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The algorithms of constructing the continued fractions for any rations of the hypergeometric Gaussian functions

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An algorithm for constructing recurrence relations of geometric Gaussian functions, in which the displacement of parameters is equal to 0, 1 or -1, is described. On the basis of such recurrence relations, the expansion for the ratio of Gaussian functions into continued fractions is developed. The obtained continued fractions are the development of the corresponding hypergeometric Gaussian functions in the case when the parameters of the function are integers.

Keywords: *Gaussian hypergeometric series, hypergeometric function, continued fraction, recurrence relation, expansion, ratio, algorithm, approximant.*

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1. Introduction

In recent decades, interest in the theory of special functions has increased, since these functions play an important role in various fields of natural and engineering sciences, particularly in solving problems of mathematical physics, aerodynamics, nuclear physics, astrophysics, acoustics, quantum field theory, theory of probability and mathematical statistics, biomedicine and others.

Among the special functions, hypergeometric functions stand out in a special way. Hypergeometric functions were known to be introduced by L. Euler, and eventually many domestic and foreign scientists have been involved in their study, research and generalization [1–4].

Gaussian hypergeometric function is one of the main classes of special functions, being a component of the solution of various problems of mathematical physics, aerodynamics, aeromechanics, astrophysics, quantum mechanics, astronomy, biomedicine etc [5, 6].

Theoretical and practical significance of hypergeometric functions causes the need and expediency of deeper study and research of their properties, methods of approximation and algorithms of computation. Recent domestic and foreign publications [5, 7–10] confirmed the significant interest in this issue.

Continued fractions are an effective method of approximation of the hypergeometric Gaussian functions [9–12]. However, software has been used only for calculating the values of the approximations or analysis of their domains of convergence until now. While the method of approximation constructing (finding the general formulas of continued fraction coefficients) remained troublesome routine work for scientists who built approximations with simplifying and transforming expressions manually.

Today, the algorithmic mathematics makes it possible not only to perform accurate calculations or compare the rate of the convergence of different algorithms, but to build new formulas and recurrent structures that is a significant contribution to the development of analytic theory.

One of the examples of using the algorithmic mathematics in constructing of the rational approximations of the Gaussian hypergeometric function is described in this article.

2. Gaussian hypergeometric series and its analytic continuation

The main object for investigation is Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$, which is depicted as a hypergeometric series inside the unit circle [2]

$$ {}_2F_1(a,b;c;z) \equiv F(a,b;c;z) \equiv \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}, \tag{1} $$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, i.e. $(a)_0 = 1$, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, $n = 1,2,3,\ldots$ and $c \neq 0,-1,-2,\ldots$, $a,b,c,z \in \mathbb{C}$.

The series converges at $z = 1$ if $\operatorname{Re}(c-a-b) > 0$. The function to which it converges can be extended analytically to the cut plane

$$ D = \{ z \in C \colon |\arg(1-z)| < \pi \}. \tag{2} $$

It is known as the hypergeometric function, or more precisely, the principal branch of the hypergeometric function, and we use the same notation ${}_2F_1(a,b;c;z)$ for this function as for the series.

From the definition (1) we have got the equation

$$ F(a,b;c;z) = F(b,a;c;z). $$

The next six functions $F(a\pm 1,b;c;z)$, $F(a,b\pm 1;c;z)$, $F(a,b;c\pm 1;z)$ are called contiguous to the function $F(a,b;c;z)$. Between $F\,(a,\ b;\ c;\ z)$ and any two functions contiguous to it there exists a linear relation with coefficients which are linear functions of z. There are 15 relations of this type which have been found by Gaussian [2]. Any of these relations can be proved by expansion of hypergeometric functions in power series by the formula (1).

If l, m, n are integers, then $F(a\pm l, b\pm m; c\pm n; z)$ can be expressed by repeated applications of these relations as a linear combination of $F(a,b;c;z)$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, z [1,2,4].

For the definition of Gaussian hypergeometric function $F(a,b;c;z)$ outside the unit disk, its analytic continuation, particularly integrals [1–4] or continued fractions, are used [10–12]. Continued fractions are an effective tool for analytic continuation of the ratio of two Gaussian hypergeometric functions and the hypergeometric Gaussian function with integer parameters itself [10–12]. Continued fraction is defined as the fraction:

$$ \mathop{K}_{n=1}^{\infty} \left(\frac{a_n}{b_n} \right) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots}}}. \tag{3} $$

The n-th approximant of continued fraction (3) is the finite fraction:

$$ \mathop{K}_{n=1}^{m} \left(\frac{a_n}{b_n} \right) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ldots + \cfrac{a_{m-1}}{b_{m-1} + \cfrac{a_m}{b_m}}}}. $$

Continued fraction is considered to be convergent if the sequence of its n-th approximants is convergent. The n-th approximants are used for approximate calculation of continued fractions.

