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ON THE CHRAPLYVY TRANSFORMATION AND SOME FEATURES OF ITS APPLICATION FOR HIGHER-ORDER EXPANSIONS. II

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Using a sequence of canonical transformations with the generating functions for the case of particles with arbitrary masses, valid for particles of equal masses as well, the transformation of the relativistic two-body Hamiltonian containing even–even, even–odd, odd–even, and odd–odd terms into an even–even form is carried out up to the order $1/c^4$. It is shown that, similarly to the transformation based on the set of generators excluding the case of equal masses of the particles, the final form of the obtained approximate Hamiltonian is not uniquely defined; namely, it depends on the order of application of the initial generators in the procedure, and can involve certain extra terms, which are eliminated with an additional unitary transformation.

Key words: Chraplyvy transformation, transformed Hamiltonian, higher orders.

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In our recent paper [1], we considered the Chraplyvy transformation with the set of generating functions used in the sequence of canonical transformations, which is not applicable to the singular case of equal masses of the particles, and which is known as the "radical" transformation. The starting point in this procedure is the Hamiltonian of a relativistic two-body wave equation, which is represented in the following general form:

$$H = \beta_1 m_1 c^2 + \beta_2 m_2 c^2 + (\mathcal{E}\mathcal{E}) + (\mathcal{E}\mathcal{O}) + (\mathcal{O}\mathcal{E}) + (\mathcal{O}\mathcal{O}), \tag{1}$$

and involves even–even, even–odd, odd–even, and odd–odd terms, respectively (here for definiteness, it is assumed that $(\mathcal{E}\mathcal{E})$, $(\mathcal{O}\mathcal{O})$ are of order c^0, and $(\mathcal{O}\mathcal{E})$, $(\mathcal{E}\mathcal{O})$ of the order c^1). We obtained the higher-order transformed Hamiltonian which, in fact, was a continuation of the transformation in [2] for the case when the even–odd and odd–even terms from (1) commuted. At once, it was found there that the procedure under consideration had some ambiguity which consisted in the fact that the form of the transformed Hamiltonian depended on the order of application of the initial generators, and besides the "regular" terms, it could involve certain extra terms. We showed that the last ones could be removed by an additional unitary transformation.

Nevertheless, the case of equal masses is not less important and interesting for many problems. For example, the Chraplyvy transformation applied for higher-order expansions may be of some interest for the study of fine and hyperfine structure of the spectra of helium and positronium.

In his article [3], Chraplyvy proposed and justified the whole class of generating functions for the arbitrary-mass case, which included the special case of only different masses as a partial one. Using one set of them, which, for the first iteration, is

$$S_{oe} = -\frac{i\beta_1(1 \pm \beta_2)}{4m_1c^2}(\mathcal{O}\mathcal{E}), \tag{2a}$$

$$S_{eo} = -\frac{i\beta_2(1 \pm \beta_1)}{4m_2c^2}(\mathcal{E}\mathcal{O}), \tag{2b}$$

$$S_{oo} = -\frac{i(\beta_1 + \beta_2)}{4(m_1 + m_2)c^2}(\mathcal{O}\mathcal{O}), \tag{2c}$$

a transformation called the "least change" transformation, was performed to the second order in detail. It is well seen that these functions do not contain any mass differences in the denominators, and thus can be applicable to wave equations for two particles with arbitrary masses (see also [4, 5]). One can easily verify that generators (2) do not commute with the large terms from Eq. (1) and thereby get, after all the iterations, the even–odd, odd–even, and odd–odd terms of the same order and higher with the factors $\frac{1}{2}(1 \mp \beta_1)$, $\frac{1}{2}(1 \mp \beta_2)$, and $\frac{1}{2}(1 - \beta_1\beta_2)$, respectively. Thus this transformation has one feature consisting in the fact that the final transformed Hamiltonian H_{tr}, in addition to the "effective" terms, involves some undesirable terms of the order c^1, c^0, and so on. However, as shown by Chraplyvy [3], such terms are acceptable in the expression because all of them vanish when both particles are in positive or negative energy states.

