MATHEMATICAL MODELING AND COMPUTING, Vol. 1, No. 2, pp. 264–283 (2014)

Interparticle interactions, general relativity effects, and critical parameters of white dwarfs

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(Received 1 December 2014)

Two methods of mass calculation of degenerate dwarfs were considered: based on (1) hydrostatic equilibrium equation as well as (2) variational principle. In this work we used model with ideal electron subsystem and one with Coulomb interaction. An instability region of massive white dwarfs was explored. For the first time, taking into account the Coulomb interaction, there were obtained critical values of mass and relativistic parameter at wich instability due to general relativity effects occured.

Keywords: *degenerate dwarf, white dwarf, Coulomb interaction, general relativity effects, instability, critical value of relativistic parameter*

2000 MSC: 85A15, 83C25, 82D35

UDC: 524.31.084, 524.384, 524.352.3

1. Introduction

This year we have 100^{th} anniversary of discovery of the first white dwarf. In 1914 american astronomer W. Adams has found the spectrum of Sirius A to be similar to that of its faint companion, Sirius B [1]. Hence he concluded that the size of the last one is tiny. At that moment there was known an estimation of mass of Sirius B [2] from orbital motion. Thus, it was discovered a new type of stars with masses $M_* \sim M_\odot$ and radii $R_* \sim 10^{-2} R_\odot$, where $M_\odot, R_\odot$ are solar ones. Average interparticle distances in such objects are of the order of 0,1Å, it means the matter is under extreme conditions (metallic state), electrons have no localized states. Such objects can be considered as metal with the simpliest structure.

On the base of quantum theory R. Fowler [3] for the first time suggested degeneration of an extreme dense nonrelativistic electron gas as crucial factor in the existence of white dwarfs. At such high densities electron momentum on the Fermi surface is of the order $m_0 c$ (m_0 is the electron mass, c is the speed of the light), therefore S. Chandrasekhar [4,5] developed two component model of a cold dwarf consisting of an ideal degenerate relativistic electron gas at $T = 0K$ (in paramagnetic state) and nuclear subsystem as unstructured classical environment. The stability of white dwarf is provided by equilibrium between electron pressure and gravity of nuclear subsystem. Energetic and structural characteristics of a star can be obtained from the equilibrium equation. Chandrasekhar's model is chemical homogeneous and doesn't take into account the Coulomb interaction, finite temperature effects (partial degeneration of electron gas), stellar rotation, magnetic field, effects of general relativity – which can affect internal stellar structure.

Chandrasekhar's model has two parameters – central density ρ_c and average chemical composition parameter $\mu_e = \left\langle \frac{A}{z} \right\rangle$, where A is the nucleon number, z is the nuclear charge. It is convenient to overwrite ρ_c in terms of dimensionless relativistic parameter

$$x_0 = p_F(0)(m_0 c)^{-1} = \hbar (3\pi^2 n_0)^{1/3} (m_0 c)^{-1}, \tag{1}$$

here $p_F(0)$ is Fermi momentum, n_0 is the electron number density in the centre of a star. In the frame of this model well-known result was yielded – the existence of maximum mass of white dwarf (Chandrasekhar limit)

$$M \leqslant M_{ch} = 2.01824\ldots M_0\mu_e^{-2}, \qquad (2)$$
$$M_0 = \left(\frac{3}{2}\right)^{1/2}(4\pi)^{-1}\left(\frac{hc}{G}\right)^{3/2} m_u^{-2} \approx 2.89\ldots M_\odot,$$

here m_u is the atomic mass unit. Another famous result obtained by Chandrasekhar is the "mass-radius" relation; according to this all the white dwarfs have to form a line on the plane $(\mu_e^2 M, \mu_e R)$. Dwarfs with masses $M < M_{ch}$ are stable in this model. The value M_{ch} corresponds to the limit $x_0 \to \infty$ and have a sense of upper bound. But in the frame of this model it is impossible to obtain a reasonable mass limit as well as to explain observed narrow distribution of white darfs by masses (radii). Only cold dwarfs with effective temperatures $T_{eff} \leqslant 10^4 K$ are in good agreement with the theoretical line on the "mass–radius" plane $M - R$. Recent observational data reveal dozens of very hot dwarfs with small masses and surface temperatures reaching $10^5 K$. They form rather a sequence of lines on the $M - R$ plane, where every line corresponds to the fixed value of T_{eff} (see Fig. 1). Generalization of the Chandrasekhar's theory is needed to describe the observed diversity of white dwarfs. There was shown in [6–8] that for the case of hot degenerate dwarfs with small and moderate masses (extended radii) taking into account the partial degeneracy of electron subsystem is necessary, as well as consideration of non-uniform chemical composition due to gravitational settling. For the first time we have solved the inverse problem of the theory of degenerate dwarfs – radial distributions of thermodynamic parameters (temperature and density) were computed for given values of mass, radius and luminosity of observed white dwarf.

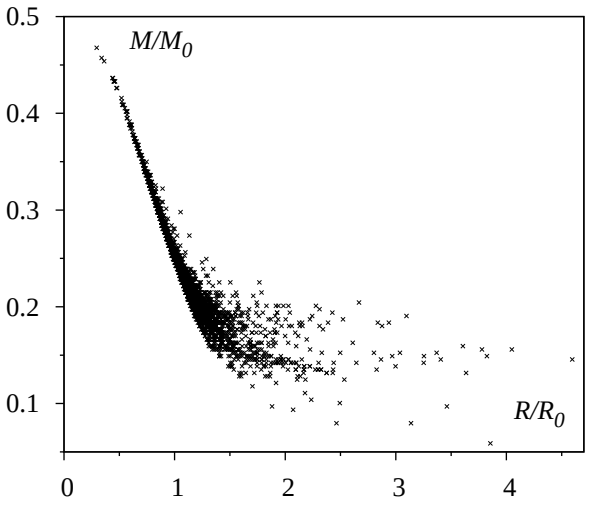

Fig. 1. Degenerate dwarfs on the mass–radius plane.

White dwarfs with masses approaching the Chandrasekhar limit are believed to formed in binary systems where the mass transfer is plausible. In such stars finite temperature effects are negligible because Fermi energy of the electron gas is of the order $m_0c^2x_0$ and significantly exceeds k_BT ($k_BT/m_0c^2 \sim 0.1$, while $x_0 > 10$). They consist of fully degenerate matter, but deviation from the ideal approximation and general relativity effects have to be considered here. Both these factors reduce the white dwarf mass limit. In the paper [9] was established that interactions and neutrinization process decrease mass and can cause an instability of massive white dwarfs. In another work [10], authors in the frame of Chandrasekhar's model pointed out on general relativity effects, which probably can lead to instability of degenerate dwarfs. However, the pioneer work with the approximate estimation of general relativity effects on the maximum mass of a degenerate dwarf belongs to S. Kaplan [11], where he used the Tollman-Oppenheimer-Volkoff equation as well as equation of state of the ideal degenerate electron gas at the absolute zero of temperature.

As can be seen, determination of the critical parameters (maximum mass and critical density at wich the instability occured) by simultaneous consideration of interactions and general relativity effects is a problem to be solved. It is believed that high mass white dwarfs are connected with such energetic

processes as recurrent novae and Ia type supernovae, therefore more precise estimation of critical values can help better to understand physics of phenomena mentioned above.

In this work we have considered model of chemical homogeneous massive white dwarf with fully degenerate interacting electron system and crystal nuclear system. Model has three parameters – x_0, μ_e, z. Equation of state was obtained using reference system approach developed in the papers [12–14] for the case of nonrelativistic metallic systems. For the determination of critical parameters we have proposed self-consistent variational principle, which generalizes method of the paper [10].

2. The equation of state of degenerated relativistic electron gas

The role of interactions in modern theory of degenerated dwarfs is poorly investigated on the contrary to metallic systems theory in terrestial conditions. The main reason is that system of electrons is significantly relativistic. This fact complicates an analytic calculation. In the same time this system is weak nonideal – the coupling parameter of electron–electron interactions (Brückner's parameter)

$$r_s \;=\; \alpha_0 \,\frac{\eta}{x} \tag{3}$$

is of the same order that fine structure constant $\alpha_0 = e^2/\hbar c$ ($\eta = (9\pi/4)^{1/3}$) and $z\alpha_0$ for electron–nuclear interactions. This fact simplifies the construction of the equation of state. In the paper [15], for the first time in the frame of many-electrons (nonrelativistic) theory it was estimated the Coulomb interaction contribution to the ground state energy and the pressure in the structureless electron-nuclear model. That work is a compilation of Wigner–Seitz model, the Thomas-Fermi approximation, the correlation energy in the random phase approximation from non-relativistic theory. It gives a possibility for authors to obtain a simple equation of state. This equation describes the reduction of pressure due to Coulomb interaction in the limit of large values of relativistic parameter. Also, in order to investigate the cooling of white dwarfs, in work [16] it was considered the interactions based on the Wigner-Seitz model.

