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Solution of Helmholtz's equation in the plane with an elliptical hole

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General approach to constructing solutions of boundary value problems for Helmholtz's equations is considered. By transforming coordinates applying conforming mappings of corresponding domains onto the circle, a set of solutions of Helmholtz's equation in different coordinate systems is obtained. Solutions of boundary value problems for this equation in the plane with an elliptical hole are constructed.

Keywords: *Helmholtz's equation, conforming mapping, basis of the space of analytical functions*

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1. Introduction

Important physical processes of hydrodynamics, electrostatics, magnetostatics, heat conduction and other applied sciences are described by the harmonic equation and Helmholtz equation. The former describes stationary processes, the latter both stationary and dynamic processes. Effective solutions to boundary value problems for harmonic equation are constructed applying methods of the theory of analytical functions [1–3]. Their solving is equivalent to finding complex potentials of corresponding fields or conforming mappings of domains (in which such solutions are sought) onto canonical domains (circle, plane, etc.). Such methods cannot be directly applied to solving boundary problems for Helmholtz's equation. Solutions to boundary problems for Helmholtz's equation are represented in the form of series by special functions and are known only for separate canonical domains [1,2,4]. General approach to constructing solutions of boundary problems for Helmholtz's equation is formulated in the paper [5]. It is based on the application of solutions of harmonic equation for corresponding domains. Substituting initial coordinates for those with the application of conforming mappings of the domains (in which the solution is sought) onto the circle, the sets of solutions of Helmholtz's equation in different systems are obtained. Solutions to boundary problems for this equation are represented in the form of series by the functions of complex variables.

In this paper applying conforming mappings of the plane with an elliptical hole onto a single circle, the basis in the space of analytical in the domain functions is constructed. Taking into account the series expansion of the functions on this basis, solutions to boundary problems for Helmholtz's equation in the plane with an elliptical hole are constructed.

2. Constructing solutions of Helmholtz's equation

Let $w = \varphi(z)$ be a conforming mapping of the simply connected domain D of the complex plane $(z = x + iy)$ onto the circle $|w| \leqslant 1$ of the complex plane $(w = u + iv)$. Simultaneously the boundary $L = \partial D$ is mapped onto the circumference $K : |w| = 1$. Let us denote inverse mapping by $z = \varphi^{-1}(w)$.

We write Helmholtz's equation using variables w, $\bar{w}$,

$$4\frac{\partial^2 U}{\partial w \partial \bar{w}} + a\left(w\frac{\partial U}{\partial w} + \bar{w}\frac{\partial U}{\partial w}\right) + \kappa U = 0, \tag{1}$$

where $a, \kappa = \text{const}$.

The solution of this equation can be written in the form

$$U = \sum_{m=0}^{\infty} c_m w^m J_m^* \left(w\bar{w}\right), \tag{2}$$

where

$$J_m^* \left(w\bar{w}\right) = J_m^* \left(|w|^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_n^m |w|^{2n}}{2^{2n+m}(n+m)!n!},$$

$$\alpha_n^m = \prod_{k=0}^{n-1} \left[(m+2k)\,\alpha + \kappa\right], \; n = 1,\; 2, ..., \; \alpha_0^m = 1;$$

c_m are arbitrary constants.

Let us pass over to new variables $z = \varphi^{-1}(w)$, $\bar{z} = \bar{\varphi}^{-1}(w)$. Since $\varphi'(z) \neq 0$, $z \in D$, the following equation is obtained

$$4\frac{\partial^2 U}{\partial z \partial \bar{z}} + a\left[\varphi(z)\,\bar{\varphi}'(z)\frac{\partial U}{\partial z} + \bar{\varphi}(z)\,\varphi'(z)\frac{\partial U}{\partial \bar{z}}\right] + \kappa\varphi'(z)\,\bar{\varphi}'(z)\,U = 0. \tag{3}$$

Problem 1. Let us write the solution of equation (1) in a single circle, if the values of the sought function are given on the circle boundary

$$U(w, \bar{w})|_K = f(t), \quad t \in K \tag{4}$$

where $f(t)$ is the value on the circumference K of the analytical function in the circle

$$f(w) = \sum_{m=0}^{\infty} d_m w^m. \tag{5}$$

It is worth noticing that Schwartz integral for a single circle allowing to find an analytical in the circle $|z| < 1$ function, the real part of which on the circumference takes the value $u(\xi)$ in every point of this function continuity, has the following form

$$f(w) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{it}\right)\frac{e^{it} + w}{e^{it} - w}\,dt + iC,$$

where C is a constant variable.

