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ABOUT FINDING A NEAREST PAIR OF POINTS ON TWO NONINTERSECTING SMOOTH CURVES IN EUCLIDEAN SPACE

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ПРО ПОШУК НАЙБЛИЖЧОЇ ПАРИ ТОЧОК НА ДВОХ ГЛАДКИХ КРИВИХ, ЩО НЕ ПЕРЕТИНАЮТЬСЯ В ЕВКЛІДОВОМУ ПРОСТОРІ

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Abstract. The paper presents results of numerical experiments of finding a nearest pair of points lying on two nonintersecting smooth curves in Euclidean space, i.e., two points on the curves with minimum distance between them. Mathematical model that proposed is based on solving motion equations describing dynamics of interacting material points. The examples of method application to the some spatial curves are given.
Keywords: smooth curves, nearest pair, algorithm, Hamilton equation.

Анотація. В роботі представлено результати чисельних експериментів з пошуку найближчої пари точок, які лежать на двох гладких кривих, що не перетинаються в евклідовому просторі, тобто двох точок на кривих з мінімальною відстані між ними. Запропонована математична модель основана на розв'язку рівнянь руху, які описують динаміку взаємодії матеріальних точок. Наведено приклади застосування методу до деяких просторових кривих.
Ключові слова: гладкі криві, найближча пара, алгоритм, рівняння Гамільтона.

1. Notation and problem statement

Let A, B — subsets of Euclidean space $\mathbb{R}^n$. The search for the closest pairs or calculate the minimum distance between the elements of this sets is the popular optimization problem

$$\text{find} \;\; a_0 \in A, \; b_0 \in B: \;\; \|a_0 - b_0\| = \min_{a\in A,\, b\in B} \|a - b\|. \tag{1}$$

The effective solve of problem (1) is a key element in many applications (for example, when creating route planners for robotics). Therefore, it is natural that many investigators paid attention to the problem (1) for different classes of sets [1–8].

We consider the problem (1) for two nonintersecting smooth curves in $\mathbb{R}^3$

$$\vec{x}(\alpha) = (x_1(\alpha), x_2(\alpha), x_3(\alpha)), \quad \vec{y}(\beta) = (y_1(\beta), y_2(\beta), y_3(\beta)),$$

where radius vectors $\vec{x}$, $\vec{y}: \mathbb{R} \to \mathbb{R}^3$ — continuously differentiable vector functions of scalar parameters α, β.

A minimization problem has a form

$$\|\vec{x}(\alpha) - \vec{y}(\beta)\| \to \min_{\alpha,\,\beta}.$$

2. Mathematical model

In the paper [7] we propose a method for solving this problem that uses a model of physical interaction of material points with a potential energy of general form.

Hamilton motion equations for our mathematical model are

$$\begin{cases} \dot{\alpha} = \frac{1}{\|\vec{x}'_\alpha\|^2}\frac{p_\alpha}{m_\alpha}, \\ \dot{\beta} = \frac{1}{\|\vec{y}'_\beta\|^2}\frac{p_\beta}{m_\beta}, \\ \dot{p}_\alpha = \frac{p_\alpha^2}{m_\alpha}\frac{\langle \vec{x}'_\alpha, \vec{x}''_\alpha\rangle}{\|\vec{x}'_\alpha\|^4} + \frac{1}{r}\frac{\partial U}{\partial r}\langle \vec{r}, \vec{x}'_\alpha\rangle - \lambda p_\alpha, \\ \dot{p}_\beta = \frac{p_\beta^2}{m_\beta}\frac{\langle \vec{y}'_\beta, \vec{y}''_\beta\rangle}{\|\vec{y}'_\beta\|^4} - \frac{1}{r}\frac{\partial U}{\partial r}\langle \vec{r}, \vec{y}'_\beta\rangle - \lambda p_\beta, \end{cases} \tag{2}$$

where $\vec{r}(\alpha,\beta) = \vec{y}(\beta) - \vec{x}(\alpha)$ — radius vector between two points on given curves; $r = \|\vec{r}\|$, $m_\alpha > 0$, $m_\beta > 0$, α, β, p_α, p_β — independent variables (coordinates and impulses); λ — positive constant.

