РОЗДІЛ IV

ЕКОНОМІЧНА СТАТИСТИКА, БУХГАЛТЕРСЬКИЙ ОБЛІК ТА АУДИТ

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NUMERICAL METHODS OF SOLVING THE INITIAL VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS WITH EVALUATION OF THE MAIN MEMBER OF THE LOCAL ERROR

One of the modern scientific methods for investigating phenomena and processes is mathematical modeling. A mathematical modeling is the effective method of study of economic processes, in many important cases allows to replace the real process, and also gives an opportunity to get both quality and quantitative picture of the designed problems. Since the exact solutions of the investigated models can be got only in very partial cases, then it is necessary to use numerical methods. In the design of economic problems there is a necessity not only to find the numeral solutions of such models but also research of the estimations of their local and global errors. Continuous fractions are widely used in computational mathematics. They make it possible to obtain monotonic and two-sided approximations, have a weak sensitivity to rounding errors, and also correctly reflect the basic properties of the problems studied. A research object is initial value problem for ordinary differential equations. The purpose of the study is to develop methods and algorithms to build computational methods for the numerical solution of the Cauchy problem for ordinary differential equations. Formulas of the Runge-Kutta type of the fourth order of accuracy of the solution of the initial problem for ordinary differential equations based on continued fractions are derived. New two-sided numerical methods of the third order of accuracy are offered which at each node point make it possible to obtain not only upper and lower approximations to an exact solution, but also without additional computational costs give information about the magnitude of the leading term of the local error of the approximate solution. Two-sided formulas at each step of integration use less number of calculations of the right-hand side of the differential equation than the known bilateral methods of Runge-Kutta type. The proposed formulas, using only four calculations of the right-hand side of the differential equation, allow at each step to obtain a method of fourth-order accuracy method and two bilateral formulas of the third order of accuracy

Keywords: *initial value problem, continued fractions, methods of Runge-Kutta type, two-sided approximations, nonlinear numerical methods.*

INTRODUCTION

A mathematical modelling is the effective method of study of physical processes, in many important cases allows to replace the real process, and also gives an opportunity to get both quality and quantitative picture of the designed process. Since the exact solutions of the investigated models can be got only in very partial cases, then it is necessary to use numerical methods.

Continuous fractions were widely used in computational mathematics. They make it possible to obtain monotonic and two-sided approximations, have a weak sensitivity to rounding errors, and also correctly reflect the basic properties of the problems studied [1-5]. The process of calculating continued fractions is cyclical and can be easily programmed.

Problems involved in the construction and investigation of mathematical models of physical and chemical, biological and economic processes, problems of multidimensional optimization, electronic, kinetics, etc lead to the need to solve nonlinear differential equations and its systems [6-8].

Objects and methods of investigation. A research object is Cauchy problem for ordinary differential equations. A problem of construction of numerical methods for the solution of initial value problem for ordinary differential equations, that are based on the continued fractions, is the actual problem of computing modeling.

The purpose of the study is to develop new computational methods of the third and fourth order of accuracy for the numerical solution of the Cauchy problem for ordinary differential equations**.**

Formulation of the problem. Many applied problems lead to the need to solve systems of nonlinear differential equations.

A new application of continued fractions to the development of numerical methods for the solution of differential equations is proposed

RESULT AND ITS DISCUSSION

Consider the initial value problem for ordinary differential equations

$$y'(x) = f\left(x, y(x)\right),\ y(x) = y_0,\ 0 \le x \le X, \quad (1)$$

where $y(x)$ is a real m-component vector and f – a real vector function of a dependent and an independent variable; moreover, it is assumed that the function f is differentiable as many times as necessary for numerical analysis.

We consider the scalar case, since on the systems of differential equations this technique is carried over componentally.