Substituting expression (2) into condition (4) taking into account picture (5) and equality $w\,\bar{w} = 1$, fulfilled for the points of the circumference K, we receive

$$\sum_{m=0}^{\infty} c_m e^{im\psi} J_m^*(1) = \sum_{m=0}^{\infty} d_m e^{im\psi}.$$

Hence, we find $c_m = d_m / J_m^*(1)$ and write the solution of the problem

$$U = \sum_{m=0}^{\infty} \frac{d_m}{J_m^*(1)} w^m J_m^*\left(w\bar{w}\right).$$

Problem 2. Let us construct solution of equation (3) in the simply connected domain D, which satisfies the following condition

$$U(z,\bar{z})\big|_L = f(t), \quad t \in L, \tag{6}$$

where $f(t)$ – is the value on L function $f(z)$, analytical in D, $\lim\limits_{z\to\infty} f(z) = 0$.

Substituting the expression of conforming mapping into formula (2), we write equation (3) in the form

$$U(z,\bar{z}) = \sum_{m=0}^{\infty} c_m \varphi^m(z)\, J_m^*\left[\varphi(z)\,\bar{\varphi}(z)\right]. \tag{7}$$

We find values of the constants in correlation (7) from the equation received by substituting this correlation into condition (6) taking into consideration the equality $\varphi(z)\,\bar{\varphi}(z) = 1$, $z \in L$,

$$\sum_{m=0}^{\infty} c_m \varphi^m(t)\, J_m^*(1) = f(t), \quad t \in L. \tag{8}$$

A) Let the known basis of the space of the functions, analytical in domain D, be a system of functions $\{g_n(z)\}$, constructed by applying mapping $w = \varphi(z)$ and it is such that

$$\frac{1}{2\pi i}\int_{L^*} g_n(z)\,\omega_m(z)\,dt = \delta_{nm},$$

where $L^* \subset D$ is the mapping of the circumference $|w| = \delta$, $0 < \delta \leqslant 1$; $\{\omega_m(z)\}$ is a system of functions related to their basis. By developing the functions $f(z)$ and $\varphi^m(z)$ by this basis, from equation (8) we will obtain a system of linear equations for defining unknown coefficients.

B) Another approach to solving this problem is based on the application of the expression of mapping boundaries L and K, $\varphi(t) = w$. Then, equation (8) for defining unknown coefficients will have the form

$$\sum_{m=0}^{\infty} c_m w^m J_m^*(1) = f\left[\varphi^{-1}(w)\right], \quad w \in L. \tag{9}$$

Problem 3. Let us write the solution of equation (3) for the plane with a circular hole of the single radius with the center at the beginning of coordinates. Thus, we have conforming mapping of the plane with the single hole onto the single circle $w = \dfrac{1}{z}$. Then correlation (7) will be as follows

$$U(z,\,\bar{z}) = \sum_{m=0}^{\infty} \frac{c_m}{z^m} J_m^*\left(\frac{1}{z\bar{z}}\right).$$

Taking into account that $t = e^{-i\psi}$, $-\pi < \psi \leqslant \pi$, is the point of the single circumference, i.e. the boundary of the domain (the plane with a hole), the boundary condition (8) will have the following form

$$\sum_{m=0}^{\infty} c_m e^{-im\psi} J_m^*(1) = \sum_{m=1}^{\infty} a_m e^{-im\psi}.$$

Hence, we will have $c_0 = 0$, $c_m = \dfrac{a_m}{J_m^*(1)}$, $m \geqslant 1$, and will write the solution of the problem

$$U(z,\,\bar{z}) = \sum_{m=1}^{\infty} \frac{a_m}{J_m^*(1)} \frac{1}{z^m} J_m^*\left(\frac{1}{z\bar{z}}\right). \tag{10}$$

3. Solution of Helmholtz's equation for the plane with an elliptical hole

Problem 4. Let us find the solution of equation (3) in domain D – the plane with an elliptical hole with the center at the beginning of coordinates and half-axes a, b fulfilling condition (6) in the form (8). The mapping of domain D into the single circle and the mapping inverse to it are as follows:

$$w = \varphi(z) = \frac{2R}{z + \sqrt{z^2 - c^2}}, \quad z = \varphi^{-1}(w) = \frac{c^2}{4R} w + \frac{R}{w}, \tag{11}$$

where $\lim\limits_{z\to\infty}\left(z + \sqrt{z^2 - c^2}\right) = \infty$; $2R = a + b$; $c^2 = a^2 - b^2$.