Examples with analytical solution. Any two curves in plane are suitable for test the method, if we can easy find an analytical solution.

Let one of curves be a parabola symmetric respect to y axis and passing through the origin, and second one the line in xy plane that does not intersect parabola

$$\begin{cases} y &= & ax^2, a > 0; \\ y &= & bx + c. \end{cases} \tag{3}$$

Lets construct a tangent line to parabola that simultaneously parallel to the line. Then drop a perpendicular from tangency point to our line. Obviously that distance between tangency point and point of projection is the minimum distance.

First get the tangency point as a solution of quadratic equation

$$ax^2 - bx - c = 0, \tag{4}$$

where

$$x_{1,2} = \frac{b \pm \sqrt{b^2 + 4ac}}{2a}. \tag{5}$$

Two roots $x_1 \neq x_2$ means that parabola and line are crossed that corresponds to $b^2 + 4ac > 0$.

We search for the tangency point with only one root, i.e. $x_1 = x_2$, that corresponds to $b^2 + 4ac = 0$ or $c = -\frac{b^2}{4a}$.

When roots are absent or all them imaginary then the line is parallel to the tangent line, i.e. $k^2 + 4ab < 0$.

Then for the tangency point we have

$$c = -\frac{b^2}{4a}. \tag{6}$$

For (3) the radius vectors have the form

$$\begin{cases} \vec{x} &= (\alpha, b\alpha + c, 0); \\ \vec{y} &= (\beta, a\beta^2, 0). \end{cases} \tag{7}$$

Respectively $\vec{r} = \vec{y} - \vec{x} = (\beta - \alpha, a\beta^2 - b\alpha - c, 0)$. Then

$$r^2 = (\beta - \alpha)^2 + (a\beta^2 - b\alpha - c)^2 \tag{8}$$

and have equation

$$\frac{\partial(r^2)}{\partial\alpha} = -ba\beta^2 - \beta + \alpha(b^2 + 1) + bc = 0. \tag{9}$$

From (9) the projection of the tangency point on the line is

$$\alpha_0 = \frac{\beta + ba\beta^2 - bc}{1 + b^2}. \tag{10}$$

Similarly

$$\frac{\partial(r^2)}{\partial\beta} = (1 + 2ab\beta)\alpha - 2a^2\beta^3 - (1 - 2ac)\beta = 0. \tag{11}$$

Solution for β_0 get from (11) after substitution α_0

$$\begin{cases} \beta = \frac{b}{2a}; \\ \beta = \frac{b + \sqrt{b^2+4ac}}{2a}; \\ \beta = -\frac{-b + \sqrt{b^2+4ac}}{2a}. \end{cases} \tag{12}$$

