Stress and elastic displacement in doubly-connected rectangular contact areas of machine pieces

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Abstract
The space problem of contact interaction of doubly-connected rectangular area is considered. Analytical and numerical-analytical methods using decomposition of simple-layer potential are developed. The contact tension and deformations are found. Influence of roughness coefficient on the distribution of normal pressures is investigated.

Key words: SPACE CONTACT PROBLEM, TENSION, DEFORMATION, METHOD OF SMALL PARAMETER, CONTACT AREA

Introduction

During design of cars and mechanisms, it is necessary to provide the contact durability and rigidity of interacting details. Contact interaction of construction units and machine pieces is always present during operation of equipment. As a rule, areas of contact are characterized by the high level of concentration of tension that often leads to partial or final fracture of construction elements. In balance of elastic movements of lathes the contact movements, led to cutting zone, make up to 50% of elastic movements of the system machine-tool-product, in open-side coordinate boring machines and vertical milling machines – up to 70%, in double-frame boring-and-turning mills – up to 40% [1].

Improvement of modern techniques of calculations on durability requires detailed studying of stress-strain state of construction units taking into account features of their form and size, and also conditions of contact [2]. Therefore the solution of contact tasks is one of the priority directions for development of calculation methods in mechanical engineering.

The analysis of publications shows that the mathematical theory of solution of contact tasks is rather well developed for circular, elliptic and ring areas of contact [2–7]. However, machine pieces often have complicated configuration. Tasks taking into account real geometry of the contacting bodies and roughness of surfaces, are insufficiently investigated. This makes the work actual. In many practical cases there is a need of solution of tasks for the contacting details having doubly-connected areas of contact in the form of similar rectangles.

In the conditions of ideal and smooth contact, approximate solutions of tasks for stamps with a form of the basis close to rectangular in the plan in the work [7, 8] and with the basis close to a doubly-connected square [9] were obtained. With account of roughness, friction and initial stresses [5] it is possible to make the model more adequate to real conditions of contact. The numerical and analytical solution of task on a stamp with the doubly-connected basis considering friction forces and roughness of an elastic half-space is found in work [10].

Work objective is consideration of a similar task for doubly-connected rectangular area taking into account roughness, finding of the analytical and numerical-analytical solution of tasks on determination of contact pressure and deformation, using decomposition of potential of a simple layer.

Problem statement

Let us consider space task about pressing into homogeneous elastic half-space of rigid stamp limited in the plan by the lines close to rectangles. The scheme of a task is represented in fig. 1.

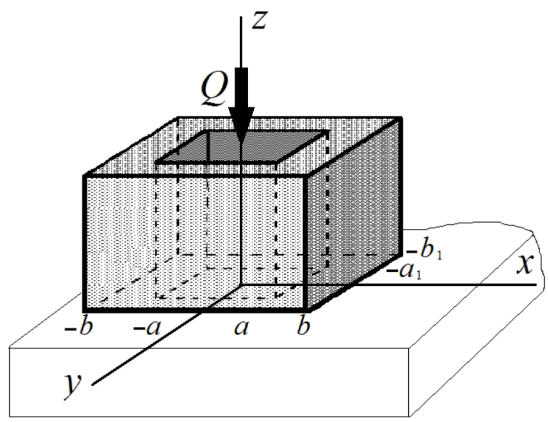

Figure 1. Loading pattern of the stamp

The stamp is affected by the vertical force *Q*, where the line of action passes through the beginning of coordinate system, and the direction matches the direction of *z* axis, perpendicular to the plane of elastic half-space. At the same time *x* axis is parallel to longer party of rectangles, and the beginning of coordinates, matches the center of symmetry of contact area.

Under the action of forces, the stamp will move progressively on δ value.

During mathematical modeling of contact interaction of elastic bodies with rough surface, full vertical motions of stamp are presented as superposition of movements of the points of elastic half-space caused by the application of normal pressure and additional deformations, which are defined by a surface roughness, which is of local character and therefore depends only on pressure applied in this point [2, 11-13]. Such representation leads to two-dimensional integral equation of boundary condition for vertical movements, which contains integrals of potentials of simple layer type [2]

$$\delta = \tilde{\varphi}\left(p\left(\rho_0,\theta_0\right)\right) + \frac{1-\nu^2}{\pi E}\iint\limits_S \frac{p\left(\rho,\theta\right)}{r}\,\mathrm{d}S\,, \qquad (1)$$

where δ – is unknown deepening of a stamp; E – elastic modulus of half-space; v – is the Poisson's ratio; $p(\rho,\theta)$ – unknown function of distribution of normal pressures; $r^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\theta-\theta_0)$, (ρ,θ) – polar coordinates, $x = \rho\cos\theta$, $y = \rho\sin\theta$; $(\rho_0,\theta_0) \in S$; S – the area of contact limited by two concentric lines Γ_1 and Γ_2, close to rectangles with sides that are equal to $2a$, $2a_1$ at internal respectively and to $2b$, $2b_1$ at external, herein $a/a_1 = b/b_1 = 1{,}2$. Function $\tilde{\varphi}(p(\rho,\theta))$ characterizes the law of change of local deformations caused by a surface roughness of an elastic half-space. Unknown value δ is defined from equilibrium equation of the stamp $Q = \iint\limits_S p(\rho,\theta) dS$, $M_1 = 0$, $M_2 = 0$, where Q, M_1, M_2 are the main vector and main moments.