In the theory of continued fractions the expansion of the next ratio into the Gaussian fraction is well known [11]:

$$ \frac{F(a,b;c;z)}{F(a,b+1,c+1,z)} = 1 + \mathop{K}_{n=1}^{\infty} \left(\frac{a_n z}{1} \right), \tag{4} $$

where

$$a_{2n+1} = -\frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)}, \quad a_{2n+2} = -\frac{(b+n+1)(c-a+n+1)}{(c+2n+1)(c+2n+2)}, \quad n = 0, 1, 2, \ldots .$$

Such a fraction evenly converges to meromorphic function which is analytic continuation of the function in the right side of equation (4) in the whole complex plane with the exception of incision along the real axis $D = [z : 0 < \arg(z-1) < 2\pi]$ [11].

This result is obtained by using the following recurrence relations for hypergeometric series (1):

$$F(a, b; c; z) = F(a, b+1; c+1; z) - \frac{a(c-b)}{c(c+1)} zF(a+1, b+1; c+2; z), \tag{5}$$

$$F(a, b+1; c+1; z) = F(a+1, b+1; c+2; z) - \frac{(b+1)(c-ba+1)}{(c+1)(c+2)} z\, F(a+1, b+2; c+3; z).$$

Another less known expansion of the Gaussian hypergeometric functions ratio in a continuous fraction of Norlund is built for the ratio

$$\frac{F(a, b; c; z)}{F(a+1, b+1; c+1; z)} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots}}} = b_0 + \mathop{K}_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right), \tag{6}$$

where

$$a_n = \frac{(a+n)\,(b+n)}{(c+n-1)(c+n)} z(1-z), \quad n = 1, 2, 3, \ldots ,$$

$$b_n = \left(1 - \frac{a+b+2n+1}{c+n} z\right), \quad n = 0, 1, 2, \ldots .$$

Such a fraction evenly converges to function $\frac{F(a,b;c;z)}{F(a+1,b+1;c+1;z)}$ if it terminates, or if $\mathrm{Re}(z) < 1/2$, or if $z = 1/2$ and $|\mathrm{Im}(a+b)| < \mathrm{Re}(2c-a-b-1)$ [12].

This expansion was obtained from the following correlation:

$$F(a, b; c; z) = \left(1 - \frac{a+b+1}{c} z\right) + F(a+1, b+1; c+1; z) \frac{(a+1)(b+1)}{c(c+1)} z(1-z) F(a+2, b+2; c+2; z). \tag{7}$$

A number of another expansions of the hypergeometric Gaussian functions ratios to the continued T-fractions was researched in the monograph [10].

These results prove the relevance of building new expansions of Gaussian hypergeometric functions ratios in continued fractions of different types for their further studies and using them for solving of applied problems.

3. The algorithm of building the recurrent ratios for the hypergeometric Gaussian function

The generalized algorithm of deriving ratios for the hypergeometric series (1) is developed. It's to have such type that will allow building new expansions of hypergeometric Gaussian functions into continued fractions. Any recurrent ratio for three hypergeometric functions can be represented as follows

$$\begin{aligned} &G(a+\delta_a, b+\delta_b; c+\delta_c; z) \times F(a+\delta_a, b+\delta_b; c+\delta_c; z) \\ &\qquad + G'(a+\lambda_a, b+\lambda_b; c+\lambda_c; z) \times F(a+\lambda_a, b+\lambda_b; c+\lambda_c; z) \\ &\qquad\qquad + G''(a+\mu_a, b+\mu_b; c+\mu_c; z) \times F(a+\mu_a, b+\mu_b; c+\mu_c; z) = 0, \end{aligned} \tag{8}$$

where $\delta_a, \delta_b, \delta_c, \lambda_a, \lambda_b, \lambda_c, \mu_a, \mu_b, \mu_c \in \mathbb{Z}$ and $\delta \neq \lambda \wedge \delta \neq \mu \wedge \lambda \neq \mu$ (where $\delta \neq \lambda$ means that $\delta = (\delta_a, \delta_b; \delta_c)$, $\lambda = (\lambda_a, \lambda_b; \lambda_c)$, $\delta_a \neq \lambda_a \vee \delta_b \neq \lambda_b \vee \delta_c \neq \lambda_c$), F is hypergeometric Gaussian function with relevant parameters, G, G', G'' are coefficients, which generally are different functions of these parameters and can be represented as polynomial from z. For convenience, further we shall use short notation of the equations

$$G_\delta F_\delta + G_\lambda F_\lambda + G_\mu F_\mu = 0.$$

The algorithm of building the new recurrent ratios consists in the application to the known ratios the transformations of two following types:

A) Change of the parameters displacement of all components of equation (offset):

$$G'_o F_o + G_{\delta-\lambda} F_{\delta-\lambda} + G''_{\mu-\lambda} F_{\mu-\lambda} = 0 \quad \Rightarrow \quad G'_\lambda F_\lambda + G_\delta F_\delta + G''_\mu F_\mu = 0,$$

$$G_\delta F_\delta + G'_\lambda F_\lambda + G''_\mu F_\mu = 0,$$

where $o = (0, 0; 0)$ is zero offset, that is $F_o = F(a, b; c; z)$.

