the quantities $e_M(f)_q$

for the concrete functions D^{ψ}_{β}, that are of interest themselves. And besides, by using established results, we have written down the order relations for classes $L^{\psi}_{\beta,1}$.

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