Our main scope here is to convert the Hamiltonian (1) into an even–even form up to the order $1/c^4$ for the case of particles with arbitrary masses and therefore to find the higher-order transformed Hamiltonian. Yet, the use of the generating functions written above for this purpose encounters great difficulties and excessively cumbersome calculations because of the structure of the first two of them. In the paper, we will apply the generators in a simpler form, which, for the first iteration, read

$$S_{oe} = -\frac{i\beta_1}{2m_1c^2}(\mathcal{OE}), \tag{3a}$$

$$S_{eo} = -\frac{i\beta_2}{2m_2c^2}(\mathcal{EO}), \tag{3b}$$

$$S_{oo} = -\frac{i(\beta_1+\beta_2)}{4(m_1+m_2)c^2}(\mathcal{OO}). \tag{3c}$$

Here the former two of them are similar to those from the "radical" transformation, and the latter coincides with the generator (2c). We note that this set of generators is a partial case of the ones proposed in [3] and was also considered by Pursey in [6].

First, we consider the transformation of the Hamiltonian (1) to the order $1/c^2$. After the calculations we get the following expression:

$$\begin{aligned} H_{\rm tr} &= \beta_1 m_1 c^2 + \beta_2 m_2 c^2 + (\mathcal{EE}) + \frac{\beta_1}{2m_1c^2}(\mathcal{OE})^2 + \frac{\beta_2}{2m_2c^2}(\mathcal{EO})^2 + \frac{1}{8m_1^2c^4}[(\mathcal{OE}),[(\mathcal{EE}),(\mathcal{OE})]] \\ &+ \frac{1}{8m_2^2c^4}[(\mathcal{EO}),[(\mathcal{EE}),(\mathcal{EO})]] - \frac{\beta_1}{8m_1^3c^6}(\mathcal{OE})^4 - \frac{\beta_2}{8m_2^3c^6}(\mathcal{EO})^4 \\ &+ \frac{\beta_1\beta_2}{8m_1m_2c^4}\Big\{[(\mathcal{OE}),[(\mathcal{EO}),(\mathcal{OO})]_+]_+ + [(\mathcal{EO}),[(\mathcal{OE}),(\mathcal{OO})]_+]_+\Big\} + \frac{\beta_1+\beta_2}{4(m_1+m_2)c^2}(\mathcal{OO})^2 \\ &+ \frac{(\beta_1-\beta_2)(m_1^2-m_2^2)+2(\beta_1+\beta_2)m_1m_2}{32m_1^2m_2^2(m_1+m_2)c^6}[(\mathcal{OE}),(\mathcal{EO})]^2 - \frac{\beta_1m_1+\beta_2m_2}{16m_1^2m_2^2c^6}[(\mathcal{OE})^2,(\mathcal{EO})^2]_+ \\ &+ \frac{\beta_1}{8m_1m_2^2c^6}(\mathcal{EO})(\mathcal{OE})^2(\mathcal{EO}) + \frac{\beta_2}{8m_1^2m_2c^6}(\mathcal{OE})(\mathcal{EO})^2(\mathcal{OE}) \\ &+ \frac{(1+\beta_1\beta_2)(m_1-m_2)}{16m_1m_2(m_1+m_2)c^4}[[(\mathcal{EO}),(\mathcal{OE})],(\mathcal{OO})] \qquad (4a) \\ &+ \frac{1}{2}(1-\beta_1\beta_2)\left\{(\mathcal{OO})+\sum\mathfrak{N}_{oo}\right\}. \qquad (4b) \end{aligned}$$

It is easy to see that due to the form of S_{oe} and S_{eo} in (3) the obtained transformation is considerably simpler than the one based on the generating functions (2). It consists of the "effective" terms (4a), and because of the form of S_{oo}, of the odd–odd terms (4b), which, however, disappear when $\beta_1 = \beta_2 = \pm 1$ (here it is meant that ± 1 is the unit or the minus unit matrix of the fourth rank), i.e. when both particles are in positive or negative energy states; but the expression (4) can be applicable only for these cases, and it leads to the same reduction of wave equations as the "least change" transformation. We should note that, though we used S_{oo} in the form (3c), many terms in this expression remain unchanged and coincide with the ones from the "radical" transformation (see [2]).

In general, the procedure of continuation of the transformation to higher orders is rather laborious and leads to tedious calculations. But we can simplify considerably the calculations provided that the $(\mathcal{EO})$ and $(\mathcal{OE})$ terms commute with one another. Thus, let us consider the transformation of the Hamiltonian (1) to the order $1/c^4$ subjected to satisfaction of the following commutation relation:

$$[(\mathcal{OE}),(\mathcal{EO})] = 0. \tag{5}$$

Recall that for this case, we calculated the transformed Hamiltonian up to the fourth order for the "radical" transformation in [1].

