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SOLVING LYAPUNOV-SYLVESTER OPERATOR EQUATIONS BY CONVEX PROGRAMMING METHODS

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РОЗВ'ЯЗУВАННЯ ОПЕРАТОРНОГО РІВНЯННЯ ЛЯПУНОВА-СІЛЬВЕСТРА МЕТОДАМИ ОПУКЛОГО ПРОГРАМУВАННЯ

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Abstract. In this paper necessary and sufficient conditions are given for solvability of Lyapunov-Sylvester operator equations. We explain how to solve these equations by the methods of convex programming. Our purpose is to investigate some properties of stochastic differential equations in Hilbert spaces. These objects arise in diverse areas of applied mathematics as models for various natural phenomena, in particular, the evolution of complex systems with infinitely many degrees of freedom. It is not trivial to carry over the results concerning stochastic differential equations in finite-dimensional spaces to the infinite dimensional case.
Keywords: Lyapunov-Sylvester operator equations, stochastic differential equation, Hilbert space.

Резюме. У роботі наведено необхідні та достатні умови розв'язання операторного рівняння Ляпунова-Сільвестра. Обгрунтовано застосування методів опуклого програмування для розв'язування таких рівнянь.
Ключові слова: операторне рівняння Ляпунова-Сільвестра, стохастичне диференціальне рівняння, гільбертів простір.

Introduction

We investigate of some properties of Ito-Skorohod stochastic differential equations in Hilbert spaces. In particular, we are interested in solvability of Lyapunov-Sylvester operator equations. Existence of a solution to such an equation provides stability of certain system of Ito-Skorohod stochastic differential equation in a Hilbert space [3].

1. Main result

Let X be a separable Hilbert space with the inner product (x, y) and the norm $|x|$. We investigate the stochastic differential equation

$$\mathrm{d}X_t = X_t A \mathrm{d}t + \sum_{k\geq 1}\left(X_t B_k \mathrm{d}W_k(t) + \int_U C_k(u) X_t \tilde{\nu}_k(\mathrm{d}t, \mathrm{d}u)\right) \tag{1}$$

defined on a fixed filtered probability space. Here A, B_k and $C_k(u)$ are some unbounded linear operators defined on a dense set D in X satisfying

$$\sum_{k\geq 1} |B_k x|^2 < \infty \quad \text{and} \quad \sum_{k\geq 1} |C_k(u)x|^2 < \infty$$

for $x \in D$; $W_1(\cdot)$, $W_2(\cdot), \ldots$ are independent Wiener processes; $\tilde{\nu}_1(\cdot,\cdot)$, $\tilde{\nu}_2(\cdot,\cdot), \ldots$ are independent centered Poisson measures. Furthermore, it is assumed that X_0 is independent of $\{W_k,\ k = 1, 2, \ldots\}$ and $\{\tilde{\nu}_i,\ i = 1, 2, \ldots\}$. We are interested in the case when equation (1) has a solution with finite second moment. By a solution to (1) is meant a strong operator process X_t such that, for $x \in D$, $X_t x$ has a stochastic differential obtained by the application of the right-hand side of (1) to x

$$A^* H + HA + \sum_{k\geq 1}\left(B_k^* H B_k + \int_U C_k^*(u) H C_k(u) \Pi_k(\mathrm{d}u)\right) = -G. \tag{2}$$

In the following we shall use an approach based on the methods of convex programming. Instead of dealing with the Lyapunov-Sylvester operator equation we shall solve an equivalent optimization problem.

Define the objective function

$$\begin{gathered}\varphi_0(H) \equiv -\lambda_{\min}\Big(-A^*H - HA - \\ - \textstyle\sum_{k\geq 1}\left(B_k^* H B_k - \int_U C_k^*(u) H C_k(u)\Pi_k(\mathrm{d}u)\right)\Big), \\ H \in \bar{L},\end{gathered} \tag{3}$$

$$\bar{L} = \{H:\ \lambda_{\min}(H) \geq 0,\ \lambda_{\max} \leq 1\}, \tag{4}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are lower and upper bound of the operator spectrum respectively.

Now we consider the optimization problem

$$H^* = \arg\min_{H\in\bar{L}} \varphi_0(H). \tag{5}$$

Lemma 1. *Optimization problem (5) has a solution.*

Proof. Let $\bar{L}$ be the intersection of the set of positive semidefinite operators and the ball of radius 1. Note that $\bar{L}$ is a compact set. The function $\varphi_0(H)$ is continuous and attains its minimal value by the Weierstrass theorem. The proof of Lemma 1 is complete.