I. Construction of numerical methods based on continuous fractions

Using the theory of constructing one-step methods [9–14] and continuous fractions, we seek the approximate solution of the Cauchy problem (1) in the form

$$y_{n+1}^{[k,l]} = {y_n}\big/{D_n}, \quad (2)$$

$$D_n = \sum_{i=0}^{k-1} d_{i,0} + \cfrac{d_{k,0}}{1+\cfrac{d_{k,1}}{1+\ddots+\cfrac{d_{k,l-1}}{1+d_{k,l}}}}. \quad (3)$$

Expressions $d_{k,l}$ for $k+l=\overline{1,4}$, ($k=\overline{1,4}$; $l=\overline{0,3}$) have the form

$$d_{0,0}=1,\ d_{i,0} = -\sum_{m=1}^{i} d_{i-m,0}\cdot\frac{\sigma_m}{y_n},\ i=\overline{1,4},$$

$$d_{\nu,1} = -\frac{d_{\nu+1,0}}{d_{\nu,0}},\ \nu=\overline{1,3},$$

$$d_{\mu,2} = d_{\mu+1,1} - d_{\mu,1},\ \mu = 1,2;$$

$$d_{1,3} = d_{1,2}\frac{d_{2,2}}{d_{1,2}},\ \sigma_m = h\sum_{i=1}^{q} a_{mi}k_i,\ q=k+l,$$

$$k_i = f(x_n+\alpha_i h,\ y_n + h\sum_{j=1}^{q}\beta_{ij}k_j),$$

$$\alpha_i = \sum_{j=1}^{q}\beta_{ij}. \quad (4)$$

Here h – the step of integration ($h = x_{n+1}-x_n$, $n=0,1,2,3,...$), a_{ij}, α_i, β_{ij} – the parameters. These formulas allow to construct both explicit ($\beta_{ij}=0$, at $i \le j$) and implicit numerical methods. It is convenient to write parameter a_{ij}, α_i, β_{ij} values in the form of a table:

α_1	β_{11} Λ $\beta_{1\nu}$	a_{11} Λ $a_{1\nu}$
M	M M	M M
α_ν	$\beta_{\nu 1}$ Λ $\beta_{\nu\nu}$	$a_{\nu 1}$ Λ $a_{\nu\nu}$

In this paper, explicit one-step methods of the fourth order of accuracy are proposed. The approximate solution of the problem (1) is sought in the form (2)–(3) for $k=4$, $l=0$ i.e.

$$y_{n+1}^{[4,0]} = \frac{y_n}{\sum_{k=0}^{4} d_{k,0}},\ n=0,1,2,..., \quad (5)$$

$$d_{0,0}=1,\ d_{k,0} = -\sum_{m=1}^{k} d_{k-m}\cdot\frac{\sigma_m}{y_n},\ k=\overline{1,4},$$

$$\sigma_m = h\cdot\sum_{i=1}^{4} a_{mi}k_i,$$

$$k_i = f\left(x_n+\alpha_i h, y_n + h\sum_{j=1}^{4}\beta_{ij}k_j\right),$$

$$\alpha_i = \sum_{j=1}^{4}\beta_{ij},\ y_n \ne 0. \quad (6)$$

Substituting expressions for $d_{k,0}$ ($k=\overline{0,4}$) in (5) we get

$$y_{n+1}^{[4,0]} = \frac{P_{[4,0]}}{Q_{[4,0]}},$$

where

$$P_{[4,0]} = y_n^5,$$

$$Q_{[4,0]} = y_n^4 - hy_n^3\left\{\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}k_i\right\} +$$

$$+h^2y_n^2\left\{\left(\sum_{i=1}^{4} a_{1i}k_i\right)^2 + \right.$$

$$+2\left(\sum_{i=1}^{4} a_{1i}k_i\right)\left(\sum_{i=1}^{4} a_{2i}k_i+\sum_{i=1}^{4} a_{3i}k_i\right)+$$