B) Combining the two ratios, in which the two terms have equal offset:

$$G'_o F_o + G'_\delta F_\delta + G'_\lambda F_\lambda = 0 \quad \Rightarrow \quad G'_o G''_\delta F_o + \left[G'{}_\delta G''{}_\delta F_\delta\right] + G'_\lambda G''_\delta F_\lambda = 0,$$

$$G''_o F_o + G''_\delta F_\delta + G''_\mu F_\mu = 0 \quad \Rightarrow \quad G''_o G'_\delta F_o + \left[G''{}_\delta G'{}_\delta F_\delta\right] + G''_\mu G'_\delta F_\mu = 0,$$

$$\left(G'{}_o G''{}_\delta - G''{}_o G'{}_\delta\right) F_o + \left(G'{}_\lambda G''{}_\delta\right) F_\lambda + \left(-G''{}_\mu G'{}_\delta\right) F_\mu = 0, \quad G_o F_o + G_\lambda F_\lambda + G_\mu F_\mu = 0.$$

In A) transformation the corresponding value λ is added to all parameters of equation, i.e. in a traditional form the transformation looks like a change of variables $a' \to a + \lambda_a$, $b' \to b + \lambda_b$, $c' \to c + \lambda_c$.

In B) transformation the equations are multiplied to the corresponding coefficient to get a common term (which is highlighted with square brackets), and substitution is made.

The initial input data for the algorithm are 15 known recurrent ratios for the hypergeometric Gaussian function ([2], pp.111–112). The transformation of either type A (with $\delta = (0, 0; 0)$) or B (with $o = (0, 0; 0)$) is to be used for building the new recurrent ratios.

Let's introduce the following notations: o, α, β, γ – the displacement of the vector $(a, b; c)$, where $o = (0, 0; 0)$, $\alpha, \beta, \gamma \in \{(+1, 0; 0), (-1, 0; 0), (0, +1; 0), (0, -1; 0), (0, 0; +1), (0, 0; -1)\}$, $\alpha \neq \beta \wedge \alpha \neq -\beta \wedge \beta \neq \gamma \wedge \beta \neq -\gamma \wedge \gamma \neq \alpha \wedge \gamma \neq -\alpha$.

While building the continued fractions the ratios containing $F(a, b; c; z)$ are chosen:

$$\begin{aligned} G(a, b; c; z) \times F(a, b; c; z) + G'(a + \delta_a, b + \delta_b; c + \delta_c; z) \times F(a + \delta_a, b + \delta_b; c + \delta_c; z) \\ + G''(a + \lambda_a,\ b + \lambda_b; c + \lambda_c; z) \times F(a + \lambda_a, b + \lambda_b; c + \lambda_c; z) = 0, \end{aligned}$$

where $\delta_a, \delta_b,\ \delta_c, \lambda_a, \lambda_b, \lambda_c \in \{-1, 0, +1\}$ and $\delta \neq \lambda$ ($\delta_a \neq \lambda_a \vee \delta_b \neq \lambda_b \vee \delta_c \neq \lambda_c$). That is

$$G_o F_o + G_\delta F_\delta + G_\lambda F_\lambda = 0. \tag{9}$$

The Tabl. 1 below is a summary table of the building of all recurrent ratios for the hypergeometric Gaussian function of the form (9).

Each line indicates a final ratio type, type of transformation, offset parameters (λ, μ, δ), types of the initial ratios, and formulas that can be obtained by using this method.

In the result of programming of this algorithm 325 ratios of (9) type were obtained. Among them there are the known recurrent formulas (5) and (7). All obtained new ratios were tested for reliability with the program "Wolfram Mathematica" by using the function of expressions decomposition in Taylor-Maclaurin series in the point $z_0 = 0$.

Table 1. Summary table of the building of all recurrent ratios for the hypergeometric Gaussian function.