From (6) follows that last two roots are imaginary. Then

$$\beta_0 = \frac{b}{2a} \tag{13}$$

and corresponding functions are

$$\begin{cases} \vec{x}'_\alpha = (1, b, 0); \\ \vec{y}'_\beta = (1, 2a\beta, 0); \\ \vec{x}''_\alpha = (0, 0, 0); \\ \vec{y}''_\beta = (0, 2a, 0); \\ \vec{x}'^2_\alpha = \langle \vec{x}'_\alpha, \vec{x}'_\alpha \rangle = 1 + b^2; \\ \langle \vec{x}'_\alpha, \vec{x}''_\alpha \rangle = 0; \\ \vec{y}'^2_\beta = \langle \vec{y}'_\beta, \vec{y}'_\beta \rangle = 1 + 4a^2\beta^2; \\ \langle \vec{y}'_\beta, \vec{y}''_\beta \rangle = 4a^2\beta; \\ |\vec{x}'_\alpha|^4 = \langle \vec{x}'_\alpha, \vec{x}'_\alpha \rangle^2 = (1 + b^2)^2; \\ |\vec{y}'_\beta|^4 = \langle \vec{y}'_\beta, \vec{y}'_\beta \rangle^2 = (1 + 4a^2\beta^2)^2; \\ \vec{r} = \vec{y}(\beta) - \vec{x}(\alpha) = (\beta - \alpha, a\beta^2 - b\alpha - c, 0); \\ \langle \vec{r}, \vec{r} \rangle = (\beta - \alpha)^2 + (a\beta^2 - b\alpha - c)^2; \\ r = \sqrt{(\beta - \alpha)^2 + (a\beta^2 - b\alpha - c)^2}; \\ \langle \vec{r}, \vec{x}'_\alpha \rangle = (\beta - \alpha) + ab\beta^2 - b^2\alpha - cb; \\ \langle \vec{r}, \vec{y}'_\beta \rangle = (\beta - \alpha) + 2a\beta(a\beta^2 - b\alpha - c). \end{cases} \tag{14}$$

Let the masses and coefficients of friction for material points be

$$\begin{cases} m_\alpha = m_\beta = m; \\ \lambda = \gamma/m. \end{cases} \tag{15}$$

Potential energy of spring it derivative and force

$$\begin{cases} U = \kappa \frac{r^2}{2}; \\ \frac{\partial U}{\partial r} = \kappa r; \\ F_r = -\frac{\partial U(r)}{\partial r} = -\kappa r. \end{cases} \tag{16}$$

From (2) with (14), (15), (16) we have

$$\begin{cases} \dot{\alpha} = \frac{p_\alpha}{m}; \\ \dot{\beta} = \frac{1}{(1+4a^2\beta^2)} \frac{p_\beta}{m}; \\ \dot{p_\alpha} = 2\kappa(\beta - \alpha) - \gamma \frac{p_\alpha}{m}; \\ \dot{p_\beta} = \frac{4a^2\beta}{(1+4a^2\beta^2)^2} \frac{p_\beta^2}{m} - 2\kappa(\beta - \alpha + 2a\beta(a\beta^2 - b\alpha - c)) - \gamma \frac{p_\beta}{m}. \end{cases} \tag{17}$$

3. Features of algorithm implementation

When we go to the minimum there are two interesting points: a local minimum and a point of turn.

That points have three key characteristics.

Let the minimum be set by an interval with indices i, $i-1, i-2$. Similarly, the point of turn be set by an interval with indices k, $k-1, k-2$. It is evident that $k > i$.

Thus we have a neighborhood that include the minimum if next conditions are fulfillment

if $(r_{i-1} - r_{i-2}) < 0$ and $(r_i - r_{i-1}) > 0$ **then**
 $isMin := 1$. {set flag of minimum that was passed}

Further, motion will continue by inertia until the velocity falls to zero at a point of turn. Under action of gravity the movement will start in the opposite direction, that is, again to minimum. Such oscillatory process can reduces amplitude only due to the presence of frictional force. So the number of such iterations depends on the coefficient of friction γ.

Remark 1. Potential (16) automatically reduces the speed when draw near the minimum of distance or a points of turn. This is due to the next factors, namely:

1) due to the nature of the force of attraction and
2) due to the friction force.

Therefore it is important to be able to distinguish these points in the calculation process.

When system enters in oscillation process for fixing the turning point we have conditions

if $(r_{k-1} - r_{k-2}) > 0$ and $(r_k - r_{k-1}) < 0$ **then**
  $ReverCount := ReverCount + 1$; {counter of a point of turn}
  $isMin := 0$. {remove the flag of the minimum}

As a criterion for stopping it is logical to use the speed at the minimum point.