The modern microscopic theory of non-relativistic many-electron systems allows a simple generalization to the case of relativistic model. We have adapted mathematical methods of reference system approach developed for description of the strongly non-ideal nonrelativistic electron liquid model [12–14].

The reference system approach is a case of perturbation theory formulated on the base of many particles dynamic correlation functions of zero order approximation model (reference system). For the homogeneous electron liquid the reference system is the model of ideal electrons, in the case of the electron-nuclear model this is the model of a homogeneous electron liquid.

We consider electrically neutral macroscopically homogeneous electron-nuclear model. It consists of N_e electrons and $N_n = z^{-1}N_e$ nuclei per volume V in the thermodynamic limit $N_e, V \to \infty$, $N_e/V = \text{const}$ at low temperatures (much lower than temperature of degeneration) in the adiabatic approximation assuming that the nuclear subsystem forms a lattice. In the second quantization representation this model has the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} + V_{nn}, \tag{4}$$

where

$$\hat{H}_0 = \sum_{\mathbf{k},s} E_k \, a^+_{\mathbf{k},s} \, a_{\mathbf{k},s} \tag{5}$$

corresponds to the Hamiltonian of free relativistic electrons,

$$\hat{H}_{int} = \hat{V}_{ee} + \hat{V}_{en} \tag{6}$$

is the sum of interelectrons and electron–nuclear interaction operators,

$$\hat{V}_{ee} = (2V)^{-1} \sum_{\mathbf{q}\neq 0} V_q \sum_{\mathbf{k}_1,\mathbf{k}_2} \sum_{s_1,s_2} a^+_{\mathbf{k}_1+\mathbf{q},s_1}\, a^+_{\mathbf{k}_2-\mathbf{q},s_2}\, a_{\mathbf{k}_2,s_2}\, a_{\mathbf{k}_1,s_1}, \tag{7}$$

$$\hat{V}_{en} = -zV^{-1} \sum_{\mathbf{q}\neq 0} V_q\, S_{\mathbf{q}} \sum_{\mathbf{k},s} a^+_{\mathbf{k}+\mathbf{q},s}\, a_{\mathbf{k},s}.$$

The component

$$V_{nn} = z^2 (2V)^{-1} \sum_{\mathbf{q}\neq 0} V_q \{S_{\mathbf{q}} S_{-\mathbf{q}} - N_n\} \tag{8}$$

describes the Coulomb nuclear interaction. Here $S_{\mathbf{q}} = \sum\limits_{j=1}^{N_n} \exp[i(\mathbf{q}, \mathbf{R}_j)]$ is the structural factor of nuclear subsystem; $V_q = 4\pi e^2/q^2$ is frequency domain representation of the Coulomb potential; $a^+_{\mathbf{k},s}$, $a_{\mathbf{k},s}$ are operators of creation and annihilation of electron in the quantum states of a given wave vector $\mathbf{k}$ and spin projection s, they are described by Fermi statistics.

The calculations for the partition function for the electron variables in the grand canonical ensemble, the thermodynamic potential and energy of the ground state of the model are performed according to the general procedure described in the works [12,13], where the non-relativistic model of electron liquid was considered.

Deviations of the results are caused only by the calculation of n–particles dynamic correlation functions of type "density–density" for the model of non-interacting electrons with spectrum $E_k = \{(m_0c^2)^2 + \hbar^2k^2c^2\}^{1/2} - m_0c^2$ instead of $\hbar^2k^2/2m_0$.

Traditional representation of the ground state energy of electron–nuclear model was obtained by using the electron liquid model as reference system.

$$E = E_e + V_{nn} - \sum_{n\geqslant 2} \frac{z^n}{n!} V^{-n} \sum_{\mathbf{q}_1,\ldots,\mathbf{q}_n\neq 0} V_{\mathbf{q}_1} \cdots V_{\mathbf{q}_n} S_{-\mathbf{q}_1} \cdots S_{-\mathbf{q}_n} \tilde{\mu}_n(\mathbf{q}_1,\ldots,\mathbf{q}_n|0). \tag{9}$$

Here E_e is the energy of reference system,

$$E_e = E_0 + (2\beta V)^{-1} \sum_{\mathbf{q}\neq 0} V_q \sum_{\nu} \int_0^1 d\lambda\, \tilde{\mu}_2^{\lambda}(y,-y), \tag{10}$$

where E_0 is the energy of ideal electrons system and $\tilde{\mu}_2^{\lambda}(y,-y)$ is two-particle dynamic correlation function of the auxiliary model, which is the electron liquid model, where the frequency domain representation of interactions potential is λV_q. Here was used momentum–frequency representation, where $y \equiv (\mathbf{q},\nu)$; $\nu = 2\pi n\beta^{-1}$ is the Bose-Matsubara frequency. The function $\tilde{\mu}_n(\mathbf{q}_1,\ldots,\mathbf{q}_n|0)$ represents statical limit of n-particles dynamic correlation functions of reference system.

2.1. The reference systems correlation functions

The system of ideal electrons (without interaction) can be used as statistical basis in the calculation of the correlation functions of the electron liquid model. The correlation functions of the ideal electron gas can be rewritten in the form of convolution of Green's functions of Matsubara,

$$\tilde{\mu}_2^0(y,-y) = -\beta^{-1} \sum_{\mathbf{k};s;\nu_*} G_{\mathbf{k},s}(\nu_*) G_{\mathbf{k}+\mathbf{q},s}(\nu_*+\nu);$$

$$\tilde{\mu}_3^0(y_1,y_2,y_3) = 2\beta^{-1} \delta_{\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3,0}\, \delta_{\nu_1+\nu_2+\nu_3,0} \sum_{\mathbf{k},s;\nu_*} G_{\mathbf{k},s}(\nu_*) G_{\mathbf{k}+\mathbf{q}_1,s}(\nu_*+\nu_1)\, G_{\mathbf{k}-\mathbf{q}_2,s}(\nu_*-\nu_2); \tag{11}$$

$$\tilde{\mu}_4^0(y_1,-y_1,y_2,-y_2)=\beta^{-1}\sum_{\mathbf{k},s;\nu_*}G_{\mathbf{k},s}(\nu_*)\,G_{\mathbf{k}-\mathbf{q}_1,s}(\nu_*-\nu_1)\times$$
$$\times\sum_{\sigma=\pm1}G_{\mathbf{k}-\sigma\mathbf{q}_2,s}(\nu_*-\sigma\nu_2)\{2G_{\mathbf{k},s}(\nu_*)+G_{\mathbf{k}+\mathbf{q}_1+\sigma\mathbf{q}_2,s}(\nu_*+\nu_1+\sigma\nu_2)\},$$

where

$$G_{\mathbf{k}s}(\nu_*)=\{i\nu_*-E_k+\mu\},\tag{12}$$

μ is the chemical potential of the model; $\nu_*=(2n+1)\pi\beta^{-1}$ is the Bose-Matsubara frequency. The static limit of functions (11) for non-relativistic model are well known for $n=2,3,4$ [17,18]. The dynamic function $\tilde{\mu}_2^0(y,-y)$ at $T=0K$ was calculated in [19]. The functions $\tilde{\mu}_3^0(y_1,y_2,y_3)$ and $\tilde{\mu}_4^0(y_1,-y_1,y_2,-y_2)$ are represented via elementary functions in the paper [12]. For the model with one-particle spectrum $E_k=[(m_0c^2)^2+\hbar^2k^2c^2]^{1/2}-m_0c^2$ it is unable to get exact analytical expressions for $\tilde{\mu}_n^0(y_1,\ldots,y_n)$ in terms of elementary functions. The approximate analytical representation is possible [20], another way is numerical calculations.