Now we can reformulate a criterion of asymptotic stability in the mean square.

Theorem 1. *System (1) is asymptotically stable in the mean square if, and only if, there exists operator H_0 which is a solution to problem (5) for $\varphi_0(H_0) < 0$.*

Further, we investigate the structure and the domain of function $\varphi_0(H)$.
Lemma 2. *The function $\varphi_0(H)$ is convex.*
Proof. The function $\lambda_{\min}$ is concave because $H \in \bar{L}$. Therefore, for an arbitrary $\xi \in (0,1)$

$$\begin{gathered}
\varphi_0(\xi H_1 + (1-\xi)H_2) = -\lambda_{\min}\Bigg[- A^*(\xi H_1 + (1-\xi)H_2) - \\
-(\xi H_1 + (1-\xi)H_2)A - \sum_{k\geq 1}\Bigg(B_k^*(\xi H_1 + (1-\xi)H_2)B_k - \\
- \int_U C_k^*(u)(\xi H_1 + (1-\xi)H_2)C_k(u)\Pi_k(du)\Bigg)\Bigg] = \\
= -\lambda_{\min}\Bigg[- A^*\xi H_1 - A^*(1-\xi)H_2 - \xi H_1 A - (1-\xi)H_2 A - \\
- \sum_{k\geq 1}\Bigg(B_k^*\xi H_1 B_k - B_k^*(1-\xi)H_2 B_k - \\
- \int_U C_k^*(u)\xi H_1 C_k(u)\Pi_k(du) - \int_U C_k^*(u)(1-\xi)H_2 C_k(u)\Pi_k(du)\Bigg)\Bigg] = \\
= -\lambda_{\min}\Bigg[\xi(-A^*H_1 - H_1 A - \sum_{k\geq 1}\Bigg(B_k^* H_1 B_k - \\
- \int_U C_k^*(u)H_1 C_k(u)\Pi_k(du)\Bigg) + (1-\xi)\Bigg(- A^*H_2 - H_2 A - \\
- \sum_{k\geq 1}\Bigg(B_k^* H_2 B_k - \int_U C_k^*(u)H_2 C_k(u)\Pi_k(du)\Bigg)\Bigg)\Bigg] \leq \\
\leq -\xi\lambda_{\min}\Bigg[- A^*H_1 - H_1 A - \sum_{k\geq 1}\Bigg(B_k^* H_1 B_k - \\
- \int_U C_k^*(u)H_1 C_k(u)\Pi_k(du)\Bigg)\Bigg] - (1-\xi)\lambda_{\min}\Bigg[- A^*H_2 - H_2 A - \\
- \sum_{k\geq 1}\Bigg(B_k^* H_2 B_k - \int_U C_k^*(u)H_2 C_k(u)\Pi_k(du)\Bigg)\Bigg] = \\
= \xi\varphi_0(H_1) + (1-\xi)\varphi_0(H_2).
\end{gathered} \tag{6}$$

The proof of Lemma 2 is complete.
Lemma 3. *The set $\bar{L}$ defined in (4) is convex.*
Proof. This follows from the fact that the set $\bar{L}$ is the intersection of a convex cone of positive semidefinite operators and the ball of radius 1.
Definition 1. We call

$$\langle H_1, H_2\rangle \equiv \sum_{i\geq 1}\sum_{j\geq 1} h_{ij}^1 h_{ij}^2 \tag{7}$$

inner product of operators $\{H_1, H_2\} \subset \bar{L}$ where $H_1 = \{h_{ij}^1\}$, $H_2 = \{h_{ij}^2\}$, $i, j \geq 1$.
Definition 2. The operator Φ satisfying

$$\varphi(H) - \varphi(H_0) \geq \langle \Phi(\varphi(H_0)), H - H_0\rangle$$

is called generalized gradient of the convex function $\varphi(H)$ at point $H_0 \in \bar{L}$.