<table>
<tr><th>No</th><th>Type</th><th>λ</th><th>μ</th><th>δ</th><th colspan="2">Used</th><th>Number of formulas</th></tr>
<tr><td>0.0</td><td rowspan="2">–</td><td>$-\alpha$</td><td rowspan="2">α</td><td rowspan="2">–</td><td colspan="2">Given</td><td>3</td></tr>
<tr><td>0.1</td><td>β</td><td colspan="2">Given</td><td>12</td></tr>
<tr><td>1.0</td><td>A</td><td>α</td><td rowspan="4">$\alpha+\beta$</td><td>–</td><td colspan="2">0.1</td><td>24</td></tr>
<tr><td rowspan="2">1.1</td><td rowspan="3">B</td><td rowspan="2">$-\alpha$</td><td>α</td><td>0.0</td><td rowspan="3">1.0</td><td>24</td></tr>
<tr><td>β</td><td rowspan="2">0.1</td><td>24</td></tr>
<tr><td>1.2</td><td>γ</td><td>α</td><td>24</td></tr>
<tr><td>2.0</td><td>A</td><td>α</td><td rowspan="3">$\alpha+\beta+\gamma$</td><td>–</td><td colspan="2">1.2</td><td>24</td></tr>
<tr><td rowspan="2">2.1</td><td rowspan="2">B</td><td rowspan="2">$-\alpha$</td><td>α</td><td>0.0</td><td rowspan="2">2.0</td><td>24</td></tr>
<tr><td>β</td><td>0.1</td><td>24</td></tr>
<tr><td>3.0</td><td rowspan="5">B</td><td>$\alpha-\beta$</td><td rowspan="5">$\alpha+\beta$</td><td rowspan="4">α</td><td>1.0</td><td rowspan="5">1.0</td><td>12</td></tr>
<tr><td>3.1</td><td>$-\alpha-\beta$</td><td>1.1</td><td>6</td></tr>
<tr><td>3.2</td><td>$\alpha+\gamma$</td><td>1.0</td><td>24</td></tr>
<tr><td rowspan="2">3.3</td><td rowspan="2">$-\alpha+\gamma$</td><td>1.1</td><td>24</td></tr>
<tr><td>β</td><td>1.2</td><td>24</td></tr>
<tr><td rowspan="2">4.0</td><td rowspan="2">A</td><td rowspan="2">$\alpha+\beta$</td><td rowspan="6">$\alpha+\beta+\gamma$</td><td rowspan="2">–</td><td colspan="2">1.2</td><td>24</td></tr>
<tr><td colspan="2">2.0</td><td>24</td></tr>
<tr><td>4.1</td><td rowspan="4">B</td><td>$\alpha-\beta$</td><td>α</td><td rowspan="2">1.0</td><td>2.0</td><td>48</td></tr>
<tr><td rowspan="3">4.2</td><td rowspan="3">$-\alpha-\beta$</td><td>$-\alpha$</td><td>2.1</td><td>24</td></tr>
<tr><td>α</td><td>1.1</td><td rowspan="2">2.0</td><td>24</td></tr>
<tr><td>γ</td><td>1.2</td><td>24</td></tr>
<tr><td rowspan="2">5.0</td><td rowspan="4">B</td><td rowspan="2">$\alpha+\beta-\gamma$</td><td rowspan="4">$\alpha+\beta+\gamma$</td><td>α</td><td>2.0</td><td>2.0</td><td>12</td></tr>
<tr><td>$\alpha+\beta$</td><td>4.0</td><td>4.0</td><td>12</td></tr>
<tr><td>5.1</td><td>$\alpha-\beta-\gamma$</td><td rowspan="2">α</td><td>2.0</td><td rowspan="2">2.0</td><td>12</td></tr>
<tr><td>5.2</td><td>$-\alpha-\beta-\gamma$</td><td>2.1</td><td>4</td></tr>
</table>

4. The expansion of the ratio of the geometric Gaussian function into continued fractions

After analyzing the known examples of expansion of the ratios of hypergeometric functions into continued fractions — Gaussian fraction (4) and Norlund fraction (6) — the generalized algorithm of building of the continued fractions based on recurrent ratios for hypergeometric Gaussian function was developed. A software implementation of the algorithm in order to automate building of relevant fractions and to do their further research was done.

4.1. A formula of the fraction, derived from the one ratio

Consider the ratio (of 0.0, 3.1, 5.2 group) in a shortened form

$$G_o F_o + G'_\delta F_\delta + G''_{-\delta} F_{-\delta} = 0, \tag{10}$$

where $\{\delta_a, \delta_b, \delta_c\} \subseteq \{-1, 0, 1\}$.

Divide it by $G''_{-\delta} F_o$

$$\frac{F_{-\delta}}{F_o} = -\frac{G_o}{G''_{-\delta}} - \frac{G'_\delta}{G''_{-\delta}} \times \frac{F_\delta}{F_o},$$

and we'll get the recurrent formula for the fraction

$$\frac{F_{-\delta}}{F_o} = -\frac{G_o}{G''_{-\delta}} + \frac{-\dfrac{G'_\delta}{G''_{-\delta}}}{\dfrac{F_o}{F_\delta}}.$$