Expanding products of Green's functions into prime factors and using relation

$$\beta^{-1}\sum_{\nu_*}G_{\mathbf{k},s}=n_{\mathbf{k},s}=\{1+\exp[\beta(E_k-\mu)]\}^{-1},\tag{13}$$

we obtain following representations:

$$\tilde{\mu}_2(y,-y)=\gamma_2(y)=-2\mathrm{Re}\sum_{\mathbf{k},s}n_{\mathbf{k},s}\{i\nu+\tilde{E}_{\mathbf{k}}-\tilde{E}_{\mathbf{k}+\mathbf{q}}\}^{-1},\tag{14}$$
$$\tilde{\mu}_3^0(y_1,y_2,y_3)=\delta_{y_1+y_2+y_3,0}\{\gamma_3(y_1,-y_2)+\gamma_3(y_2,-y_3)+\gamma_3(y_3,-y_1)\};$$
$$\gamma_3(y_1,y_2)=2\mathrm{Re}\sum_{\mathbf{k},s}n_{\mathbf{k},s}\{i\nu_1+\tilde{E}_{\mathbf{k}}-\tilde{E}_{\mathbf{k}+\mathbf{q}_1}\}^{-1}\{i\nu_2+\tilde{E}_{\mathbf{k}}-\tilde{E}_{\mathbf{k}+\mathbf{q}_2}\}^{-1},\ldots;$$
$$\tilde{E}_k=[(m_0c^2)^2+\hbar^2k^2c^2]^{1/2}.$$

As an example we consider the function $\tilde{\mu}_2^0(y,-y)$, which is important for weakly nonideal systems. The representation in the one-dimensional integral form of $\tilde{\mu}_2^0(y,-y)$ can be obtained via replacing the vector's $\mathbf{k}$ sum by the integral and afterward integrating in the spherical coordinate system by the angular variables of the vector $\mathbf{k}$:

$$\tilde{\mu}_2^0(y,-y)=\frac{3N_e}{m_0c^2x^2}J_2(q_*,\nu_*|x),$$
$$J_2(q_*,\nu_*|x)=(xq_*)^{-1}\int_0^\infty dk_*\;k_*\;n(k_*)A(k_*|q_*,\nu_*),\tag{15}$$
$$A(k_*|q_*,\nu_*)=\sum_{\sigma=\pm1}\sigma\left\{[1+(k_*+\sigma q_*)^2]^{1/2}-\nu_*\,\mathrm{arctg}[\nu_*^{-1}\eta_\sigma(k_*,q_*)]\right.$$
$$\left.+\frac{1}{2}(1+k_*^2)^{1/2}\ln[\nu_*^2+\eta_\sigma^2(k_*,q_*)]\right\},$$
$$\eta_\sigma(k_*,q_*)=[1+(k_*+\sigma q_*)^2]^{1/2}-[1+k_*^2]^{1/2},$$
$$n(k_*)=\left\{1+\exp\left[\beta_*\left(\left[1+k_*^2\right]^{1/2}-1-\mu_*\right)\right]\right\}^{-1}.$$

Here were used dimensionless variables

$$k_* = \frac{|\mathbf{k}|\hbar}{m_0 c} = \frac{|\mathbf{k}|}{k_F} x, \quad q_* = \frac{|\mathbf{q}|\hbar}{m_0 c} = \frac{|\mathbf{q}|}{k_F} x, \quad \nu_* = \frac{\nu}{m_0 c^2} = \frac{\nu}{2\varepsilon_F} x^2, \tag{16}$$
$$\beta_* = \beta m_0 c^2; \quad \mu_* = \frac{\mu}{m_0 c^2} = \frac{\mu}{\varepsilon_F} \frac{x^2}{2}; \quad \varepsilon_F \equiv \frac{\hbar^2 k_F^2}{2m_0}.$$

At absolute zero of temperature $n(k_*) = 1$ in the range $0 \leqslant k_* \leqslant x$ and $n(k_*) = 0$ at $k_* > x$. The integration by variable k_* is fulfilled from zero to x.

The dependence of the static function $J_2(q_*, 0|x)$ on wave vector and relativistic parameter is shown in the Figure 2. The Figure 3 illustrates the function $J_2(q_*, \nu_*|x)$ for the case of small frequency $(\nu/2\varepsilon_F = 0{,}15)$.

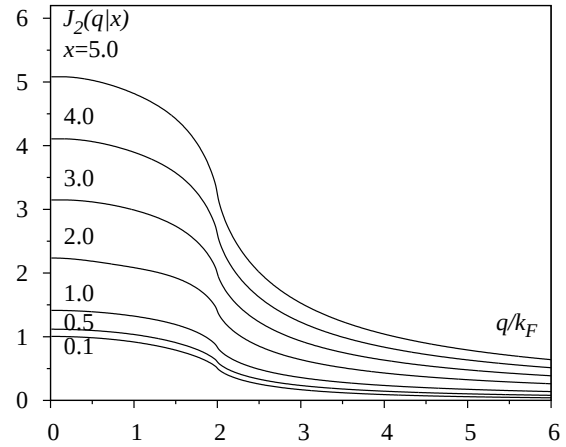

Fig. 2. The static function $J_2(q_*, 0|x)$ as function of wave vector and relativistic parameter.

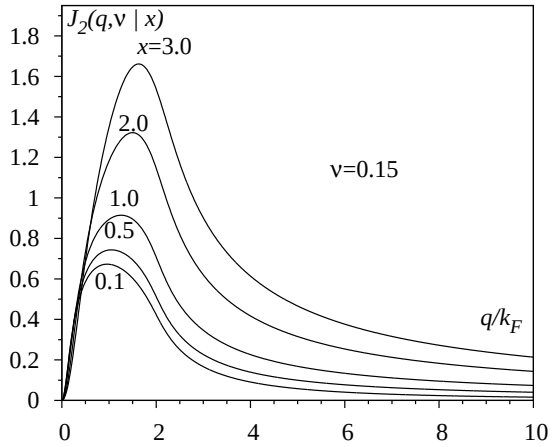

Fig. 3. The dynamic function $J_2(q_*, \nu_*|x)$ as function of wave vector and relativistic parameter.

The static correlation function $\mu_2^0(\mathbf{q}, -\mathbf{q}|0)$ has the next asymptotic behavior:

$$\tilde{\mu}_2^0(\mathbf{q}, -\mathbf{q}|0) \Rightarrow \begin{cases} 3N_e(m_0 c^2 x^2)^{-1}(1+x^2)^{1/2} & \text{for} \quad q \to 0; \\ 2N_e(m_0 c^2 q)^{-1} = 2N_e(c\hbar q) & \text{for} \quad q \to \infty. \end{cases} \tag{17}$$

The well known correlation function of non-relativistic theory [19] can be obtained from $\tilde{\mu}_2^0(y, -y)$ by transformation to non-relativistic variables $\tilde{q} = |\mathbf{q}|k_F^{-1}$, $\tilde{k} = |\mathbf{k}|k_F^{-1}$, $\tilde{\nu} = \nu/2\varepsilon_F$ using (16) and formally passing to limit $x \to 0$.

In the static case it can be obtained an approximate analytical representation via elementary functions by following identity transformation in (14) and representing the function $\tilde{\mu}_2^0(\mathbf{q}, -\mathbf{q}|0)$ in the form

$$\tilde{\mu}_2^0(\mathbf{q}, -\mathbf{q}|0) = 2 \sum_{\mathbf{k},s} n_{\mathbf{k},s} \{\tilde{E}_{\mathbf{k}+\mathbf{q}} + \tilde{E}_{\mathbf{k}}\} \{\tilde{E}_{\mathbf{k}+\mathbf{q}}^2 - \tilde{E}_{\mathbf{k}}^2\}^{-1}. \tag{18}$$

In the case of variables (16) and at $T = 0K$

$$J_2(q_*, 0|x) = (2xq_*)^{-1} \int_0^x dk_* \, k_* \int_{-1}^1 dt (t + q_*/2k_*)^{-1} \, \{(1 + k_*^2 + q_*^2 + 2k_* q_* t)^{1/2} + (1 + k_*^2)^{1/2}\}. \tag{19}$$

The convergent sum

$$J_2(q_*,0|x)=\sum_{n\geqslant 0}J_{2,n}(q_*|x), \tag{20}$$

was obtained by expanding into series $[1+k_*^2+q_*^2+2q_*k_*t]^{1/2}$ via t. The terms are expressed in elementary functions. The convergence of this series is illustrated in Figure 4.

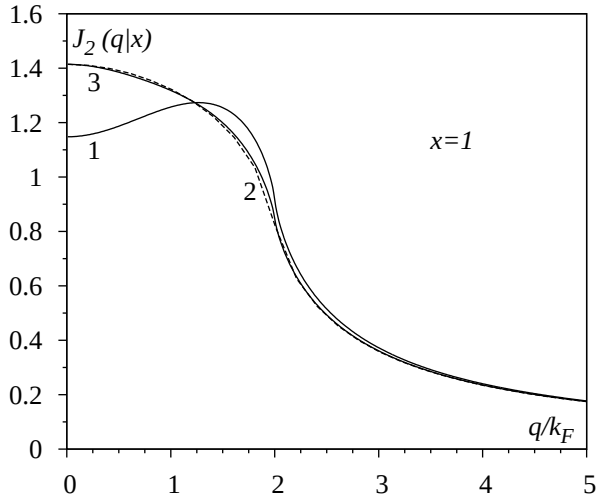

Fig. 4. The sequence of functions $\sum_{n\geqslant 0}J_{2,n}(q_*|x)$ (see eq. (20)). Curve 1 corresponds to $n_0=0$; 2 – $n_0=1$; 3 – $n_0=2$.