$$+\left(\sum_{i=1}^{4} a_{2i}k_i\right)^2\Bigg\}-h^3 y_n+$$

$$+\left\{\left(\sum_{i=1}^{4} a_{1i}k_i\right)^3+3\left(\sum_{i=1}^{4} a_{1i}k_i\right)^2\cdot\left(\sum_{i=1}^{4} a_{2i}k_i\right)\right\}+$$

$$+h^4\left(\sum_{i=1}^{4} a_{1i}k_i\right)^4+O\left(h^5\right).$$

Developing

$$k_i = f\left(x_n+\alpha_i h, y_n + h\sum_{j=1}^{4}\beta_{ij}k_j\right), \left(i=\overline{1,4}\right)$$

in the Taylor series in degrees h, we find

$$\sum_{i=1}^{4} a_{mi}k_i=\left(\sum_{i=1}^{4} a_{mi}\right)f+h\left(\sum_{i=1}^{4} a_{mi}\alpha_i\right)Df+$$

$$+h^2\left\{\left(\sum_{i=1}^{4} a_{mi}\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)\frac{\partial f}{\partial y}\cdot Df+\right.$$

$$+\left(\sum_{i=1}^{4} a_{mi}\sum_{j=1}^{4}\beta_{ij}\frac{\alpha_j^2}{2}\right)\frac{\partial f}{\partial y}\cdot D^2 f+$$

$$\left.+\left(\sum_{i=1}^{4} a_{mi}\frac{\alpha_i^2}{2}\right)D^2 f\right\}+h^3\left\{\left(\sum_{i=1}^{4} a_{mi}\frac{\alpha_i^3}{6}\right)D^3 f+\right.$$

$$+\left(\sum_{i=1}^{4} a_{mi}\alpha_i\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)\cdot Df\cdot D\left(\frac{\partial f}{\partial y}\right)+$$

$$\left.+\left(\sum_{i=1}^{4} a_{mi}\sum_{l=1}^{4}\beta_{il}\sum_{j=1}^{4}\beta_{lj}\alpha_j\right)\cdot\left(\frac{\partial f}{\partial y}\right)^2\cdot Df\right\}+$$

$$+O\left(h^4\right), \qquad (7)$$

where is marked

$$D^{\nu}=\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)^{\nu}.$$

Substituting the value for $\sum_{i=1}^{4} a_{mi}k_i$ in the formula (7) for determination $Q_{[4,0]}$, we obtain

$$Q_{[4,0]}=y_n^4-hy_n^3\cdot f\left\{\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\right\}+h^2\left\{y_n^2 f^2\left[\left(\sum_{i=1}^{4} a_{1i}\right)^2+\left(\sum_{i=1}^{4} a_{2i}\right)^2+\right.\right.$$

$$\left.\left.+2\left(\sum_{i=1}^{4} a_{1i}\right)\left(\sum_{i=1}^{4} a_{2i}+\sum_{i=1}^{4} a_{3i}\right)\right]-y_n^3\cdot Df\left(\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\alpha_i\right)\right\}+O\left(h^3\right).$$

Find the difference

$$y\left(x_{n+1}\right)\cdot Q_{[4,0]}-P_{[4,0]}=\left(1-\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\right)h\cdot y_n^4\cdot f+$$

$$+h^2\left(\frac{1}{2}-\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\alpha_i\right)\cdot y_n^4\cdot Df+h^2\left[\left(\sum_{i=1}^{4} a_{1i}\right)^2+\left(\sum_{i=1}^{4} a_{2i}\right)^2+\right.$$

$$\left.+2\left(\sum_{i=1}^{4} a_{1i}\right)\left(\sum_{i=1}^{4} a_{2i}+\sum_{i=1}^{4} a_{3i}\right)-\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\right]y_n^3 f^2+h^3 y_n^4\cdot D^2 f\times$$

$$\times\left(\frac{1}{6}-\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\frac{\alpha_i^2}{2}\right)+h^3\cdot y_n^4\cdot\frac{\partial f}{\partial y}\cdot Df\left(\frac{1}{6}-\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)+$$

