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NONUNIFORM QUANTUM ZENO EFFECT

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We study the quantum Zeno effect in the case when the time intervals between two consecutive measurements are different. It can be called a nonuniform quantum Zeno effect. Explicit examples of the nonuniform quantum Zeno effect are presented. We also show that maximal quantum Zeno effect in the case of a large number of measurements can be achieved when the time intervals between consecutive measurements are equal.

Key words: quantum evolution, Zeno effect.

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I. INTRODUCTION

Quantum Zeno effect is connected with paper [1] where for the first time this effect was called by such a name. But in fact its history is very long and closely related works were done much earlier [2, 3]. Since its discovery, quantum Zeno effect has attracted much attention and there are many papers on this subject. Recent reviews on quantum Zeno effect and its historical aspect can be found in [4, 5].

The quantum Zeno effect can be formulated as slowing the quantum evolution because of frequent observations. Let a quantum system at the time $t = 0$ be prepared in the pure state $|\psi_0\rangle$. Now we carry out N measurements at the time interval $\tau = t/N$ in order to check whether the system is still in the initial state. The probability to find the system in the initial state after the first measurement is

$$p(\tau) = \left|\langle\psi_0|e^{-iHt/\hbar}|\psi_0\rangle\right|^2 = 1 - \tau^2/\tau_z^2 + \dots, \qquad (1)$$

where $\tau_z^{-2} = \langle\psi_0|(\Delta H)^2|\psi_0\rangle/\hbar^2$ and τ_z is called the Zeno time, $\Delta H = H - \langle\psi_0|H|\psi_0\rangle$. If the measurement has a positive outcome, the wave function collapses to the initial one and the evolution starts anew from $|\psi_0\rangle$. When every time the measurement has a positive outcome the survival probability after N measurements for large N reads

$$p^{(N)}(t) = p^N(\tau) \sim \left(1 - \tau^2/\tau_z^2\right)^N \sim \exp\left(-t^2/N\tau_z^2\right). \qquad (2)$$

In the limit $N \to \infty$ we find $p^{(N)}(t) \to 1$. It means that infinitely frequent measurements halt the quantum evolution. Note that traditionally quantum Zeno effect is formulated for the case when the time intervals between two consecutive measurements are the same. There are also papers where the authors study general properties Zeno or anti-Zeno evolution considering different time intervals but do not use it for concrete calculations (see for instance [6, 7]).

In this paper we study quantum Zeno effect in the case when the time intervals between two consecutive measurements are different. We call it a nonuniform quantum Zeno effect. In the frame of this studies we ask the question when to carry out N measurements in order to obtain a maximal quantum Zeno effect, i. e. maximal probability to find the system in the initial state.

II. QUANTUM ZENO EFFECT WITH DIFFERENT TIME INTERVALS BETWEEN TWO CONSECUTIVE MEASUREMENTS

Now we carry out consecutively N measurements at the time intervals τ_i. It is convenient to write $\tau_i = \tau f(i)$, where we find τ from the condition $\sum_{i=1}^N \tau_i = t$ and thus

$$\tau_i = t\frac{f(i)}{\sum_{i=1}^{N} f(i)}, \qquad (3)$$

where $f(i)$ is such a positive function for which τ_i tends to 0 when N tends to infinity. This restriction on functions $f(i)$ gives us the possibility to obtain a continuous Zeno effect in the limit of the infinitely large N.

The probability of a positive outcome in each measurement for large N now reads

$$p^{(N)}(t) \sim \prod_{i=1}^{N}(1 - \tau_i^2/\tau_z^2) \sim \exp(-\alpha t^2/\tau_z^2), \qquad (4)$$

where

$$\alpha = \frac{\sum_{i=1}^{N} f^2(i)}{\left(\sum_{i=1}^{N} f(i)\right)^2}, \qquad (5)$$

or we can rewrite

$$\frac{1}{\alpha} = N - \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}\left(f(i) - f(j)\right)^2}{2\sum_{i=1}^{N} f^2(i)}. \qquad (6)$$

From (6) we see that the maximal value of $1/\alpha$ that corresponds to the minimal value of α given by (5) is achieved for $f(i) = f(j) = \text{const}$. So, we find

$$\alpha_{\min} = \frac{1}{N}. \tag{7}$$

The minimal value of α corresponds to the maximal quantum Zeno effect, i. e. to the maximal probability to find the system in the initial state. Thus we can state that the maximal quantum Zeno effect in the limit of a large number of measurements N is achieved when the time intervals between two consecutive measurements are equivalent. In this case probability (4) coincides with (2).