After relative cumbersome calculations we get

$$\begin{aligned} H_{\rm tr} &= \beta_1 m_1 c^2 + \beta_2 m_2 c^2 + (\mathcal{EE}) + \frac{\beta_1}{2m_1c^2}(\mathcal{OE})^2 + \frac{\beta_2}{2m_2c^2}(\mathcal{EO})^2 + \frac{\beta_1+\beta_2}{4(m_1+m_2)c^2}(\mathcal{OO})^2 \\ &+ \frac{1}{8m_1^2c^4}[(\mathcal{OE}),[(\mathcal{EE}),(\mathcal{OE})]] + \frac{1}{8m_2^2c^4}[(\mathcal{EO}),[(\mathcal{EE}),(\mathcal{EO})]] + \frac{\beta_1\beta_2}{4m_1m_2c^4}[(\mathcal{OE}),[(\mathcal{EO}),(\mathcal{OO})]_+]_+ \\ &- \frac{\beta_1}{8m_1^3c^6}(\mathcal{OE})^4 - \frac{\beta_2}{8m_2^3c^6}(\mathcal{EO})^4 \qquad (6a) \end{aligned}$$

$$- \frac{\beta_1}{8m_1^3c^6}[(\mathcal{OE}),(\mathcal{EE})]^2 - \frac{\beta_2}{8m_2^3c^6}[(\mathcal{EO}),(\mathcal{EE})]^2 \tag{6b}$$

$$+ \frac{\beta_1}{8m_1m_2^2c^6}[(\mathcal{EO}),(\mathcal{OO})]_+^2 + \frac{\beta_2}{8m_1^2m_2c^6}[(\mathcal{OE}),(\mathcal{OO})]_+^2 \tag{6c}$$

$$- \frac{\beta_1+\beta_2}{32m_1^2(m_1+m_2)c^6}[(\mathcal{OO}),[(\mathcal{OE}),[(\mathcal{OE}),(\mathcal{OO})]_+]_+]_+ - \frac{\beta_1+\beta_2}{32m_1(m_1+m_2)^2c^6}[(\mathcal{OO}),[(\mathcal{OO}),(\mathcal{OE})^2]_+]_+$$

$$- \frac{\beta_1+\beta_2}{32m_2^2(m_1+m_2)c^6}[(\mathcal{OO}),[(\mathcal{EO}),[(\mathcal{EO}),(\mathcal{OO})]_+]_+]_+ - \frac{\beta_1+\beta_2}{32m_2(m_1+m_2)^2c^6}[(\mathcal{OO}),[(\mathcal{OO}),(\mathcal{EO})^2]_+]_+ \tag{6d}$$

$$+ \frac{\beta_1+\beta_2}{16m_1m_2(m_1+m_2)c^6}[(\mathcal{OO}),[(\mathcal{OE}),[(\mathcal{EO}),(\mathcal{EE})]]]_+$$

$$+ \frac{\beta_1}{8m_1m_2^2c^6}[[(\mathcal{EO}),(\mathcal{EE})],[(\mathcal{OE}),(\mathcal{OO})]_+] + \frac{\beta_2}{8m_1^2m_2c^6}[[(\mathcal{OE}),(\mathcal{EE})],[(\mathcal{EO}),(\mathcal{OO})]_+] \tag{6e}$$

$$+ \frac{1+\beta_1\beta_2}{16(m_1+m_2)^2c^4}[(\mathcal{OO}),[(\mathcal{EE}),(\mathcal{OO})]] \tag{6f}$$

$$+ \frac{1}{384m_1^4c^8}\Big\{[(\mathcal{OE}),[(\mathcal{OE}),[(\mathcal{OE}),[(\mathcal{OE}),(\mathcal{EE})]]]] + 32[(\mathcal{OE})^3,[(\mathcal{OE}),(\mathcal{EE})]]\Big\}$$

$$+ \frac{1}{384m_2^4c^8}\Big\{[(\mathcal{EO}),[(\mathcal{EO}),[(\mathcal{EO}),[(\mathcal{EO}),(\mathcal{EE})]]]] + 32[(\mathcal{EO})^3,[(\mathcal{EO}),(\mathcal{EE})]]\Big\} \tag{6g}$$

$$+ \frac{1}{64m_1^2m_2^2c^8}[(\mathcal{OE}),[(\mathcal{OE}),[(\mathcal{EO}),[(\mathcal{EO}),(\mathcal{EE})]]]] \tag{6h}$$