Next, we'll obtain general view of the n-th approximants by shifting the parameters on δ

$$\frac{F_{-\delta}}{F_o} = -\frac{G_o}{G''_{-\delta}} + \cfrac{-\dfrac{G'_\delta}{G''_{-\delta}}}{-\dfrac{G_\delta}{G''_o} + \cfrac{-\dfrac{G'_{2\delta}}{G''_o}}{-\dfrac{G_{2\delta}}{G''_\delta} + \ldots + \cfrac{-\dfrac{G'_{N\delta}}{G''_{(N-2)\delta}}}{-\dfrac{G_{N\delta}}{G''_{(N-1)\delta}}}}} = -\frac{G_o}{G''_{-\delta}} + \mathop{K}_{i=1}^{N} \frac{-\dfrac{G'_{i\delta}}{G''_{(i-2)\delta}}}{-\dfrac{G_{i\delta}}{G''_{(i-1)\delta}}}.$$

The Norlund's continued fraction (6) is built due to this principle. Also, we can obtain the expansion into relevant fractions for the following ratios:

$$\frac{F(a, b \pm 1; c; z)}{F(a, b; c; z)}, \quad \frac{F(a, b \pm 1; c \pm 1; z)}{F(a, b; c; z)},$$
$$\frac{F(a \pm 1, b \pm 1; c; z)}{F(a, b; c; z)}, \quad \frac{F(a \pm 1, b \pm 1; c \pm 1; z)}{F(a, b; c; z)}.$$

4.2. A formula of the continued fraction, derived from the m relations

Let's consider the following ratios:

$$\begin{aligned}
&G_o^{(1)} F_o + \dot{G}_{-\delta_1}^{(1)} F_{-\delta_1} + \ddot{G}_{\delta_2}^{(1)} F_{\delta_2} = 0,\\
&G_o^{(2)} F_o + \dot{G}_{-\delta_2}^{(2)} F_{-\delta_2} + \ddot{G}_{\delta_3}^{(2)} F_{\delta_3} = 0,\\
&\ldots\\
&G_o^{(k)} F_o + \dot{G}_{-\delta_k}^{(k)} F_{-\delta_k} + \ddot{G}_{\delta_{k+1}}^{(k)} F_{\delta_{k+1}} = 0,\\
&\ldots\\
&G_o^{(m-1)} F_o + \dot{G}_{-\delta_{m-1}}^{(m-1)} F_{-\delta_{m-1}} + \ddot{G}_{\delta_m}^{(m-1)} F_{\delta_m} = 0,\\
&G_o^{(m)} F_o + \dot{G}_{-\delta_m}^{(m)} F_{-\delta_m} + \ddot{G}_{\delta_1}^{(m)} F_{\delta_1} = 0,
\end{aligned} \tag{11}$$

where

$$F_o = F(a, b; c; z), \quad F_{\delta_k} = F(a + \delta_a^{(k)}, b + \delta_b^{(k)}; c + \delta_c^{(k)}; z), \quad \delta_k \neq o \wedge -\delta_k \neq \delta_{k+1},$$
$$o = (0, 0; 0), \quad \delta_k = \left(\delta_a^{(k)}, \delta_b^{(k)}; \delta_c^{(k)}\right), \quad \delta_a^{(k)}, \quad \delta_b^{(k)}, \quad \delta_c^{(k)} \in \mathbb{Z}, \quad k = \overline{1, m}, \quad \delta_{m+1} = \delta_1,$$

$$\frac{F_{-\delta_k}}{F_o} = -\frac{G_o^{(k)}}{\dot{G}_{-\delta_k}^{(k)}} + \frac{-\dfrac{\ddot{G}_{\delta_{k+1}}^{(k)}}{\dot{G}_{-\delta_k}^{(k)}}}{\dfrac{F_o}{F_{\delta_{k+1}}}}, \quad \frac{F_{-\delta_{k+1}}}{F_o} = -\frac{G_o^{(k+1)}}{\dot{G}_{-\delta_{k+1}}^{(k+1)}} + \frac{-\dfrac{\ddot{G}_{\delta_{k+2}}^{(k+1)}}{\dot{G}_{-\delta_{k+1}}^{(k+1)}}}{\dfrac{F_o}{F_{\delta_{k+2}}}},$$

$$\frac{F_o}{F_{\delta_{k+1}}} = -\frac{G^{(k+1)}_{\delta_{k+1}}}{\dot{G}^{(k+1)}_o} + \cfrac{-\frac{\ddot{G}^{(k+1)}_{\delta_{k+1}+\delta_{k+2}}}{\dot{G}^{(k+1)}_o}}{\cfrac{F_{\delta_{k+1}}}{F_{\delta_{k+1}+\delta_{k+2}}}},$$