3.1. System of functions

$$\left\{ g_n(z) = \varphi^{n+1}(z) = \frac{(2R)^{n+1}}{\left(z + \sqrt{z^2 - c^2}\right)^{n+1}} \right\}_{n=0}^{\infty} \tag{12}$$

is the basis of the space of functions, analytical in the plane with an elliptical hole and such that they take zero values in the infinitely distant point.

We have the development of function $g_n(z)$ by negative powers of the variable

$$g_n(z) = R^{n+1} \sum_{l=0}^{\infty} \frac{(n+1)\, C_{n+2l}^{l}}{2^{2l}(l+n+1)} \frac{c^{2l}}{z^{n+2l+1}}, \quad \text{where} \quad C_n^m = \frac{n!}{(n-m)!\, m!}.$$

System of functions $\{\omega_m(z)\}_{m=0}^{\infty}$, where

$$\omega_m(z) = \frac{1}{R^{m+1}} \sum_{k=0}^{[m/2]} \frac{(-1)^k C_{m-k}^{k} c^{2k}}{2^{2k}} z^{m-2k}, \tag{13}$$

is biorthogonal with system (12),

$$\frac{1}{2\pi i} \int\limits_{\Gamma} g_n(z)\, \omega_m(z)\, dz = \delta_{nm},$$

where Γ is a closed contour comprising ellipsis. Indeed,

$$\frac{1}{2\pi i} \int\limits_{\Gamma} g_n(z)\, \omega_m(z)\, dz = \frac{1}{2\pi i} \int\limits_{\Gamma} \left(\frac{A_2}{2}\right)^{n-m} \sum_{l=0}^{\infty} \sum_{k=0}^{[m/2]} \frac{(-1)^k C_{m-k}^{k} (n+1)\, C_{n+2l}^{l}}{2^{2k+2l}(l+n+1)} \frac{c^{2l+2k}}{z^{n-m+2l+2k+1}} dz$$

or separately for even and odd values of indexes taking into account corresponding combinatory identities,

$$\frac{1}{2\pi i} \int\limits_{\Gamma} g_{2n}(z)\, \omega_{2m}(z)\, dz = \frac{1}{2\pi i} \int\limits_{\Gamma} R^{2n-2m} \sum_{l=0}^{\infty} \sum_{k=0}^{[m/2]} \frac{(-1)^{m-k} (2n+1)\, C_{m+k}^{m-k} C_{2l}^{l-n}}{2^{2l-2n+2m-2k}(l+n+1)} \frac{c^{2l-2n+2m-2k}}{z^{2l-2k+1}} dz =$$
$$= \left(\frac{R}{c}\right)^{2n-2m} \sum_{k=n}^{m} \frac{(-1)^{m-k} (2n+1)\, C_{m+k}^{m-k} C_{2k}^{k-n}}{(k+n+1)} = \delta_{nm},$$

$$\frac{1}{2\pi i}\int\limits_{\Gamma} g_{2n+1}(z)\,\omega_{2m+1}(z)\,dz = \frac{1}{2\pi i}\int\limits_{\Gamma} R^{2n-2m}\sum_{l=n}^{\infty}\sum_{k=0}^{m}\frac{(-1)^{m-k}(2n+2)\,C_{m+k+1}^{m-k}C_{2l+1}^{l-n}}{2^{2l-2n+2m-2k}(l+n+2)}\frac{c^{2l-2n+2m-2k}}{z^{2l-2k+1}}dz =$$
$$= \left(\frac{R}{c}\right)^{2n-2m}\sum_{k=n}^{m}\frac{(-1)^{m-k}(2n+2)\,C_{m+k+1}^{m-k}C_{2k+1}^{k-n}}{(k+n+2)} = \delta_{nm}.$$

Notice that functions $\omega_m(z)$ on condition that $a=b=1$ are Chebishov's polynoms of the second type [8]. If $F(z)$ is an analytical function in domain D and $F(\infty)=0$, then

$$F(z) = \sum_{m=0}^{\infty} F_m g_m(z), \quad \text{where} \quad F_m = \frac{1}{2\pi i}\int\limits_{\Gamma} F(z)\,\omega_m(z)\,dz.$$