Now let us consider the nonuniform quantum Zeno effect for $f(i) = i^\gamma$, where in order to satisfy the condition $\tau_i \to 0$ when $N \to \infty$ we require that $\gamma \geq -1$. It is necessary to consider the cases $\gamma = -1$ and $\gamma = -1/2$ separately.

1) $\gamma = -1/2$. In the asymptotic of a large N we find

$$\sum_{i=1}^{N} f(i) = \sum_{i=1}^{N} \frac{1}{\sqrt{i}} \sim \int_1^N \frac{dx}{\sqrt{x}} = 2\sqrt{N} - 2 \sim 2\sqrt{N}, \tag{8}$$

$$\sum_{i=1}^{N} f^2(i) = \sum_{i=1}^{N} \frac{1}{i} \sim \int_1^N \frac{dx}{x} = \ln N. \tag{9}$$

Then according to (5)

$$\alpha = \frac{\ln N}{4N}. \tag{10}$$

2) $\gamma = -1$.

For a large N we have

$$\begin{aligned} &\sum_{i=1}^{N} f(i) = \sum_{i=1}^{N} \frac{1}{i} \sim \ln N, \\ &\sum_{i=1}^{N} f^2(i) = \sum_{i=1}^{N} \frac{1}{i^2} \sim \frac{\pi^2}{6} - \frac{1}{N}. \end{aligned} \tag{11}$$

In this case

$$\alpha = \frac{\pi^2}{6(\ln N)^2}. \tag{12}$$

3) $-1 < \gamma < -1/2$.

$$\begin{aligned} &\sum_{i=1}^{N} f(i) = \sum_{i=1}^{N} i^\gamma \sim \frac{N^{\gamma+1}}{\gamma+1}, \\ &\sum_{i=1}^{N} f^2(i) = \sum_{i=1}^{N} i^{2\gamma} \sim \zeta(-2\gamma), \end{aligned} \tag{13}$$

$$\alpha = \frac{(\gamma+1)^2 \zeta(-2\gamma)}{N^{2(\gamma+1)}}, \tag{14}$$

here ζ is Riemann's zeta-function.

4) $-1/2 < \gamma$.

$$\begin{aligned} &\sum_{i=1}^{N} f(i) = \sum_{i=1}^{N} i^\gamma \sim \frac{N^{\gamma+1}}{\gamma+1}, \\ &\sum_{i=1}^{N} f^2(i) = \sum_{i=1}^{N} i^{2\gamma} \sim \frac{N^{2\gamma+1}}{2\gamma+1}. \end{aligned} \tag{15}$$

$$\alpha = \frac{(\gamma+1)^2}{(2\gamma+1)N}. \tag{16}$$

Comparing this result with (7) we can state that in the case $\gamma \neq 0$ the quantum Zeno effect is less pronounced than in a uniform case. Note also that in the case $\gamma = 0$ we reproduce result (7) and obtain the maximal quantum Zeno effect.

III. CONCLUSIONS

We study a nonuniform quantum Zeno effect which is a quantum Zeno effect in the case of different time intervals between two consecutive measurements. As an explicit example of the nonuniform quantum Zeno effect we consider time interval function $f(i) = i^\gamma$. The asymptotic behavior of the survival probability for a large number of measurements appeared to depend essentially on the time interval function, namely, on γ. We ask the question when to carry out N measurements (what is function $f(i)$) in order to obtain maximal quantum Zeno effect, i.e. the maximal probability to find the system in the initial state. It was shown that the maximal quantum Zeno effect for the case of a large number of measurements is achieved for equivalent time intervals between two consecutive measurements, $f(i) = \text{const}$.

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НЕОДНОРІДНИЙ КВАНТОВИЙ ЕФЕКТ ЗЕНОНА

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Ми вивчаємо квантовий ефект Зенона у випадку, коли часові інтервали між двома послідовними вимірюваннями різні. Його можна назвати неоднорідним ефектом Зенона. Представлено явні приклади неоднорідного ефекту Зенона. Також показано, що максимальний ефект Зенона при великій кількості вимірів досягається, коли часові інтервали між послідовними вимірюваннями однакові.