Definition 3. The set of operators Φ which are generalized gradients of a function $\varphi(H)$ at point $H_0 \in \bar{L}$ is called gradient set (notation $G_\varphi[H]$).
Theorem 2. *The generalized gradient of the function*

$$\varphi_0(H) = -\lambda_{\min}\Bigg(A^*H - HA- \\ - \sum_{k\geq 1}\Bigg(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\Bigg)\Bigg), \quad H \in \bar{L} \tag{8}$$

at the point $H_0 \in L$ is determined by the operator

$$\Phi = \{\varphi_{ij}\}, \ \varphi_{ij} = -y^*_{\min}\Bigg(- A^*\Delta_{ij} - \Delta_{ij}A- \\ - \sum_{k\geq 1}\Bigg(B_k^*\Delta_{ij}B_k - \int_U C_k^*(u)\Delta_{ij}C_k(u)\Pi_k(du)\Bigg)\Bigg)y_{\min}, \ i,j = 1,2,..., \tag{9}$$

where Δ_{ij} is an infinite matrix such that 1 is located at the intersection of the ith row and the jth column, while the other elements are 0, $y_{\min}$ is a unit vector such that the quadratic form

$$y^*\Bigg(- A^*H_0 - H_0A - \sum_{k\geq 1}\Bigg(B_k^*H_0B_k - \int_U C_k^*(u)H_0C_k(u)\Pi_k(du)\Bigg)\Bigg)y$$

attains minimal value.
Proof. Plainly, there is the equality of symmetric positive semi-definite operators

$$\lambda_{\min}\big[- A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big] = \\ = \min_{|y|=1}\big\{y^*(-A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)y\big\}.$$

Further,

$$\varphi_0(H) - \varphi_0(H_0) = -\lambda_{\min}\big(- A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB_k- \\ - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big) + \lambda_{\min}\big(- A^*H_0 - H_0A- \\ - \sum_{k\geq 1}\big(B_k^*H_0B_k - \int_U C_k^*(u)H_0C_k(u)\Pi_k(du)\big)\big) = \\ = \min_{|y|=1}\big\{y^*\big(- A^TH - HA - \sum_{k\geq 1}\big(B_k^*HB_k- \\ - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big)y\big\} + \min_{|y|=1}\big\{y^*(-A^TH_0 - H_0A- \\ - \sum_{k\geq 1}\big(B_k^*H_0B_k - \int_U C_k^*(u)H_0C_k(u)\Pi_k(du)\big)\big)y\big\}. \tag{10}$$

Assume that the first summand of (10) attains the minimal value at the unit sphere when $y = y_1$, and the second summand does so when

$y = y_{\min}$. We have

$$\varphi_0(H) - \varphi_0(H_0) = -y_1^*\big(-A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB- \\ - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big)y_1 + y_{\min}^*\big(-A^*H_0 - H_0A- \\ - \sum_{k\geq 1}\big(B_k^*H_0B_k - \int_U C_k^*(u)H_0C_k(u)\Pi_k(du)\big)\big)y_{\min} = \\ = y_1^*\big(-A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big)y_1 + \\ + y_{\min}^*\big(-A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big)y_{\min} - \\ - y_{\min}^*\big[-A^*(H-H_0) - (H-H_0)A - \sum_{k\geq 1}\big(B_k^*(H-H_0)B_k - \\ - \int_U C_k^*(u)(H-H_0)C_k(u)\Pi_k(du)\big)\big]y_{\min}.$$

Since the quadratic form

$$y^*\Bigg(-A^*H - HA - \sum_{k\geq 1}\Bigg(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\Bigg)\Bigg)y$$

attains its minimum at y_1, then

$$-y_1^*\big(-A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big)y_1 + \\ + y_{\min}^*\big(-A^*H - HA - \sum_{k\geq 1}\big(B_k^*HB_k - \int_U C_k^*(u)HC_k(u)\Pi_k(du)\big)\big)y_{\min} \geq 0.$$

It follows that

$$\varphi_0(H) - \varphi_0(H_0) \geq -y_{\min}^*\big(-A^*(H-H_0) - (H-H_0)A- \\ - \sum_{k\geq 1}\big(B_k^*(H-H_0)B_k - \int_U C_k^*(u)(H-H_0)C_k(u)\Pi_k(du)\big)\big)y_{\min}. \tag{11}$$