$$\frac{F_{-\delta_k}}{F_o} = -\frac{G^{(k)}_o}{\dot{G}^{(k)}_{-\delta_k}} + \cfrac{-\frac{\ddot{G}^{(k)}_{\delta_{k+1}}}{\dot{G}^{(k)}_{-\delta_k}}}{-\frac{G^{(k+1)}_{\delta_{k+1}}}{\dot{G}^{(k+1)}_o} + \cfrac{-\frac{\ddot{G}^{(k+1)}_{\delta_{k+1}+\delta_{k+2}}}{\dot{G}^{(k+1)}_o}}{\cfrac{F_{\delta_{k+1}}}{F_{\delta_{k+1}+\delta_{k+2}}}}},$$

$$\frac{F_{\delta_{k+1}}}{F_{\delta_{k+1}+\delta_{k+2}}} = -\frac{G^{(k+2)}_{\delta_{k+1}+\delta_{k+2}}}{\dot{G}^{(k+2)}_{\delta_{k+1}}} + \cfrac{-\frac{\ddot{G}^{(k+2)}_{\delta_{k+1}+\delta_{k+2}+\delta_{k+3}}}{\dot{G}^{(k+2)}_{\delta_{k+1}}}}{\cfrac{F_{\delta_{k+1}+\delta_{k+2}}}{F_{\delta_{k+1}+\delta_{k+2}+\delta_{k+3}}}}.$$

The continued fraction has been obtained:

$$\frac{F_{-\delta_k}}{F_o} = -\left(\frac{G^{(k)}_o}{\dot{G}^{(k)}_{-\delta_k}}\right)_o + \cfrac{-\left(\frac{\ddot{G}^{(k)}_{\delta_{k+1}}}{\dot{G}^{(k)}_{-\delta_k}}\right)_o}{-\left(\frac{G^{(k+1)}_o}{\dot{G}^{(k+1)}_{-\delta_{k+1}}}\right)_{\delta_{k+1}} + \cfrac{-\left(\frac{\ddot{G}^{(k+1)}_{\delta_{k+2}}}{\dot{G}^{(k+1)}_{-\delta_{k+1}}}\right)_{\delta_{k+1}}}{-\left(\frac{G^{(k+2)}_o}{\dot{G}^{(k+2)}_{-\delta_{k+2}}}\right)_{\delta_{k+1}+\delta_{k+2}} + \cfrac{-\left(\frac{\ddot{G}^{(k+2)}_{\delta_{k+3}}}{\dot{G}^{(k+2)}_{-\delta_{k+2}}}\right)_{\delta_{k+1}+\delta_{k+2}}}{\left(\frac{F_{-\delta_{k+3}}}{F_o}\right)_{\delta_{k+1}+\delta_{k+2}+\delta_{k+3}}}}},$$

i.e.

$$\frac{F_{-\delta_k}}{F_o} = b_0 + \underset{i=1}{\overset{\infty}{K}} \frac{a_i}{b_i},$$

where

$$a_i = -\left(\frac{\ddot{G}^{(k+i-1)}_{\delta_{k+i}}}{\dot{G}^{(k+i-1)}_{-\delta_{k+i-1}}}\right)_{\sum\limits_{j=1}^{i-1}\delta_{k+j}}, \quad b_i = -\left(\frac{G^{(k+i)}_o}{\dot{G}^{(k+i)}_{-\delta_{k+i}}}\right)_{\sum\limits_{j=1}^{i}\delta_{k+j}}, \quad \delta_{nm+k} = \delta_k,$$

$$a_{nm+i} = -\left(\frac{\ddot{G}^{(k+i-1)}_{\delta_{k+i}}}{\dot{G}^{(k+i-1)}_{-\delta_{k+i-1}}}\right)_{\sum\limits_{j=1}^{i-1}\delta_{k+j}+n\sum\limits_{j=1}^{m}\delta_j}, \quad b_{nm+i} = -\left(\frac{G^{(k+i)}_o}{\dot{G}^{(k+i)}_{-\delta_{k+i}}}\right)_{\sum\limits_{j=1}^{i}\delta_{k+j}+n\sum\limits_{j=1}^{m}\delta_j}, \; k=\overline{1,m}, \quad n \in \mathbb{Z}.$$

5. Software implementation of the Gaussian function expansion into continued fractions

To improve the efficient research of analytic continuation of the Gaussian series using continued fractions the program with the automatic process of building of the appropriate fraction was designed. The program is intended to obtain recurrent ratios (9), to build the continued fractions and to get the most simplified coefficients in general terms.

Program description

The program consists of two modules:

1) Module "LeoLibProj.exe" contains the algorithms described in the second and third parts. It outputs the recurrent ratios, builds the continued fractions and simplifies obtained coefficients. Thus, all the logic of the program without visual design is implemented in this module;

2) "Form2F1.exe" visual shell (UI). This module provides the correct input and output of data using the first module.

To run the program one must have Windows operating system with the installed virtual machine "NET.Framework", version 2.0 or higher, or Linux with the similar virtual machine.

The program is designed with the parameters that allow to run it on the computers with architecture x86 (32 bit) and x64 (64 bit).