$$- \frac{\beta_1\beta_2}{96m_1^3m_2c^8}\Big\{[(\mathcal{OE}),[(\mathcal{OE}),[(\mathcal{OE}),[(\mathcal{EO}),(\mathcal{OO})]_+]_+]_+]_+ + 8[(\mathcal{OE})^3,[(\mathcal{EO}),(\mathcal{OO})]_+]_+\Big\}$$

$$- \frac{\beta_1\beta_2}{96m_1m_2^3c^8}\Big\{[(\mathcal{OE}),[(\mathcal{EO}),[(\mathcal{EO}),[(\mathcal{EO}),(\mathcal{OO})]_+]_+]_+]_+ + 8[(\mathcal{OE}),[(\mathcal{EO})^3,(\mathcal{OO})]_+]_+\Big\} \tag{6i}$$

$$+ \frac{\beta_1}{16m_1^5c^{10}}(\mathcal{OE})^6 + \frac{\beta_2}{16m_2^5c^{10}}(\mathcal{EO})^6 \tag{6j}$$

$$+ \frac{1}{2}(1-\beta_1\beta_2)\Big\{(\mathcal{OO}) + \sum \mathfrak{R}'_{oo}\Big\}. \tag{6k}$$

The group of terms which we put together in (6a) represents the transformation that is correct to the order $1/c^2$. One can easily verify that the expression (4) goes over into this part under the commutation relation (5). The terms (6b…j) are of the order $1/c^4$ and form the part in $H_{\rm tr}$ which we called the higher-order transformed Hamiltonian. Furthermore, all the conclusions that we made for $H_{\rm tr}$ in the case of only different masses of particles (see [1]) are also correct for the case under consideration. As to the terms (6k), where we put together all the odd–odd terms arisen in the procedure, they, obviously, do not play any role due to the factor $\frac{1}{2}(1-\beta_1\beta_2)$.

We remark that the number of the "effective" terms from (6) (and also from (4)) coincides with the number of the ones from the "radical" transformation; moreover, each term has its analog in that transformation, in contrast to the "least change" transformation. All of the terms which are linear in $(\mathcal{EE})$ and $(\mathcal{OO})$ coincide with the analogous ones from the "radical" transformation, though S_{oo} has a different form. The terms (6b, c) and the last two terms from (6e) saved their forms too. In general, a comparison of the expression (6) with that transformation shows that the changes affected only those members which involved the difference of masses in their denominators. Provided that $\beta_1 = \beta_2 = \pm 1$, only these cases are our main point of interest, the obtained transformation leads to the same expressions for the reduced Hamiltonians as the "radical" transformation.

As in the case of the "radical" transformation, the form of the transformed Hamiltonian (6) depends on the order of application of the initial generating functions. Indeed, the expression (6) is obtained provided that the generator S_{oe} (or the same S_{eo}, as we put that the relationship (5) is satisfied) was used first in the sequence of the unitary transformations canceling the undesirable terms from the Hamiltonian (1), but if we destroy the $(\mathcal{OO})$ terms first, in addition to the terms from the higher-order Hamiltonian (6b…j), we will get the following ones:

$$
\begin{aligned}
&-\frac{\beta_1+\beta_2}{16m_1m_2(m_1+m_2)c^6}[[(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})],(\mathcal{E}\mathcal{E})]-\frac{\beta_1-\beta_2}{16m_1m_2(m_1+m_2)c^6}[(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{O})(\mathcal{O}\mathcal{E})-(\mathcal{O}\mathcal{E})(\mathcal{O}\mathcal{O})(\mathcal{E}\mathcal{O}),(\mathcal{E}\mathcal{E})]\\
&-\frac{1+\beta_1\beta_2}{32m_1^2m_2(m_1+m_2)c^8}[[(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})],(\mathcal{O}\mathcal{E})^2]-\frac{1+\beta_1\beta_2}{32m_1m_2^2(m_1+m_2)c^8}[[(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})],(\mathcal{E}\mathcal{O})^2]\\
&-\frac{1-\beta_1\beta_2}{32m_1^2m_2(m_1+m_2)c^8}[(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{O})(\mathcal{O}\mathcal{E})-(\mathcal{O}\mathcal{E})(\mathcal{O}\mathcal{O})(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{E})^2]\\
&+\frac{1-\beta_1\beta_2}{32m_1m_2^2(m_1+m_2)c^8}[(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{O})(\mathcal{O}\mathcal{E})-(\mathcal{O}\mathcal{E})(\mathcal{O}\mathcal{O})(\mathcal{E}\mathcal{O}),(\mathcal{E}\mathcal{O})^2] \qquad \text{(7a)}\\
&+\frac{\beta_1-\beta_2}{16m_1^2(m_1+m_2)c^6}\Big\{(\mathcal{O}\mathcal{E})(\mathcal{O}\mathcal{O})(\mathcal{O}\mathcal{E})(\mathcal{O}\mathcal{O})+(\mathcal{O}\mathcal{O})(\mathcal{O}\mathcal{E})(\mathcal{O}\mathcal{O})(\mathcal{O}\mathcal{E})\Big\}\\
&-\frac{\beta_1-\beta_2}{16m_2^2(m_1+m_2)c^6}\Big\{(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{O})(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{O})+(\mathcal{O}\mathcal{O})(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{O})(\mathcal{E}\mathcal{O})\Big\}. \qquad \text{(7b)}
\end{aligned}
$$