Any operator $H - H_0$ can be represented as

$$H - H_0 = \sum_{i\geq 1}\sum_{j\geq 1}(h_{ij} - h_{ij}^0)\Delta_{ij}.$$

Indeed,

$$-y_{\min}^*\big(-A^*(H-H_0) - (H-H_0)A - \sum_{k\geq 1}\big(B_k^*(H-H_0)B_k- \\ - \int_U C_k^*(u)(H-H_0)C_k(u)\Pi_k(du)\big)\big)y_{\min} = \\ = -y_{\min}^*\big[-A^*\sum_{i\geq 1}\sum_{j\geq 1}(h_{ij}-h_{ij}^0)\Delta_{ij} - \sum_{i\geq 1}\sum_{j\geq 1}(h_{ij}-h_{ij}^0)\Delta_{ij}A- \\ - \sum_{k\geq 1}\big(B_k^*\sum_{i\geq 1}\sum_{j\geq 1}(h_{ij}-h_{ij}^0)\Delta_{ij}B_k- \\ - \int_U C_k^*(u)\sum_{i\geq 1}\sum_{j\geq 1}(h_{ij}-h_{ij}^0)\Delta_{ij}C_k(u)\Pi_k(du)\big)\big]y_{\min} = \\ = \sum_{i\geq 1}\sum_{j\geq 1}\big\{ -y_{\min}^*\big(-A^*\Delta_{ij} - \Delta_{ij}A - \sum_{k\geq 1}\big(B_k^*\Delta_{ij}B_k-$$

$$-\int_U C_k^*(u)HC_k(u)\Pi_k(du))y_{\min}\}(h_{ij}-h_{ij}^0)=\sum_{i\geq 1}\sum_{j\geq 1}\varphi_{ij}(h_{ij}-h_{ij}^0).$$

According to Definition 2 we have

$$\varphi_0(H)-\varphi_0(H_0)\geq\langle\Phi,H-H_0\rangle,$$

where

$$\Phi=\{\varphi_{ij}\},\ \varphi_{ij}=-y_{\min}^*\big(-A^*\Delta_{ij}-\Delta_{ij}A-$$
$$-\sum_{k\geq 1}\big(B_k^*\Delta_{ij}B_k-\int_U C_k^*(u)\Delta_{ij}C_k(u)\Pi_k(du)\big)\big)y_{\min}.$$

The proof of Theorem 2 is complete.

Now we replace problem (5) by a problem of unconditional minimization by virtue of introducing the Lagrange function

$$L(H,\beta)=\varphi_0(H)+\beta_1\varphi_1(H)+\beta_2\varphi_2(H),\ \beta_1\geq 0,\ \beta_2\geq 0. \tag{12}$$

Here, $\varphi_1(H)=-\lambda_{\min}(H)$, $\varphi_2(H)=\lambda_{\max}(H)-1$. Since the functions $\varphi_0(H)$, $\varphi_1(H)$, $\varphi_2(H)$ are convex, we have a problem of convex programming.

Theorem 3. *A point $H_0\in\bar L$ is a solution to problem (5) if, and only if, there exists a vector (β_1^0,β_2^0), such that the the triple $(H_0,\beta_1^0,\beta_2^0)$ is a saddle point of Lagrange function (12) on the set*

$$\bar L\times\{\beta_1\geq 0,\ \beta_2\geq 0\}.$$

The proof is a consequence of the Kuhn-Tucker theorem.

Denote by $G_{\varphi_1}[H]$ the gradient set of the fuction $\varphi_1(H)$ at point H_0 and $\Phi=\{\varphi_{ij}^1\}$, $i,j\geq 1$ the set of operators with elements $\varphi_{ij}^1=-x_{\min}^T\Delta_{ij}x_{\min}$. Let $G_{\varphi_2}[H]$ be the gradient set of the function $\varphi_2(H)$ at point H_0, where $\Phi=\{\varphi_{ij}^2\}$, $i,j=\overline{1,n}$, $\varphi_{ij}^2=-x_{\max}^T\Delta_{ij}x_{\max}$, and $x_{\min}$, $x_{\max}$ are unit vectors at which the quadratic form x^THx attains minimum and maximum value, respective.

The linear combination

$$G_L[H]=G_{\varphi_0}[H]+\beta_1G_{\varphi_1}[H]+\beta_2G_{\varphi_2}[H]$$

is a gradient set of the Lagrange function $L(H,\beta)$. With this at hand the theorem can be reformulated in terms of the generalized gradients.

Theorem 4. *$H_0\in\bar L$ is a solution to problem (5) if, and only if, there exist vector (β_1^0,β_2^0), such that the gradient set*

$$G_L[H]=G_{\varphi_0}[H]+\beta_1G_{\varphi_1}[H]+\beta_2G_{\varphi_2}[H]$$

contains zero operator, i.e., $0\in G_L(H)$.

Thus, when solving optimization problem (5) to build gradient and set it to check for zero operator H_0, where the generalized gradient becomes zero, gives the solution of the problem (5).

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