For instance, for power functions we have

$$\frac{1}{z^{n+1}} = \frac{1}{R^{n+1}}\sum_{m=0}^{\infty}\frac{(-1)^m C_{m+n}^{m}c^{2m}}{(2R)^{2m}}g_{2m+n}(z), \quad n=0,1,\ldots.$$

Theorem 1. *If function $f(z)$ is analytical in domain D (the plane with an elliptical hole), $f(\infty)=0$ and on the ellipsis Γ: $\left|\frac{2R}{z+\sqrt{z^2-c}}\right| = 1$ is bounded so that $|f(z)| \leqslant M$, then the series of this function by the system of functions (12),*

$$f(z) = \sum_{n=0}^{\infty} b_n g_n(z) \tag{14}$$

is converging uniformly to it on the set D^*: $\left|\frac{2R}{z+\sqrt{z^2-c^2}}\right| < r$, $0<r<1$.

Proof. Coefficients of series (14) are defined by the formula

$$b_n = \frac{1}{2\pi i}\int\limits_{\Gamma^*} f(z)\,\omega_n(z)\,dz, \tag{15}$$

where Γ^*: $\left|\frac{2R}{z+\sqrt{z^2-c^2}}\right| = r_0$, $r<r_0<1$, or changing $z=\frac{c^2r_0}{4R}e^{i\varphi}+Rr_0^{-1}e^{-i\varphi}$, $-\pi<\varphi\leqslant\pi$, we obtain

$$b_n = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f\left(\frac{c^2r_0}{4R}e^{i\varphi}+Rr_0^{-1}e^{-i\varphi}\right)\omega_n\left(\frac{c^2r_0}{4R}e^{i\varphi}+Rr_0^{-1}e^{-i\varphi}\right)\left(\frac{c^2r_0}{4R}e^{i\varphi}-Rr_0^{-1}e^{-i\varphi}\right)d\varphi$$

Here is an estimate for polynoms (13),

$$\omega_m(z)\left|\sqrt{z^2-c^2}\right| \leqslant \left(\frac{c}{2R}\right)^{m+1}\left[\left(\frac{cr_0}{2R}\right)^{m+1}+\left(\frac{2R}{cr_0}\right)^{m+1}\right],$$

where $\left|\sqrt{z^2-c^2}\right| = \frac{c}{2}\left|\frac{cr_0}{2R}e^{i\varphi}-\frac{2R}{cr_0}e^{-i\varphi}\right|$. Taking it into account as well as limited character of function $f(z)$ in expressing coefficient (15), we find

$$|b_m| \leqslant \frac{M}{2\pi}\left(\frac{c}{2R}\right)^{m+1}\left[\left(\frac{cr_0}{2R}\right)^{m+1}+\left(\frac{2R}{cr_0}\right)^{m+1}\right]\int\limits_{-\pi}^{\pi}d\varphi \leqslant M\left(\frac{c}{2R}\right)^{m+1}\left[\left(\frac{cr_0}{2R}\right)^{m+1}+\left(\frac{2R}{cr_0}\right)^{m+1}\right].$$

Since parameter R is constant, and $r_0 < 1$, we have

$$|b_m| \leqslant M \left(\frac{c}{2R}\right)^{m+1} 2 \left(\frac{2R}{cr_0}\right)^{m+1} \frac{r_0}{2R} = \frac{M}{Rr_0^m}.$$

For the function $g_n(z)$ we have such an estimate, taking into consideration formula (11): $|g_n(z)| \leqslant r^{n+1}$. If $r < r_0$, $z \in D^*$, then series (14) converges with the speed of geometrical progression. ∎

3.2. Taking into consideration formulas (11) and (12), we write solution of the problem (3), (8) with (7) in the form

$$U(z, \bar{z}) = \sum_{m=1}^{\infty} c_m g_{m-1}(z) J_m^* [\varphi(z) \bar{\varphi}(z)]$$

We also present condition (8) for defining constants c_m as follows

$$\sum_{m=0}^{\infty} c_{m+1} J_{m+1}^* (1) g_m(t) = f(t),$$

where $f(t) = \sum\limits_{m=0}^{\infty} a_m g_m(t)$.