$$+y_n^3\cdot f\cdot Df\cdot h^3\left[2\left(\sum_{i=1}^{4} a_{1i}\right)\left(\sum_{m=1}^{3}\sum_{i=1}^{4} a_{mi}\alpha_i\right)+2\left(\sum_{i=1}^{4} a_{1i}\alpha_i\right)\left(\sum_{i=1}^{4} a_{2i}+\right.\right.$$

$$\left.\left.+\sum_{i=1}^{4} a_{3i}\right)+2\left(\sum_{i=1}^{4} a_{2i}\right)\left(\sum_{i=1}^{4} a_{2i}\alpha_i\right)-\sum_{m=1}^{3}\sum_{i=1}^{4} a_{mi}\alpha_i-\frac{1}{2}\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\right]+$$

$$+h^3\cdot y_n^2\cdot f^3\left[\left(\sum_{i=1}^{4} a_{1i}\right)^2+2\sum_{i=1}^{4} a_{1i}\left(\sum_{i=1}^{4} a_{2i}+\sum_{i=1}^{4} a_{3i}\right)+\left(\sum_{i=1}^{4} a_{2i}\right)^2-\right.$$

$$\left.-\left(\sum_{i=1}^{4} a_{1i}\right)^3-3\left(\sum_{i=1}^{4} a_{1i}\right)^2\left(\sum_{i=1}^{4} a_{2i}\right)\right]+h^4\cdot y_n^4\cdot D^3 f\left(\frac{1}{24}-\sum_{m=1}^{4}\sum_{i=1}^{4} a_{mi}\frac{\alpha_i^3}{6}\right)+$$

$$+h^4\cdot y_n^4\cdot\frac{\partial f}{\partial y}\cdot D^2 f\left(\frac{1}{24}-\sum_{m=1}^{4}\sum_{i=1}^{4}a_{mi}\sum_{j=1}^{4}\beta_{ij}\frac{\alpha_j^2}{2}\right)+$$

$$+h^4\left(\frac{1}{24}-\sum_{m=1}^{4}\sum_{l=1}^{4}a_{ml}\sum_{i=1}^{4}\beta_{li}\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)y_n^4\cdot\left(\frac{\partial f}{\partial y}\right)^2\cdot Df+$$

$$+h^4\left(\frac{3}{24}-\sum_{m=1}^{4}\sum_{i=1}^{4}a_{mi}\alpha_i\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)y_n^4\cdot Df\cdot D\left(\frac{\partial f}{\partial y}\right)+$$

$$+h^4\left[2\left(\sum_{i=1}^{4}a_{1i}\right)\left(\sum_{m=1}^{3}\sum_{i=1}^{4}a_{mi}\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)+2\left(\sum_{i=1}^{4}a_{2i}\right)\left(\sum_{i=1}^{4}a_{2i}\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)+\right.$$

$$+2\left(\sum_{i=1}^{4}a_{1i}\sum_{j=1}^{4}\beta_{ij}\alpha_j\right)\left(\sum_{i=1}^{4}a_{2i}+\sum_{i=1}^{4}a_{3i}\right)-\sum_{m=1}^{3}\sum_{i=1}^{4}a_{mi}\sum_{j=1}^{4}\beta_{ij}\alpha_j-$$

$$\left.-\frac{1}{6}\left(\sum_{m=1}^{4}\sum_{i=1}^{4}a_{mi}\right)\right]y_n^3\cdot f\cdot\left(\frac{\partial f}{\partial y}\right)\cdot Df+h^4\left[2\left(\sum_{i=1}^{4}a_{2i}\right)\left(\sum_{i=1}^{4}a_{2i}\frac{\alpha_i^2}{2}\right)+\right.$$