The same series can be obtained for static limit of high-order correlation functions $\mu_n^0(\mathbf{q}_1,\ldots,\mathbf{q}_n|0)$ at $(n=3;4)$.

For the correlation functions of the electron liquid model we will use the local field approximation [14]:

$$\mu_2(y,-y)=\mu_2^0(y,-y)\left\{1+(1-G(y))\frac{V_q}{V}\mu_2^0(y,-y)\right\}^{-1}, \tag{21}$$

$$\mu_3(y_1,y_2,y_3)=\mu_3^0(y_1,y_2,y_3)\prod_{i=1}^{3}\left\{1+(1-G(y_i))\frac{V_{q_i}}{V}\mu_2^0(y_i,-y_i)\right\}^{-1},$$

etc. As we consider the electron–nuclear model for weak coupling it is sufficient to use the local field correction in the lowest approximation (Heldart-Taylor approximation) [14]:

$$G_{id}(y)\equiv-(2\beta V_q)^{-1}\{\tilde{\mu}_2^0(y,-y)\}^{-2}\sum_{\mathbf{q}_1,\nu_1}V_{q_1}\tilde{\mu}_4^0(y,-y,y_1,-y_1). \tag{22}$$

According to expressions (12) and (13)

$$G_{id}(y)=-(2V_q)^{-1}\{\tilde{\mu}_2^0(y,-y)\}^{-2}\mathrm{Re}\sum_s\sum_{\mathbf{k}_1,\mathbf{k}_2}V(\mathbf{k}_1-\mathbf{k}_2)\times \tag{23}$$

$$\times\{n_{\mathbf{k}_1,s}-n_{\mathbf{k}_1+\mathbf{q},s}\}\{n_{\mathbf{k}_2,s}-n_{\mathbf{k}_2+\mathbf{q},s}\}\{[i\nu+\tilde{E}_{\mathbf{k}_1}-\tilde{E}_{\mathbf{k}_1+\mathbf{q}}]^{-1}-[i\nu+\tilde{E}_{\mathbf{k}_2}-\tilde{E}_{\mathbf{k}_2+\mathbf{q}}]^{-1}\}^2.$$

The expression (23) can be reduced to four-dimensional integrals with respect to the variables z_1, z_2 and ρ_1, ρ_2. Then integrating by angles φ_1 and φ_2 in the cylindrical coordinate system with $\mathbf{k}=(\rho,z,\varphi)$. The final expression can by calculated numerically. The results of calculations are

presented in Figures 5. The local field correction function of the model of relativistic electron liquid is similar to the case of nonrelativistic theory but is dependent on the relativistic parameter. The local field correction function is independent of the coupling constant in the considered approach.

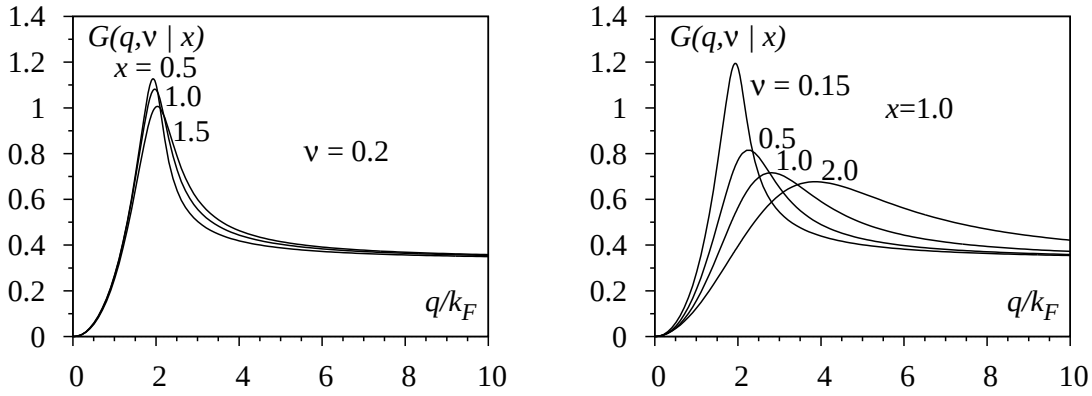

Fig. 5. The dependence of the dynamical local field function on relativistic parameter and frequence.

2.2. The ground state energy of the model

Energy of the model can be overwritten with help of separating of Hartree-Fock component in the formula (10) and from products of $S_{\mathbf{q}_1} \dots S_{\mathbf{q}_n}$ (see (9)) – unary, binary, ternary, ... by the coordinate of nuclei terms

$$E = E_0 + E_{HF} + E_c + N_n\omega + \frac{1}{2!}\sum_{j_1\neq j_2} V_2(\mathbf{R}_{j_1} - \mathbf{R}_{j_2}) + \frac{1}{3!}\sum_{j_1\neq j_2\neq j_3} V_3(\mathbf{R}_{j_1}, \mathbf{R}_{j_2}, \mathbf{R}_{j_3}) + \ldots + E_{\text{comp}}. \tag{24}$$

Here

$$E_0 = 3N_e m_0 c^2 x^{-3} \int_0^x dt\, t^2\{(1+t^2)^{1/2} - 1\} = \\ = N_e m_0 c^2\{(1+x^2)^{1/2} - 1 - (8x^3)^{-1}\mathcal{F}(x)\} \tag{25}$$

is the energy of non-interacting electrons, where

$$F(x) = x(2x^2-3)(1+x^2)^{1/2} + 3\ln[x + (1+x^2)^{1/2}]. \tag{26}$$

The term E_{HF} determines the contribution of Hartree–Fock correlations (contribution of the first order perturbation theory)

$$E_{HF} = (2\beta V)^{-1}\sum_{\mathbf{q},\nu} V_q \tilde{\mu}_2^0(y,-y) = \\ = -(2V)^{-1}\sum_{\mathbf{q};\mathbf{k},s} V_q n_{\mathbf{k}+\mathbf{q}/2,s} n_{\mathbf{k}-\mathbf{q}/2,s} = \\ = -\frac{3}{4\pi} N_e \alpha_0 m_0 c^2 x. \tag{27}$$

Correlation energy is by definition

$$E_c = (2\beta V)^{-1} \sum_{\mathbf{q}\neq 0;\nu} \int_0^1 d\lambda\, V_q \{\mu_2^\lambda(y,-y) - \mu_2^0(y,-y)\} = \tag{28}$$

$$= -(2\beta)^{-1} \sum_{\mathbf{q},\nu} (1-G(y))^{-1} \left\{ \frac{\mu_2^0(y,-y)}{V} V_q (1-G(y)) - \ln\left[1 + (1-G(y)) \frac{\mu_2^0(y,-y) V_q}{V}\right]\right\}.$$

In nonrelativistic limit expression (28) defines the correlation energy of non-relativistic model. It was approximated in the [22] from results calculated by Monte–Carlo method [21]. We perform the numerical calculations by representing the correlation energy (24) of relativistic models in the form

$$E_c = N_e\, m_0\, c^2\, \alpha_0^2 \varepsilon_c(x). \tag{29}$$

The result of the calculations was approximated via next expressions:

$$\varepsilon_c(x) = -b_0 \int_0^x dt\, \frac{(b_1 a_0 + t^{1/2})}{t^{3/2} + b_1 a_0 t + b_2 a_0^2 t^{1/2} + b_3 a_0^3} \frac{1 + a_1 t + a_2 t^2}{1 + d_1 t},$$

$$a_0 = (\alpha_0 \eta)^{1/2}; \quad a_1 = 1.21954; \quad a_2 = 1.33205; d_1 = 1.18934; \tag{30}$$

$$b_0 = 0.0621814; \quad b_1 = 9.81379; b_2 = 2.82214; \quad b_3 = 0.69699;$$

The dependence of function (30) on relativistic parameter is shown in Fig. 6 (curve 2). The curve 1 represent non-relativistic limit of expression (28). The crosses correspond to approximation of Monte–Carlo results [22].

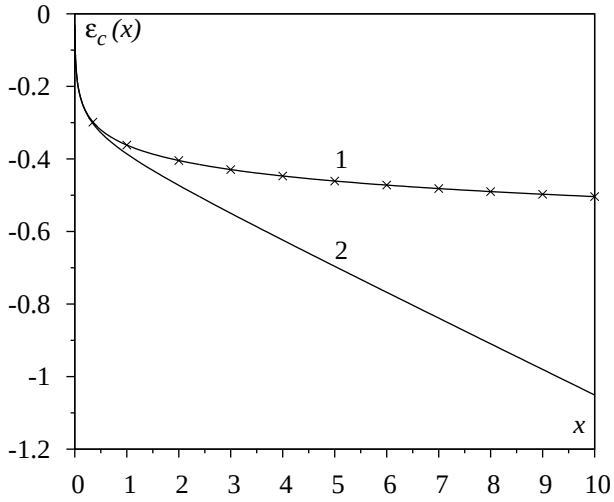

Fig. 6. The correlation energy as a function of relativistic parameter (for details see the text).