But there are two material points and two corresponding velocities vectors

$$|\dot{\vec{x}}(\alpha)| + |\dot{\vec{y}}(\beta)| = |\vec{v}(\alpha)| + |\vec{v}(\beta)| = \|\vec{x}'_\alpha \dot{\alpha}\| + \|\vec{y}'_\beta \dot{\beta}\|. \qquad (18)$$

On other hand distance between points $r(t)$ and also speed of it change are scalar functions. So, as a criterion for stopping we choose

$$\left| \frac{\Delta r}{\Delta t} \right| < \varepsilon. \qquad (19)$$

Remark 2. Despite the fact that the step of integration according to the curve parameter is constant, the step value for the $\Delta \vec{x}_i = \vec{v}(t_i)\Delta t$ coordinates is automatically adjusted by changing the speed.

Remark 3. To improve the efficiency of the method we do not exclude the control of parameter (2) in the turn points. But we do not use it to see how the method works in its pure form.

4. Numerical experiments

Let's first see how the method behaves when changing the coefficient of friction in examples that have an analytic solution.

Set the friction coefficient too large is not a good idea, because it slows down the search for a minimum and can lead to a stop in small cavities on the way to the minimum. Therefore, determining the satisfactory coefficient of friction is an important task.

First example. Let consider the curves from our example

$$\begin{cases} a = 2; \\ b = 2.5; \\ c = -1. \end{cases} \qquad (20)$$

TAB. 1. Numerical solutions[1)] of the system (17) with $\varepsilon = 1 \cdot 10^{-6}$ and $\Delta t_0 = 0.5$ for (20), (21), (22), where stop condition is (19).

N	γ/κ	$n^{2)}$	$n^{3)}_{T/2}$	r	$\|\Delta r\|^{4)}$	$\|\Delta \vec{x}\|^{4)}$	$\|\Delta \vec{y}\|^{4)}$
1	0.04	184	88	$8.124180 \cdot 10^{-2}$	$8.671 \cdot 10^{-8}$	$8.392 \cdot 10^{-5}$	$3.468 \cdot 10^{-5}$
2	0.05	123	59	$8.124401 \cdot 10^{-2}$	$2.297 \cdot 10^{-6}$	$2.262 \cdot 10^{-4}$	$3.827 \cdot 10^{-4}$
3	0.06	113	54	$8.124191 \cdot 10^{-2}$	$1.980 \cdot 10^{-7}$	$1.242 \cdot 10^{-4}$	$5.499 \cdot 10^{-5}$
4	0.07	99	46	$8.124173 \cdot 10^{-2}$	$2.084 \cdot 10^{-8}$	$5.832 \cdot 10^{-5}$	$1.299 \cdot 10^{-7}$
5	0.08	87	40	$8.124171 \cdot 10^{-2}$	$8.551 \cdot 10^{-10}$	$5.862 \cdot 10^{-7}$	$1.111 \cdot 10^{-5}$
6	0.09	75	33	$8.124186 \cdot 10^{-2}$	$1.460 \cdot 10^{-7}$	$9.357 \cdot 10^{-4}$	$1.016 \cdot 10^{-3}$
7	0.10	58	23	$8.124820 \cdot 10^{-2}$	$6.487 \cdot 10^{-6}$	$5.810 \cdot 10^{-3}$	$6.419 \cdot 10^{-3}$

1) Calculations were carried out in MatLab R2016b.
2) Total number of steps to the minimum.
3) Total number of half-periods (for movement between points of turns).
4) Denotement: $\|\Delta r\| = \|r - r^a\|$; $\|\Delta \vec{x}\| = \|\vec{x} - \vec{x}^a\|$; $\|\Delta \vec{y}\| = \|\vec{y} - \vec{y}^a\|$.