The program has been successfully executed on the base versions of the operating systems Windows XP (architecture x86) and Windows 7 (architecture x64).

When starting the program generates a list of all possible ratios of the form (9), used for building of fractions.

The user selects the recurrent relations and adds them to the bottom list (Fig. 1). The program will not allow enter the ratio, which is not suitable for the fraction building.

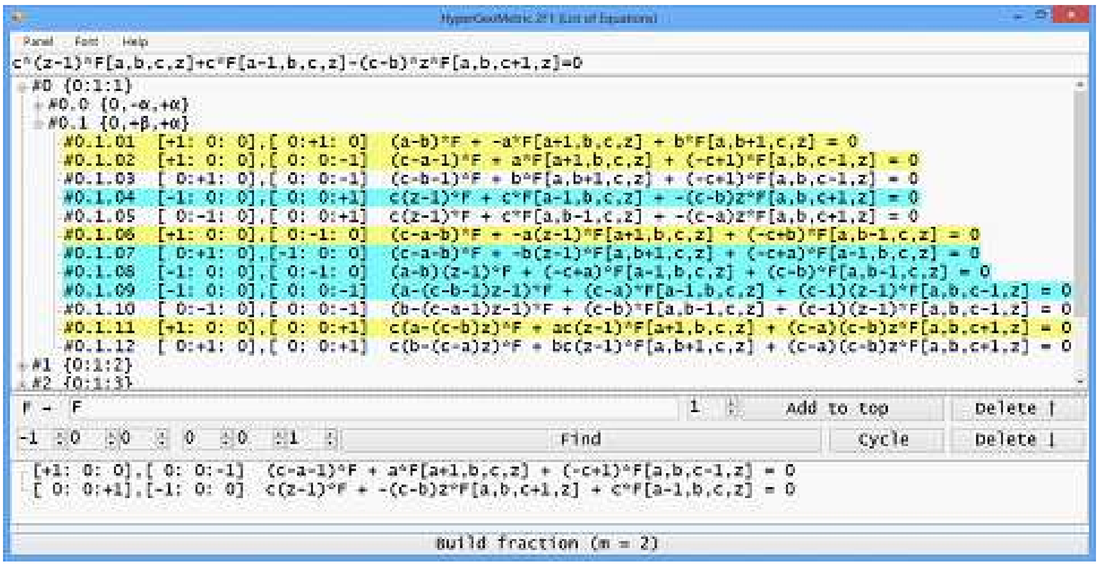

Fig. 1. Simple of selection of the recurrent relations.

The button "Build the fraction" will unblock only when a set of ratios is sufficient for the building of continued fraction.

The program simplifies all coefficients before outputting of the results. To ensure compatibility of the program with the highest number of computers, the own methods of simplifying of arithmetic expressions were designed and programmed.

Since all the coefficients in the program are presented as the tapes that are difficult to handle to obtain simplified expressions, the following methods of transformations were designed:

– string to delegate (for calculations);
– string to graph;
– graph to string;
– graph into another graph (to simplify arithmetic expressions).

When converting tape into the graph, breaking into terms, splitting on multipliers, and removing all unnecessary brackets are used.

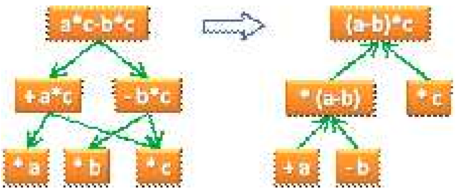

Fig. 2. Expression factorization.

When handling the graphs we use:

- converting graph into the special view, where every level of the graph contains only addition and subtraction, or multiplication and division,
- removing signs into the roof of the graph and vice versa, the brackets expansion,
- factorization (removing the common multipliers out of the brackets, as on Fig. 2),
- cancelling the opposite terms,
- sorting the vertices of a graph (converting to the consistency).

Output data are the expressions for the coefficients of the fraction in the relevant formulas. It is worth noting that by means of equivalent transformations the resulting fractions are reduced to fractions with denominators equal to one.

As the examples: the Gaussian fraction (4) in Fig. 3 and the Norlund fraction (6) in Fig. 4.

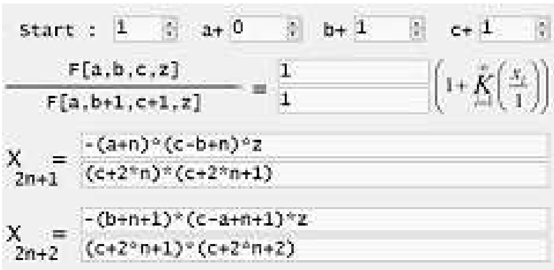

Fig. 3. Gaussian fraction.