The component $N_e\omega$ represents the intercalation nuclear energy in homogenic electron liquid (the energy of polarization)

$$\omega = -\frac{z^2}{2!V^2} \sum_{\mathbf{q}} V_q^2 \mu_2(\mathbf{q}, -\mathbf{q}|0) - \frac{z^3}{3!V^3} \sum_{\mathbf{q}_1,\mathbf{q}_2} V_{\mathbf{q}_1} V_{\mathbf{q}_2} V_{\mathbf{q}_1+\mathbf{q}_2} \mu_3(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2|0) + \ldots. \tag{31}$$

It was calculated in the approach of pair correlations and local field correction function and then approximated in the following form

$$\varepsilon_\omega(x) = -\int_0^x dt\, \frac{c_0 + c_1 t + c_2 t^2 + c_3 t^3}{1 + d_1 t + d_2 t^2 + d_3 t^3}, \tag{32}$$

$$c_0 = 4.06151; \quad c_1 = 32.6118; c_2 = -43.6587; \quad c_3 = 104.13;$$

$$d_1 = 73.8252; \quad d_2 = -67.1028; \quad d_3 = 189.781;$$

The effective potential of nuclear interactions is determined by expressions:

$$V_2(\mathbf{R}_1 - \mathbf{R}_2) = \frac{1}{V}\sum_{\mathbf{q}} \Phi^{(2)}_{eff}(\mathbf{q}, -\mathbf{q}) \exp[i(\mathbf{q}, \mathbf{R}_1 - \mathbf{R}_2)], \tag{33}$$

$$V_3(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) = \frac{1}{V^3}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \delta_{\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3,0} \Phi^{(3)}_{eff}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \exp\{i[(\mathbf{q}_1, \mathbf{R}_1) + (\mathbf{q}_2, \mathbf{R}_2) + (\mathbf{q}_3, \mathbf{R}_3)]\},$$

where

$$\Phi^{(2)}_{eff}(\mathbf{q}, -\mathbf{q}) = z^2 V_q \left\{1 - \frac{V_q}{V}\mu_2(\mathbf{q}, -\mathbf{q}|0) - \frac{1}{V^2}\sum_{\mathbf{q}_1} V_{\mathbf{q}_1} V_{\mathbf{q}+\mathbf{q}_1} \mu_3(\mathbf{q}, \mathbf{q}_1, -\mathbf{q} - \mathbf{q}_1|0) + ...\right\}; \tag{34}$$

$$\Phi^{(3)}_{eff}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = -z^3 \mu_3(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3|0) V_{\mathbf{q}_1} V_{\mathbf{q}_2} V_{\mathbf{q}_3} + \dots .$$

The component E_{comp} can be considered as energy of compensated field, because the components $\mathbf{q}_i = 0$ are missing in formula (9) when summarising by $\mathbf{q}_i$ in contrast to expression (32). Therefore

$$E_{\text{comp}} = -\frac{z^2}{2!} N_n(N_n - 1) \lim_{\mathbf{q}\to 0} \Phi^{(2)}_{eff}(\mathbf{q}, -\mathbf{q}) + \\ + \frac{1}{3!} N_n(N_n - 1)(N_n - 2) \lim_{\mathbf{q}_1,\mathbf{q}_2\to 0} \Phi^{(3)}_{eff}(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1, -\mathbf{q}_2) + \dots . \tag{35}$$

The sum of effective nuclear interactions energy and E_{comp} is the energy of the lattice of nuclei. In order to calculate that energy it was used the summing by the coordinational spheres of a simple cubic lattice. The temrs of effective interactions potential and compensating field are on the same order but they have opposite signs. In the approximation of two-particle correlations the lattice energy is

$$E_L = N_e\, m_0\, c^2 z^{0.589417}\, \alpha_0 \varepsilon_L(x|z), \tag{36}$$

$$\varepsilon_L(x|z) = -\int_0^x dt\, t \frac{a_0(z) + t a_1(z) + t^2 a_2(z)}{1 + t a_3(z) + t^2 a_4(z) + t^3 a_5(z)};$$

$$a_i(z) = \frac{a_{i0} + z a_{i1} + z^2 a_{i2}}{a_{i3} + z a_{i4} + z^2 a_{i5}}. \tag{37}$$

The coefficients a_{ij} are listed in the Table 1.

The dependence of components $\varepsilon_\omega(x)$ and $\varepsilon_L(x)$ are illustrated in the Figures 7, 8.

Table 1. Coefficients of formula (37).

a_{ij}						
i \ j	0	1	2	3	4	5
0	−128.112	−138.098	−3.30915	0	3.74936	0.882489
1	−633.899	297.304	−19.5138	1	−0.707632	1.01638
2	−1691	216.967	−8.71667	0	1.48694	0.12998
3	2.37539	1.74513	0.0417739	0	0.0212583	0.0056168
4	913.016	−452.217	30.3618	1	−0.750844	0.702212
5	5.68901	−0.704184	0.0277872	0	0.00203554	0.000234399

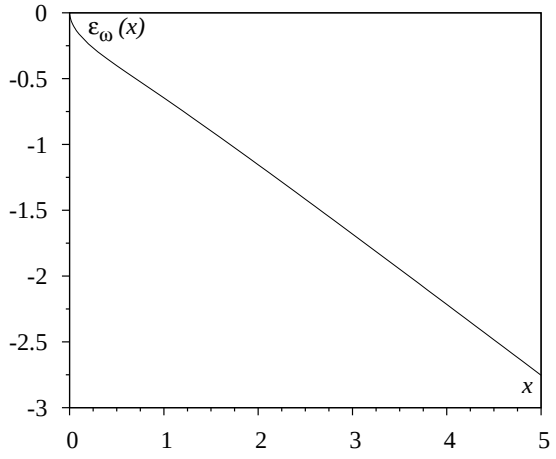

Fig. 7. The intercalation energy of nuclei as function of relativistic parameter.

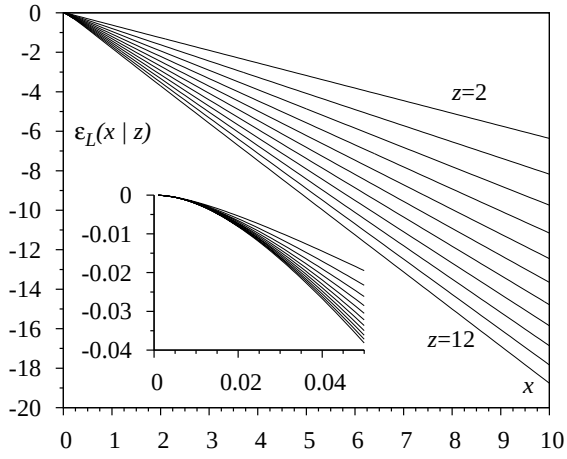

Fig. 8. The dependence of the lattice energy on relativistic parameter.

2.3. Equation of state of the model

The energy dependence of the ground state on the relativistic parameter obtained previous, allows us to write down the equation of state

$$P(x) = -\frac{dE(x)}{dV} = \frac{x^4}{N_e}\left(\frac{m_0 c}{\hbar}\right)^3 \frac{1}{9\pi^2}\frac{dE(x)}{dx}. \tag{38}$$

According to the expressions (25)–(35)

$$P(x) = \frac{\pi m_0^4 c^5}{3h^3}\left\{\mathcal{F}(x) - \frac{2\alpha_0 x^4}{\pi}\left[1 - \frac{4\pi}{3}\left(z^{0.589417}\,\frac{d\varepsilon_L}{dx} + z\,\alpha_0^{1/2}\frac{d\varepsilon_\omega}{dx} + \alpha_0\frac{d\varepsilon_c}{dx}\right)\right]\right\}, \tag{39}$$

where $\mathcal{F}(x)$ is the contribution of noninteracting electron gas. As can be seen from the Figures 6–8, all derivatives in the formula (39) are negative, i.e. they all cause the decrease of pressure of the ideal electron system. Linear dependence on parameter x of each term, $\varepsilon_c(x)$, $\varepsilon_\omega(x)$, $\varepsilon_L(x)$ at $x \gg 1$ provides the difference between asymptotes of $P(x)$ and ideal system by a constant value, therefore

$$P(x) \xrightarrow[x\gg 1]{} 2\pi\frac{m_0^4 c^5}{3h^3}x^4\left\{1 - \varphi(z)\right\}, \tag{40}$$
$$\varphi(z) = \frac{\alpha_0}{\pi}\left\{1 + \frac{4\pi}{3}\left[z^{0.589417}|\varepsilon_L'| + \alpha_0^{1/2} z|\varepsilon_\omega'| + \alpha_0|\varepsilon_c'|\right]\right\}; \quad \varepsilon_L' \equiv \frac{d}{dx}\varepsilon_L \quad \text{etc.}$$

The relative decrease of the pressure due to Coulomb interaction

$$\{P_0(x) - P(x)\}\, P_0^{-1}(x) \tag{41}$$

is shown in Figure 9. When relativistic parameter is small, $x \ll 1$, it is a large quantity, but with increasing x ($x \geqslant 2$), the relation (41) is almost constant, which depends only on the charge of nuclei and equals 1% at $z = 2$, 1.5% at $z = 6$, 2.5% at $z = 12$, 4% at $z = 26$.