Hence, we get $c_{m+1} = \dfrac{a_m}{J_{m+1}^*(1)}$, $m \geqslant 0$, and, correspondingly, we find the solution of the problem

$$U(z, \bar{z}) = \sum_{m=1}^{\infty} \frac{a_m}{J_{m+1}^*(1)} g_{m-1}(z) J_m^* [\varphi(z) \bar{\varphi}(z)]. \tag{16}$$

Taking in (16) $c = 0$, we get the solution of Helmholtz's equation for the plane with the central circular hole of radius R

$$U(z, \bar{z}) = \sum_{m=1}^{\infty} \frac{a_m R^m}{J_{m+1}^*(1)} \frac{J_m^*\left(R^2/z\bar{z}\right)}{z^m},$$

which on condition $R = 1$ coincides with solution of problem 3.

Problem 5. We will find the solution of problem 4 using boundary conditions in the form (9). Let analytical function in the plane with an elliptical hole develop by basis (12)

$$f(x) = \sum_{n=0}^{\infty} a_n g_n(z).$$

Then

$$f\left[\varphi^{-1}(w)\right] = \sum_{n=0}^{\infty} a_n g_n \left[\varphi^{-1}(w)\right] = \sum_{n=0}^{\infty} a_n g_n \left(\frac{c^2}{4R} w + \frac{R}{w}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{A_2}{A_1}\right)^{n+1} \frac{1}{w^{n+1}}.$$

Boundary condition (9) takes the form

$$\sum_{m=0}^{\infty} c_m e^{im\psi} J_m^*(1) = \sum_{n=0}^{\infty} a_n g_n \left[\frac{1}{2}\left(A_1 e^{i\psi} + A_2 e^{-i\psi}\right)\right]$$

or

$$\sum_{m=0}^{\infty} c_m e^{im\psi} J_m^*(1) = \sum_{n=0}^{\infty} c_m e^{im\psi} J_m^*(1) = \sum_{n=0}^{\infty} a_n e^{i(n+1)\psi}.$$

Hence, we get $c_0 = 0$, $c_{n+1} = \dfrac{a_n}{J^*_{n+1}(1)}$, $n = 1, 0, \ldots$, and write the solution of the problem in the form (16).

Example 1. Let us write the solution of equation (3) in domain D (the plane with an elliptical hole) in case of condition

$$U(z, \bar{z}) = t^m, \quad m = 1, 2, \ldots, t \in L, \tag{17}$$

where $t = e^{-i\varphi}$, $-\pi < \varphi \leqslant \pi$.

We find the value of the functions of system (12) on the edge of the hole,

$$g_{m-1}(t) = (2R)^m \left(t + \sqrt{t^2 - c^2}\right)^{-m} = (2R)^m \left[a\cos\varphi + ib\sin\varphi + \sqrt{(b\cos\varphi + ia\sin\varphi)^2}\right]^{-m} = e^{-im\varphi}.$$

Thus, functions of system (12), analytical in domain D, take values (17) on the edge of the domain. Taking into consideration such dependencies in (16), we get the solution of the problem

$$U(z, \bar{z}) = \frac{1}{J^*_m(1)} \varphi^m(z)\, J^*_m[\varphi(z)\,\bar{\varphi}(z)].$$

It is worth noticing that the real and imaginary parts of this solution also satisfy Helmholtz's equation.

4. Conclusions

Using conforming mappings of other simply connected domains onto the single circle (or half-plane) it is possible to construct solutions of equation (3) in these domains. An important stage in solving the problem is seeking an analytical function in the given domain, values of which coincide with the boundary values of solving equation (3). But if, applying corresponding conforming mapping, the system of functions is constructed being the basis of the space of analytical in this domain functions (as it is treated in the paper), then finding of the corresponding analytical function is considerably simplified.

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Розв'язи рівняння Гельмгольца у площині з еліптичним отвором

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Розглянуто загальний підхід до побудови розв'язків крайових задач для рівняння Гельмгольца. Перетворюючи координати з використанням конформних відображень відповідних областей на круг, одержано множини розв'язків рівняння Гельмгольца у різних системах координат. Побудовано розв'язки крайових задач для цього рівняння у площині з еліптичним отвором.

Ключові слова: *рівняння Гельмгольца, конформне відображення, базис в просторі аналітичних функцій*

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