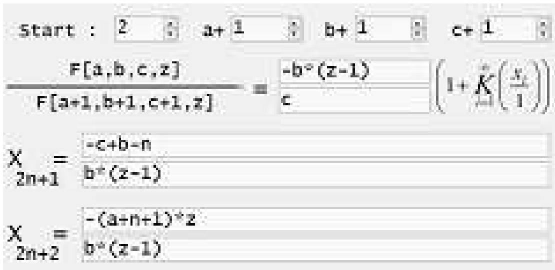

Fig. 4. Norlund fraction.

If for the building of expansion we use two ratios

$$c\,F(a,b;c;z)+a(z-1)F(a+1,b+1;c+1;z)+(a-c)F(a,b+1;c+1;z)=0,$$

$$(b-1)z\,F(a,b;c;z)+(1-c)F(a,b-1;c-1;z)+(c-1)F(a-1,b-1;c-1;z)=0,$$

we can get the continued fraction for the ratio (7) of another look (Fig. 5).

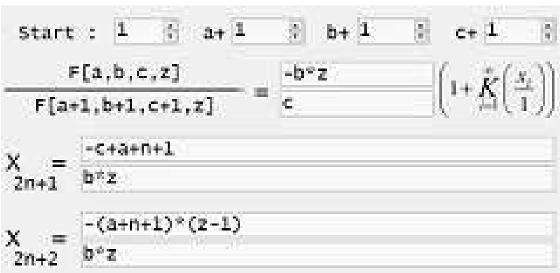

Fig. 5. Modified Norlund fraction.

Therefore, by using the proposed program one can obtain the continued fractions of different types for given ratio of the hypergeometric Gaussian functions. It provides new opportunities for development of both the analytical theory of the continued fractions and the theory of the special functions approximation.

Exspansion of ${}_2F_1(a,1;c;z)$ function into the continued fraction

The program can be used also for obtaining the expansion of the Gaussian with integer parameters into the continued fraction. For instance, for computation $F(a,1;c\,z)$ we can easily use the expansion for $\frac{F(a,b+1;c\,z)}{F(a,b;c\,z)}$ and $\frac{F(a,b;c\,z)}{F(a,b+1;c\,z)}$, where $b=0$ (Fig. 6) or $\frac{F(a,b;c\,z)}{F(a,b-1;c\,z)}$ and $\frac{F(a,b-1;c\,z)}{F(a,b;c\,z)}$ with $b=1$ (Fig. 7).

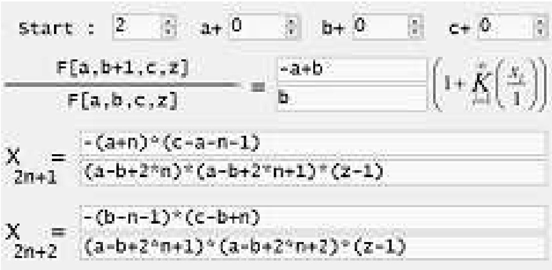

Fig. 6. Gaussian fraction.

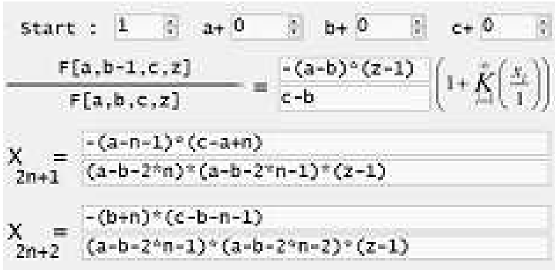

Fig. 7. Norlund fraction.

For computation $F(a,2;c\,z)$ we can find the expansion $\frac{F(a,b\pm2;c\,z)}{F(a,b;c\,z)}$, or $\frac{F(a,b;c\,z)}{F(a,b\pm2;c\,z)}$ in the same way.

6. Conclusions

The proposed algorithm allows to receive any relations of the Gaussian hypergeometric functions where displacement of parameters a, b, c is equal to 0, 1 or -1. On the base of the received recurrent ratios one can build and investigate new expansions of the ratio of Gaussian function into the continued fraction. The software with this algorithm implemented was developed. In case the parameters of the Gaussian hypergeometric function are integers, using this software one can obtain and investigate new expansions of this function into the continued fractions.

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Алгоритми побудови неперервних дробів для довільних відношень гіпергеометричних функцій Гаусса

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Описано алгоритм побудови рекурентних співвідношень гіпергеометричних функцій Гаусса, в яких зміщення параметрів a, b, c дорівнює 0, 1 або -1. На основі таких рекурентних співвідношень побудовано розвинення для відношення функцій Гаусса у неперервні дроби. Отримані неперервні дроби є розвиненням відповідних гіпергеометричних функцій Гаусса, якщо параметри функції є цілими числами.

Ключові слова: *гіпергеометричний ряд Гаусса, гіпергеометрична функція, неперервний дріб, рекурентне відношення, розвинення, відношення, алгоритм, наближення.*

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