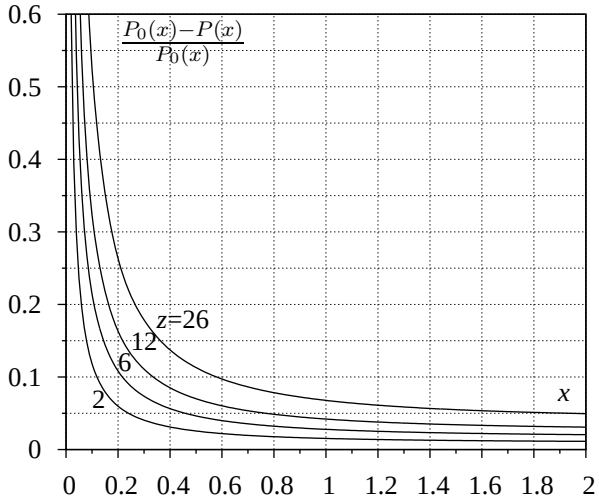

Fig. 9. The relative decrease of the pressure (see (41)).

The approximation of two-electron correlations used here, yielded equation of state similar to results of work [15]. Consideration of the contributions of three-electron correlations is an important issue to be considered later.

3. Critical mass and stability of degenerate dwarfs

3.1. Generalization of Chandrasekhar's model

The simplest improvement to the Chandrasekhar's theory is taking into account the Coulomb interaction in the equation of state, which defines inner stellar structure

$$\frac{dP(r)}{dr} = -G\frac{m(r)}{r^2}\rho(r), \quad \frac{dm}{dr} = 4\pi \int_0^r dr'(r')^2\rho(r'). \tag{42}$$

Here $P(r)$, $\rho(r)$ are pressure and mass density, respectively, on the surface with radius r, $m(r)$ is the mass inside this sphere. We can use a homogeneous model to obtain equation of state $P(r)$ substituting x with its local value $x(r) = \{3\pi^2 n(r)\}^{1/3}\hbar/m_0 c$, where $n(r)$ is the electron number density on the sphere with radius r. According to this description

$$\rho(r) = m_u \mu_e \left(\frac{m_0 c}{\hbar}\right)^3 (3\pi^2)^{-1} x^3(r). \tag{43}$$

Given the equation of state as

$$P(r) = \frac{\pi m_0^4 c^5}{3h^3}\, \mathcal{F}(x)\{1 - \varphi(x(r), z)\} \tag{44}$$

and overwriting system (42) as second order differential equation we will obtain in dimensionless form

$$\xi \;=\; r/\lambda, \quad y(\xi) = \varepsilon_0^{-1}\{[1 + x^2(r)]^{1/2} - 1\} \tag{45}$$

the next equilibrium equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left\{[1-\varphi(\xi,z)]\xi^2\frac{dy}{d\xi} - \xi^2\mathcal{F}(\xi)(8\varepsilon_0 x^3(\xi))^{-1}\frac{d\varphi(\xi,z)}{d\xi}\right\} = -\left(y^2(\xi) + \frac{2}{\varepsilon_0}y(\xi)\right)^{3/2}. \tag{46}$$

Here λ can be found from expression

$$\frac{32\pi^2 G}{3(hc)^3}\,(m_0 m_H \mu_e c^2 \lambda\,\varepsilon_0)^2 = 1, \tag{47}$$

where $\varepsilon_0 = (1 + x_0^2)^{1/2} - 1$ and $x_0 \equiv x(r = 0)$ is the relativistic parameter in the stellar centre. Equation (46) satisfies boundary conditions: $y(0) = 1$, $y'(0) = 0$ and contains two dimensionless parameters (x_0, z). In the equation were used the next denotations:

$$\begin{aligned} &x(\xi) = \varepsilon_0(y^2(\xi) + \frac{2}{\varepsilon_0}y(\xi))^{1/2}; \mathcal{F}(\xi) \equiv \mathcal{F}(x(\xi)); \\ &\varphi(\xi,z) \equiv \varphi(x(\xi),z). \end{aligned} \tag{48}$$

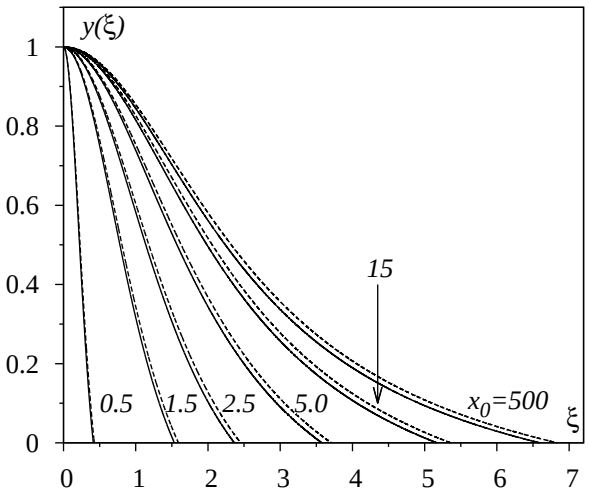

Fig. 10. The dependence of the solutions of equation (46) on parameter x_0 at $z = 6$ (solid curve). Dashed curve corresponds to Chandrasekhar's model.

In the case $\varphi(\xi, z) = 0$ equation (46) corresponds to the Chandrasekhar's theory. The solutions of the equation (46) and their dependence on the parameter x_0 are illustrated in the Figure 10. The case $\varphi(\xi, z) = 0$ is depicted with dashed lines. At given x_0, z the solution of equation will turn to zero in point $\xi = \xi_1(x_0, z)$, which is the dimensionless radius of the star. Then stellar mass can be defined as

$$M(x_0, \mu_e, z) = 4\pi \int_0^R dr\, r^2 \rho(\xi) = \frac{M_0}{\mu_e^2}\tilde{\mathcal{M}}(x_0, z), \tag{49}$$

$$\tilde{\mathcal{M}}(x_0, z) = \int_0^{\xi_1(x_0,z)} d\xi\, \xi^2 \left(y^2(\xi) + \frac{2}{\varepsilon_0}y(\xi)\right)^{3/2},$$

here M_0 is determined by (2). In the Chandrasekhar's model

$$\tilde{\mathcal{M}}(x_0,\mu_e) = \int\limits_0^{\xi_1(x_0)} d\xi\,\xi^2\left(y_0^2(\xi)+\frac{2}{\varepsilon_0}y_0(\xi)\right)^{3/2}, \tag{50}$$

where $y_0(\xi)$ is the solution of equation (46) at $\varphi(\xi,z)=0$ and $\xi_1(x_0)$ is corresponding dimensionless radius of the star.

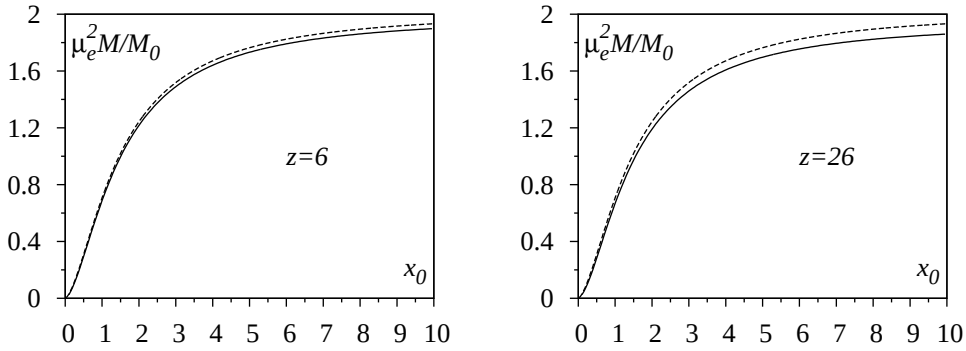

Fig. 11. Mass dependence of a dwarf on relativistic parameter: solid curve corresponds to formula (49), dashed curve – (50).