Analytical solution of the problem

Let us consider the case of smooth contact. Boundary equation of\ S area [9, 10] we will represent with the usage of wave-form analysis as follows

$$\rho_{\Gamma_1} = a \cdot f(\theta)\,;\ \rho_{\Gamma_2} = b \cdot f(\theta)\,,$$

where $f(\theta) = [1,02211 + 0,1155 \cos 2\theta - -0,1284 \cos 4\theta - 0,05499 \cos 6\theta + 0,02297 \cos 8\theta + ...]$.

Let us write down continuously differentiable function $f(\theta)$ as the series

$$f(\theta) = 1 + \sum_{i=1}^{\infty} \varepsilon^i f_i(\theta)\,,$$

where $\varepsilon = 0,3583$ and for $i = 1,2,3$:
$f_1(\theta) = 0,5 \cos 2\theta + 0,06170$;

$$f_2(\theta) = -0,4958 \cos 2\theta - \cos 4\theta\,,$$

$$f_3(\theta) = -1,1952 \cos 6\theta + 0,4992 \cos 8\theta\,.$$

The main integral equation of the task (1) will look as follows

$$\delta = \frac{1-\nu}{2\pi G} \iint\limits_S \frac{p(\rho,\theta)}{r} dS\,, \qquad (2)$$

where G is the shear modulus.

It is expected that the distribution function of normal pressure may be represented as the series in terms of powers of ε:

$$p(\rho,\theta) = \sum_{k=0}^{\infty} \varepsilon^k p_k(\rho,\theta)\,.$$

Unknown value of δ may be also presented as the series in terms of powers of ε. Than integrals in equilibrium equations will be presented as the series in terms of powers of ε [14].

Let us use one-one and continuously differentiable transformation of variables,

$$\rho = R\,[1 + f(\varepsilon,\varphi)],\quad \varphi = \theta\,,\ f(\varepsilon,\varphi) = \sum_{i=1}^{\infty} \varepsilon^i f_i(\varphi)\,, \qquad (3)$$

which transfers S area [6,15] into annulus

$$\Omega = \{\ (R,\varphi) : a \le R \le b,\quad 0 \le \varphi \le 2\pi\}\,.$$

Representing the density of simple-layer potential taking into account transformation of variables (3) in the form of

$$p(\rho(R,\varphi,\varepsilon),\varphi) = \sum_{i=0}^{\infty} P_i(R,\varphi)\varepsilon^i\ , \qquad (4)$$

it is possible to decompose the potential into the series in terms of powers of ε found in [6, 15].

Comparing expressions at identical degrees ε in equilibrium equation and (2), we obtain recurrent systems similar to them for definition of $P_k(\rho,\theta)$, δ_k, which include the integrals distributed on a circular ring, and use the solution obtained for circular ring [5]. Similarly, to the methods suggested in works [6, 9, 14, 15] we will write down expression for determination of normal pressure, being limited in decomposition (4) of the first degree ε, i.e. by two approximations. We obtain that pressure under doubly-connected stamp limited by the concentric lines close to a rectangle if $b = 2a$ and $a / a_1 = b / b_1 = 1,2$ in the points (ρ,θ_*) of the beam starting at the origin and crossing the external and internal contours of contact area in points $(\rho_{\Gamma_1},\theta_*)$ and $(\rho_{\Gamma_2},\theta_*)$, respectively, is defined by dependence:

$$p(\rho,\theta_*) = P_0(\rho) + \varepsilon \cdot P_1(\rho,\theta_*)\,,$$

where $P_0(\rho) = \tilde{Q} \cdot p_0$; $2\pi b^2 \tilde{Q} = Q$;
$P_1(\rho,\theta_*) = \tilde{Q}\big[-0,1234\sigma_0 +$
$+(0,009482\sigma_0 - 0,3384\sigma_1)\cos 2\theta_*\big]$;

$$\sigma_i = \frac{\pi\,\gamma}{2} \sum_{k,p=0}^{\infty} \left[\left(\frac{\rho}{\rho_{\Gamma_2}}\right)^{2k} \alpha_{pk}^{(i)} + \left(\frac{\rho_{\Gamma_1}}{\rho}\right)^{2k+3} \beta_{pk}^{(i)} \right] \cdot \left(\frac{a}{b}\right)^p,$$

$\varepsilon = 0,33583$, $i = 0,1,2,3$.