Such terms, but with other numerical factors, also appear in H_{tr}, if one takes the sum $S = S_{oe} + S_{eo} + S_{oo}$ as a generating function.

Similarly to the case of the “radical” transformation, the terms (7a) can be removed by an additional unitary transformation with the generating function in the form of a Hermitian even–even operator:

$$
\begin{aligned}
S_{ee} = &-\frac{i(\beta_1+\beta_2)}{16m_1m_2(m_1+m_2)c^6}[(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})] \qquad \text{(8a)}\\
&-\frac{i(\beta_1-\beta_2)}{16m_1m_2(m_1+m_2)c^6}\Big\{(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{O})(\mathcal{O}\mathcal{E})-(\mathcal{O}\mathcal{E})(\mathcal{O}\mathcal{O})(\mathcal{E}\mathcal{O})\Big\}. \qquad \text{(8b)}
\end{aligned}
$$

The same procedure of destroying this type of terms, which we called the “extra” ones, was previously described in [1] in detail.

As far as the terms (7b) are concerned, due to a difference of the beta-matrices in their numerators, they vanish for the states of the two-body system in which we are interested.

We note that the expression (8) can be represented in the same brief form as the generator S_{ee} in the case of the “radical” transformation:

$$S_{ee} = [S_{oe}, [S_{eo}, S_{oo}]], \qquad (9)$$

which probably is of a general unique nature and can be considered as a prescription for the proper choice of S_{ee} to destroy the extra terms in the Chraplyvy transformation and with other sets of generating functions as well.

It is interesting to point out the following. One can easily verify that, although $(\mathcal{O}\mathcal{E})$ and $(\mathcal{E}\mathcal{O})$ do not commute with $(\mathcal{O}\mathcal{O})$, the commutator (9) and thus S_{ee} is equal to zero for the case of a set of generating functions (2), whereby it can be assumed that there are no extra terms in the “least change” transformation (except the terms that vanish when both particles are in positive or negative energy states, like those ones from (7b)).

In conclusion we should also remark that the problem of generalization of the Foldy–Wouthuysen transformation to the three- and many-particle problems, in general, remains interesting, but, it has not been completely solved yet. Although a problem like this was considered a little bit in [6], where one of the possible ways of generalization of the generating functions to the many-particle case was proposed as well, no transformed Hamiltonians have been obtained for these cases until now even up to the order $1/c^2$, let alone higher orders. Apparently, the calculation of such approximate Hamiltonians can be carried out, with the use of a sequence of canonical transformations, with generators which are generalization of the ones from (3) because an extension of the “least change” transformation is hardly suitable for these purposes.

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ПЕРЕТВОРЕННЯ ХРАПЛИВОГО ТА ДЕЯКІ ОСОБЛИВОСТІ ЙОГО ЗАСТОСУВАННЯ ДЛЯ РОЗКЛАДІВ ДО ВИЩИХ ПОРЯДКІВ. II

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Використовуючи послідовність канонічних перетворень із ґенеруючими функціями для частинок із довільними масами, справедливими також для однакових мас, виконано перетворення релятивістського двочастинкового гамільтоніана, що містить члени різної парності, у парно-парну форму до порядку $1/c^4$ включно. Показано, що подібно до перетворення, яке ґрунтується на ґенераторах, що виключають випадок із частинками, маси яких однакові, остаточний вигляд наближеного гамільтоніана не є однозначно визначеним. А саме, його форма залежить від порядку застосування стартових ґенераторів у процедурі та може включати деякі додаткові члени, які, однак, вдається усунути за допомогою унітарного перетворення.