As can be seen from Figure 11, in the limit $x_0\to\infty$ functions $\tilde{\mathcal{M}}(x_0,\mu_e)$ tend the constant values constraining maximum mass of a dwarf. Coulomb interaction causes the decrease of mass of a dwarf for whole range of x_0, especially maximum mass corresponding to the dwarf with zero radius, since

$$R(x_0,\mu_e) = R_0\frac{\xi_1(x_0)}{\mu_e\varepsilon_0(x_0)}, \tag{51}$$

$$R(x_0,\mu_e,z) = R_0\frac{\xi_1(x_0,z)}{\mu_e\varepsilon_0(x_0)},\quad R_0=\left(\frac{3}{2}\right)^{1/2}\frac{1}{4\pi}\left(\frac{h^3}{cG}\right)^{1/2}\frac{1}{m_0m_H}.$$

Here $R_0\approx 10^{-2}R_\odot$ gives an order of the typical white dwarf size.

It should be mentioned, that taking into account of Coulomb interaction decreases mass of a dwarf, but does not change significantly the slope of the curves $\tilde{\mathcal{M}}(x_0,z)$ and $\tilde{\mathcal{M}}(x_0)$ at $x\gg 1$, which are almost parallel in this region.

3.2. Simultaneous consideration of general relativity effects and Coulomb interaction

Massive white dwarfs, for which $x_0\gg 1$, must be considered in the frames of general relativity theory. We have used the approach developed in [10] modificated by simultaneous consideration of the Coulomb interaction. In general case full stellar energy is defined by relation

$$E \;=\; (M-m_uN)\,c^2, \tag{52}$$

where M and N are full mass and number of nucleons, respectively. Mass of a spherical object

$$M \;=\; 4\pi\int\limits_0^R \rho(r)\,r^2dr \tag{53}$$

defines by its density distribution

$$\rho(r) \;=\; \rho_0(r) + \frac{\varepsilon_e(r)}{c^2}, \tag{54}$$

here $\rho_0(r) = n(r)m_u$ is the mass density of nuclear matter and $\varepsilon_e(r)$ is the nongravitational energy density, which includes kinetic energy of electrons and Coulomb interaction Full number of nucleons is

$$N = \int\limits_0^R n(r)\, dv, \tag{55}$$

where

$$dv = \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1/2} 4\pi r^2 dr \tag{56}$$

is the elementary volume in the Schwarzschild metric, $m(r)$ defines the mass of a sphere with radius r. According to (52)–(56)

$$E = c^2 \int\limits_0^R dv \left\{ \left(1 - \frac{2Gm(r)}{rc^2}\right)^{1/2} \left[\rho_0(r) + \frac{\varepsilon_e(r)}{c^2}\right] - \rho_0(r) \right\}. \tag{57}$$

Assuming $2Gm(r)/rc^2 \ll 1$, within terms of second order of smallness energy can be overwritten in the form

$$E = \int\limits_0^R dv \left\{ \varepsilon_e(r) - \rho_0(r) \left[\frac{m(r)G}{r} + \frac{1}{2}\frac{m^2(r)G^2}{r^2c^2} + \frac{\varepsilon_e(r)m(r)G}{\rho_0 rc^2} + \ldots \right] \right\}. \tag{58}$$

Zeldovich and Novikov [10] in their approach did not considered the Coulomb interaction, therefore $\varepsilon_e(r)$ is density only of electron kinetic energy. We will follow the prescription of the work [16] and introduce a full energy of the model in the Newtonian metric

$$\begin{aligned} E &= E_{Newt} + E_{GTR}, \\ E_{Newt} &= 4\pi \int\limits_0^R dr\, r^2\, \varepsilon_e(r) - G \int\limits_0^M \frac{dm\, m(r)}{r}; \\ E_{GTR} &= -\frac{G}{c^2} \int\limits_0^M \frac{dm}{r} \left\{ \frac{\varepsilon_e(r)m(r)}{\rho_0(r)} + \int\limits_0^{m(r)} \frac{dm}{\rho_0(r)}\, \varepsilon_e(r) \right\} - \\ &\quad - \frac{G^2}{c^2} \int\limits_0^M \frac{dm}{r} \left\{ \frac{1}{2}\frac{m^2(r)}{r} - \int\limits_0^m \frac{dm\, m(r)}{r} + \frac{m(r)}{r^3} \int\limits_0^r dr\, r\, m(r) \right\} + \ldots . \end{aligned} \tag{59}$$

Here components E_{GTR} can be calculated in the Newtonian approximation, when $dm = 4\pi r^2 \rho_0(r)\, dr$.

To represent full energy (59) in terms of "stellar mass", "central relativistic parameter" we will introduce dimensionless variables, let

$$r = \lambda' \eta, \tag{60}$$

here λ' is determined from

$$M = 4\pi(\lambda')^3\, m_u \mu_e \left(\frac{m_0 c}{\hbar}\right)^3 (3\pi^2)^{-1} \int\limits_0^{\eta_1(x_0)} d\eta\, \eta^2 x^3(\eta). \tag{61}$$

By defining the dimensionless mass $\tilde{\mathcal{M}} = M\, M_0^{-1}\mu_e^2$ and using the equation (60) we have

$$\frac{\lambda'}{\lambda} = \tilde{M}^{1/3}\left\{\varepsilon_0^{-3}\int\limits_0^{\eta_1(x_0)} d\eta\,\eta^2 x^3(\eta)\right\}^{-1/3}, \tag{62}$$

where $\eta_1(x_0)$ corresponds to the radius of a star in terms η and $x^3(\eta)$ is dimensionless mass density, which can be considered as trial function. Therefore $\lambda^{'}$ depends on dimensionless mass and parameter x_0.

In the new variables, the components of E_{Newt} can be overwritten: kinetic energy of electron subsystem

$$E_{kin} = \frac{E_0}{\mu_e^3}\left(\frac{\lambda'}{\lambda}\right)^3\int\limits_0^{\eta_1(x_0)} d\eta\,\eta^2\left\{x^3(r)[(1+x^2(\eta))^{1/2}-1] - \frac{1}{8}\,\mathcal{F}(x(\eta))\right\}; \tag{63}$$

energy of Coulomb interaction

$$E_{coul} = -\frac{E_0}{\mu_e^3}\left(\frac{\lambda'}{\lambda}\right)^3\alpha_0\int\limits_0^{\eta_1(x_0)} d\eta\,\eta^2\,x^3(\eta)\left\{\frac{3}{4\pi}\,x(\eta) - z^{0.6}\varepsilon_L(x(\eta)) - z\alpha_0^{1/2}\varepsilon_\omega(x(\eta)) - \alpha_0\varepsilon_c(x(\eta))\right\}; \tag{64}$$

gravitational energy

$$E_{grav} = -\frac{E_0}{\mu_e^3}\left(\frac{\lambda'}{\lambda}\right)^5\int\limits_0^{\eta_1(x_0)} d\eta\,\eta\,x^3(\eta)\int\limits_0^{\eta} d\eta_2\,\eta_2^2\,x^3(\eta_2), \tag{65}$$

where $E_0 = GM_0^2/R_0$. In the same way can be represented components of E_{GTR}. From the equation (62) functional of energy is

$$\begin{aligned} E(\tilde{M},x_0) &= \frac{E_0}{\mu_e^3}\,\varepsilon(\tilde{M},x_0), \\ \varepsilon(\tilde{M},x_0) &= \tilde{M}\{S_1(\tilde{M},x_0) - 1 - \alpha_0 S_2(\tilde{M},x_0)\} - \tilde{M}^{5/3}S_3(\tilde{M},x_0) - \\ &\quad - \frac{m_0}{\mu_e m_u}\{\tilde{M}^{5/3}P(\tilde{M},x_0) + \tilde{M}^{7/3}D(\tilde{M},x_0)\} \;+\; \ldots\,. \end{aligned} \tag{66}$$