Figure 2 shows the surface of distribution of normal pressure $P^* = 2\pi b^2 p(\rho,\theta)/Q$ along the area of contact. Corresponding pattern of lines of equal pressure is in the plane Oxy. From fig. 2 it is possible to see that the minimum values $P^* = 0,93$ are located in points on beams $\theta = \frac{\pi}{4}; \frac{3\pi}{4}; \frac{5\pi}{4}; \frac{7\pi}{4}$.

Lines of equal pressure become isolated near points with the minimum value of pressure, and when becoming estraged to border they take a form of curves, similar to borders of contact area. Extreme

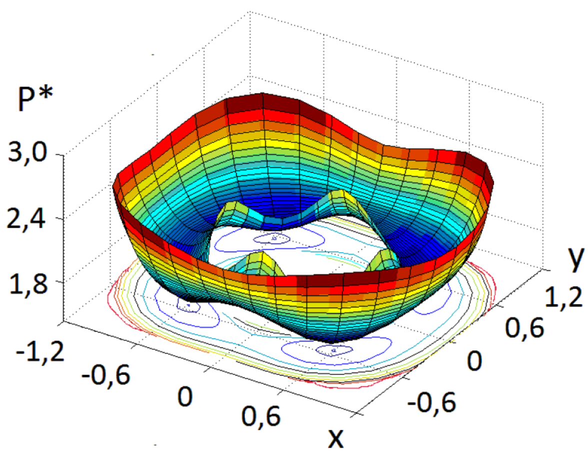

Figure 2. Distribution of normal pressure during smooth contact

lines are removed from internal and external contours on 0.01, average values of normal pressure on them are equal to $P^* = 1,77$ and $P^* = 2,51$ respectively. One may see that the pressure increases faster from outer side.

Numerical analytic solution of the problem

To compare let us introduce the roughness coefficient of an elastic half-space *B*, which may be considered as regulating parameter [2, 10].

The function characterizing law of change of local deformations caused by a surface roughness of elastic half-space [10, 15] will be registered as follows

$$\tilde{\varphi}(p(\rho,\theta)) = B \cdot p(\rho,\theta).$$

Then the equation (1) will look as follows

$$B \cdot p(\rho_0,\theta_0) + \frac{1-\nu^2}{\pi \cdot E} \iint\limits_S \frac{p(\rho,\theta)}{r(\rho_0,\theta_0,\rho,\theta)} dS = \delta, \quad (6)$$

Having applied the numerical and analytical method developed in the work [10] and having used the previous calculations and results of works [11, 15], numerical researches may be conducted.

Figure 3 shows the surface of distribution of normal pressure along the area of contact taking into account surface roughness, when the value of a roughness $B_1 = 1,57$, $B\pi E / \left(b \cdot (1-\nu^2)\right) = B_1$. The corresponding picture of lines of equal pressure is in the plane Oxy.

The greatest values normal pressures accept on external border of contact area, as well as in fig. 1. However the accounting of a surface roughness leads to final values on borders of contact area, whereas in tasks for ideally smooth contact the values of pressure increases unlimited when approaching to area borders.

On external border the value of average normal pressure has decreased up to $P^* = 2,18$, and on internal border – $P^* = 1,53$. With increase in roughness coefficient values of normal pressures on borders of contact area decrease, and in the middle they raise that confirms the conclusions made in works [10-15].

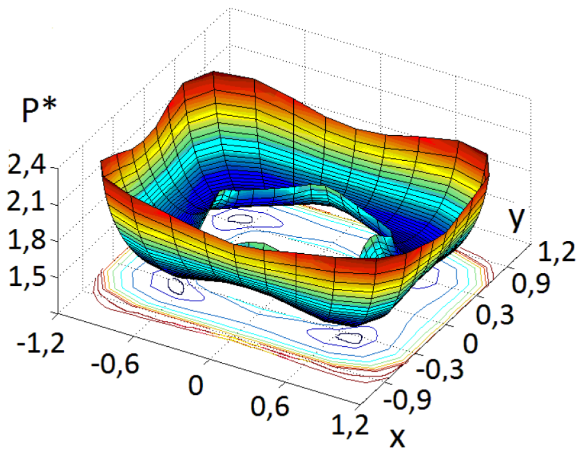

Figure 3. Distribution of normal pressure considering the roughness

Conclusions

Analytical and numerical-analytical solution of a task on pressing of a stamp with the basis in the form close to a doubly-connected rectangle is obtained. With the help of suggested methods there conducted numerical calculations of specific examples. The result of finiteness of normal pressures on borders of contact area of at the accounting of a roughness of a half-space is confirmed. Due to roughness distribution of pressure becomes more uniform. The methods offered by authors for the solution of this task may be used for stamps of other configuration and taking into account friction and adhesion.

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