Initial conditions in (17)

$$\begin{cases} \alpha(0) = 0.9; \\ \beta(0) = 0.5; \\ p_\alpha(0) = 0; \\ p_\beta(0) = 0. \end{cases} \tag{21}$$

System parameters

$$\begin{cases} m = 1; \\ \kappa = 5; \\ \gamma/\kappa \in (0.04 \ldots 0.1). \end{cases} \tag{22}$$

The analytical solution for this experiment can be find from (7), (10), (13)

$$\begin{cases} \beta_0 = \frac{b}{2a} = 0.625; \\ \alpha_0 = \frac{\beta_0 + ba\beta_0^2 - bc}{1+b^2} = 0.700431; \\ \vec{x}^a = (0.700431, 0.7510776, 0); \\ \vec{y}^a = (0.625, 0.78125, 0); \\ r^a = 8.124171 \cdot 10^{-2}. \end{cases} \tag{23}$$

Second example. Formally the radius-vectors in the first example are spatial, but actually they are on a plane. Therefore, let's consider an example with ellipses [8]

$$\begin{cases} \vec{x} = (3 \cdot \sin(\alpha), 5 \cdot \cos(\alpha), 0); \\ \vec{y} = (3 \cdot \sin(\beta), 5 \cdot \cos(\beta), 7 - 3.75 \cdot \cos(\beta)). \end{cases} \tag{24}$$

TAB. 2. Numerical solutions[1)] of the system (24) with $\varepsilon = 1 \cdot 10^{-6}$ and $\Delta t_0 = 0.5$ for (25), (26), where stop condition is (19).

N	γ/κ	$n^{2)}$	$n^{3)}_{T/2}$	r	$\|\Delta r\|^{4)}$	$\|\Delta \vec{x}\|^{4)}$	$\|\Delta \vec{y}\|^{4)}$
1	0.2	36	8	3.25	$2.191 \cdot 10^{-7}$	$5.234 \cdot 10^{-4}$	$5.111 \cdot 10^{-4}$
2	0.3	27	6	3.25	$6.391 \cdot 10^{-8}$	$8.421 \cdot 10^{-4}$	$3.011 \cdot 10^{-4}$
3	0.4	19	2	3.25	$7.811 \cdot 10^{-8}$	$9.380 \cdot 10^{-4}$	$4.241 \cdot 10^{-4}$
4	0.5	18	1	3.25	$8.512 \cdot 10^{-9}$	$3.101 \cdot 10^{-4}$	$1.350 \cdot 10^{-4}$
5	0.6	19	0	3.25	$7.593 \cdot 10^{-8}$	$9.053 \cdot 10^{-4}$	$4.816 \cdot 10^{-4}$

[1)] Calculations were carried out in MatLab R2016b.
[2)] Total number of steps to the minimum.
[3)] Total number of half-periods (for movement between points of turns).
[4)] Denotement: $\|\Delta r\| = \|r - r^a\|$; $\|\Delta \vec{x}\| = \|\vec{x} - \vec{x}^a\|$; $\|\Delta \vec{y}\| = \|\vec{y} - \vec{y}^a\|$.

Position of ellipsies were chosen so that solution is obvious. Then for (2), initial conditions

$$\begin{cases} \alpha(0) = 3.0; \\ \beta(0) = 5.0; \\ p_\alpha(0) = 2.0; \\ p_\beta(0) = -2.5 \end{cases} \tag{25}$$

and parameters

$$\begin{cases} m = 1; \\ \kappa = 5; \\ \gamma/\kappa \in (0.2 \ldots 0.6). \end{cases} \tag{26}$$

Where the analytical solution

$$\begin{cases} \vec{x}^a = (0, 5, 0); \\ \vec{y}^a = (0, 5, 3.25); \\ r^a = 3.25. \end{cases}$$

5. Conclusions

In the framework of the physical analogy and Lagrange-Hamilton formalism motion equations of two material points that lie on two given curves respectively are deduced. Such approach allows us to use arbitrary potentials in motion equations including the typical physical potentials that depend only the distance between the material points. The minimum distance between the curves is the solution of this equations. The arguments for using harmonic potential are given. It is evident that approach based on the Lagrangian-Hamilton formalism also allows us to find a minimum distance not only between the curves in 3-dimensional space, but for the surfaces of various dimensions in multidimensional Euclidean spaces. For this purpose we must assume an internal coordinates of a surface embed in a multidimensional Euclidean space as the generalized coordinates of mechanical system with the potential energy. Such a generalization will be the subject of our next publication.

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