Here were used the following denotations:

$$\begin{aligned} S_1(\tilde{M},x_0) &= \Big[C(\tilde{M},x_0)\Big]^{-1}\int\limits_0^{\eta_1(x_0)} d\eta\,\eta^2\,a(\eta); \\ S_2(\tilde{M},x_0) &= \Big[C(\tilde{M},x_0)\Big]^{-1}\int\limits_0^{\eta_1(x_0)} d\eta\,\eta^2 x^3(\eta)\,\times \\ &\quad \times\left\{\frac{3}{4\pi}\,x(\eta) - z^{0.6}\varepsilon_L(x(\eta)) - z\alpha_0^{1/2}\varepsilon_\omega(x(\eta)) - \alpha_0\varepsilon_c(x(\eta))\right\}; \end{aligned} \tag{67}$$

$$S_3(\tilde{M},x_0)=\Big[C(\tilde{M},x_0)\Big]^{-5/3}\int\limits_0^{\eta_1(x_0)}d\eta\,\eta\,x^3(\eta)\int\limits_0^{\eta}d\eta_2\,\eta_2^2x^3(\eta_2);$$

$$a(\eta)=x^3(\eta)(1+x^2(\eta))^{1/2}-\frac{1}{8}\,\mathcal{F}(x(\eta));$$

$$C(\tilde{M},x_0)=\int\limits_0^{\eta_1(x_0)}d\eta\,\eta^2x^3(\eta);$$

$$P(\tilde{M},x_0)=\Big[C(\tilde{M},x_0)\Big]^{-5/3}\Bigg\{\int\limits_0^{\eta_1(x_0)}d\eta\,\eta[a(\eta)-x^3(\eta)]\int\limits_0^{\eta}d\eta_2\eta_2^2x^3(\eta)+$$
$$+\int\limits_0^{\eta_1(x_0)}d\eta\,\eta\,x^3(\eta)\int\limits_0^{\eta}d\eta_2\,\eta_2^2\,[a(\eta_2)-x^3(\eta_2)]\Bigg\};$$

$$D(\tilde{M},x_0)=\Big[C(\tilde{M},x_0)\Big]^{-7/3}\Bigg\{\frac{1}{2}\int\limits_0^{\eta_1(x_0)}d\eta\,x^3(\eta)\Bigg[\int\limits_0^{\eta}d\eta_2\,\eta_2^2\,x^3(\eta_2)\Bigg]^2-$$
$$-\int\limits_0^{\eta_1(x_0)}d\eta\,\eta\,x^3(\eta)\int\limits_0^{\eta}d\eta_2\,\eta_2\,x^3(\eta_2)\int\limits_0^{\eta_2}d\eta_3\,\eta_3^2\,x^3(\eta_3)+ \tag{68}$$
$$+\int\limits_0^{\eta_1(x_0)}\frac{d\eta}{\eta^2}\,x^3(\eta)\int\limits_0^{\eta}d\eta_2\,\eta_2^2\,x^3(\eta_2)\Bigg[\int\limits_0^{\eta}d\eta_3\,\eta_3\int\limits_0^{\eta_3}d\eta_4\,\eta_4^2\,x^3(\eta_4)\Bigg]\Bigg\}.$$

Energy of a dwarf is the negative monotonically decreasing function of x_0. If the condition

$$\frac{d}{dx_0}\varepsilon(\tilde{M},x_0)=0 \tag{69}$$

is satisfied, then equilibrium determines such mass dependence on x_0, which corresponds to minimal energy. Let x_0^* is the value of relativistic parameter at which the curve $\mathcal{M}(x_0)$ has a bend, i.e.

$$\frac{d^2}{dx_0^2}\,\varepsilon(\tilde{M},x_0)=0, \tag{70}$$

then at this point occures an instability and for all $x_0>x_0^*$ star will turn to a neutron star. From the conditions (69), (70) we obtain system of two equations defining critical (maximum) stellar mass and value x_0^*, at which instability due to general relativity effects occured:

$$\begin{aligned}&\frac{d}{dx_0}S_{10}-\alpha_0\frac{dS_{20}}{dx_0}-\tilde{M}^{2/3}\left[\frac{dS_{30}}{dx_0}+\frac{m_0}{\mu_em_u}\frac{dP_0}{dx_0}\right]-\frac{m_0}{\mu_em_u}\tilde{M}^{4/3}\frac{dD_0}{dx_0}=0;\\ &\frac{d^2}{dx_0^2}S_{10}-\alpha_0\frac{d^2S_{20}}{d^2x_0}-\tilde{M}^{2/3}\left[\frac{d^2S_{30}}{dx_0^2}+\frac{m_0}{\mu_em_u}\frac{d^2P_0}{dx_0^2}\right]-\frac{m_0}{\mu_em_u}\tilde{M}^{4/3}\frac{d^2D_0}{dx_0^2}=0.\end{aligned} \tag{71}$$

As can be seen from Figures 12, 13, at large x_0 functions $S_{i0}(x_0) \sim x_0$ and $P_0(x_0), D_0(x_0) \sim x_0^2$. It means, there exists such value x_0^*, when energy has a bend and mass reaches its maximum.

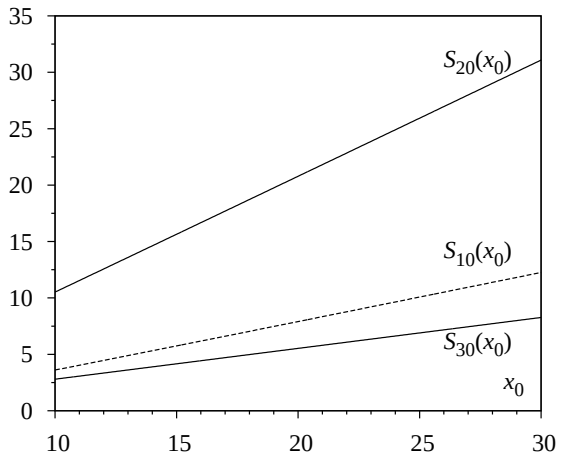

Fig. 12. Dependence $S_{i0}(x_0)$ on the relativistic parameter x_0.

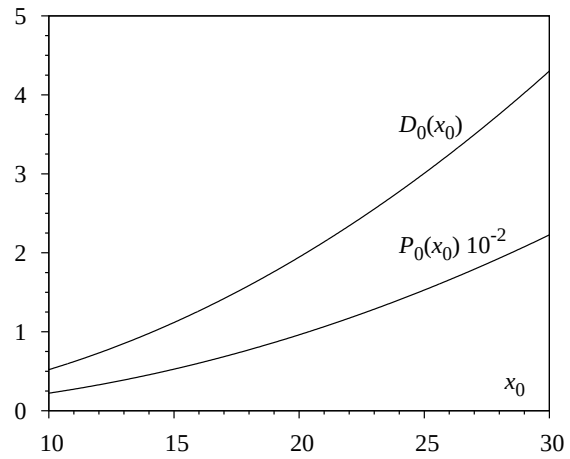

Fig. 13. Dependencies $P_0(x_0)$ and $D_0(x_0)$ on the relativistic parameter.

4. Conclusions

We have calculated the critical values of cold massive white dwarfs – dimensionless maximum mass and value of relativistic parameter x_0^*, at which an instability of a star occured. They are shown in the Table 2.

Table 2. Critical parameters of white dwarfs.

z	2	6	26
x_0^*	23.035	23.096	23.688
$\tilde{M}(x_0^*)$	1.936492	1.913637	1.823887

Deviation of dimensionless maximum mass from the Chandrasekhar's limit $\tilde{M}(\infty) = 2.01824\ldots$ reaches 4.1% at $z = 2$; 5.2% at $z = 6$; 9.6% at $z = 26$.

Coulomb interaction, as well as general relativity effects reduce the maximum mass of a dwarf. The contribution of interaction and general relativity effects are of the same order. The former one depends on the chemical composition of degenerate dwarf: at $z = 2$ the contribution of interaction is twice smaller than one of general relativity effects; at $z = 6$ they are almost equal; at $z = 26$ contribution of the interaction is four times larger then contribution of general relativity effects.

It is worth mentioning, that the influence of Coulomb interaction on the critical value of relativistic parameter is negligible. The reason is a fact, that mass dependences on x_0 in the Chandrasekhar's theory and in one with Coulomb interaction at large x_0 are almost parallel. The instability is caused mainly due to general relativity effects.

The parameter of chemical composition for pure helium dwarf is 2 and as we can see from the Table 2, mass of such dwarf can not exceed the value $1.41\ldots M_\odot$. This value decreases for another chemical compositions ($z > 2$). It means, there is an alternative:

1. Either white dwarfs in the binary systems with masses close to $1.45\ldots M_\odot$ are helium dwarfs with substantial fraction of hydrogen (average $\mu_e < 2$);

2. Or these dwarfs have to show strong evidence of magnetic field and effects of latter one exceed the effects of general relativity theory and Coulomb interaction.

We think the first case is more probable and the large mass is a result of accretion of matter (hydrogen-rich) from the star-companion on the degenerate dwarf.

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Міжчастинкові взаємодії, ефекти ЗТВ і критичні параметри вироджених карликів

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Розглянуто способи розрахунку маси вироджених карликів як на основі рівняння механічної рівноваги, так і варіаційного підходу. Для конкретних розрахунків використано модель з ідеальною електронною підсистемою та модель з кулонівськими взаємодіями. Досліджено область стійкості масивних вироджених карликів. Вперше при врахуванні міжчастинкових взаємодій визначено критичне значення маси зорі і значення параметра релятивізму у її центрі, при якому порушується стійкість за рахунок ефектів загальної теорії відносності.

Ключові слова: *вироджені карлики, кулонівські взаємодії, ефекти загальної теорії відносності, стійкість карликів, максимальна маса, критичне значення параметра релятивізму.*

2000 MSC: 85A15, 83C25, 82D35

УДК: 524.31.084, 524.384, 524.352.3