$$+2\left(\sum_{i=1}^{4}a_{1i}\right)\left(\sum_{m=1}^{3}\sum_{i=1}^{4}a_{mi}\frac{\alpha_i^2}{2}\right)+2\left(\sum_{i=1}^{4}a_{1i}\frac{\alpha_i^2}{2}\right)\left(\sum_{i=1}^{4}a_{2i}+\sum_{i=1}^{4}a_{3i}\right)-$$

$$\left.-\sum_{m=1}^{4}\sum_{i=1}^{4}a_{mi}\frac{\alpha_i^2}{2}-\frac{1}{6}\sum_{m=1}^{4}\sum_{i=1}^{4}a_{mi}\right]y_n^3\cdot f\cdot D^2 f+h^4\left[\left(\sum_{i=1}^{4}a_{1i}\alpha_i\right)^2+\right.$$

$$\left.+2\left(\sum_{i=1}^{4}a_{1i}\alpha_i\right)\left(\sum_{i=1}^{4}a_{2i}\alpha_i+\sum_{i=1}^{4}a_{3i}\alpha_i\right)+\left(\sum_{i=1}^{4}a_{2i}\alpha_i\right)^2-\frac{1}{2}\sum_{m=1}^{4}\sum_{i=1}^{4}a_{mi}\alpha_i\right]\times$$

$$\times y_n^3(Df)^2+h^4\left[\frac{1}{2}\left(\sum_{i=1}^{4}a_{1i}\right)^2+\left(\sum_{i=1}^{4}a_{1i}\right)\left(\sum_{i=1}^{4}a_{2i}+\sum_{i=1}^{4}a_{3i}\right)+\frac{1}{2}\left(\sum_{i=1}^{4}a_{2i}\right)^2-\right.$$

$$-3\left(\sum_{i=1}^{4}a_{1i}\right)^2\left(\sum_{i=1}^{4}a_{1i}\alpha_i+\sum_{i=1}^{4}a_{2i}\alpha_i\right)-6\left(\sum_{i=1}^{4}a_{1i}\right)\left(\sum_{i=1}^{4}a_{2i}\right)\left(\sum_{i=1}^{4}a_{1i}\alpha_i\right)+$$

$$+2\left(\sum_{i=1}^{4}a_{1i}\right)\left(\sum_{m=1}^{3}\sum_{i=1}^{4}a_{mi}\alpha_i\right)+2\left(\sum_{i=1}^{4}a_{1i}\alpha_i\right)\left(\sum_{i=1}^{4}a_{2i}+\sum_{i=1}^{4}a_{3i}\right)+$$

$$\left.+2\left(\sum_{i=1}^{4}a_{2i}\right)\left(\sum_{i=1}^{4}a_{2i}\alpha_i\right)\right]y_n^2\cdot f^2\cdot Df+h^4\cdot y_n\cdot f^4\left[\left(\sum_{i=1}^{4}a_{1i}\right)^4-\right.$$

$$\left.-3\left(\sum_{i=1}^{4}a_{1i}\right)^2\left(\sum_{i=1}^{4}a_{2i}\right)-\left(\sum_{i=1}^{4}a_{1i}\right)^3\right]+O\left(h^5\right) \quad . \qquad (8)$$

Equating to zero, the coefficients of the powers h^i ($i=\overline{1,4}$), we obtain the conditions that the parameters a_{ij}, α_i, β_{ij} ($i,j=\overline{1,4}$) must satisfy, so that $y(x_{n+1})-y_{n+1}^{[4,0]}=O(h^5)$.

Let's find the coefficients for explicit methods. We put $\beta_{ij}=0$ as $i\le j$, $\alpha_1=0$, and $a_{11}=1$, $a_{1i}=a_{23}=a_{24}=0$, ($i=\overline{2,4}$).

Then, from the analysis of the coefficients with $hy_n^4\cdot f$, $h^2y_n^3\cdot f$, $h^3y_n^2\cdot f^3$ and $h^4y_n\cdot f^4$ follows, that $a_{21}+a_{22}=0$, $\sum_{i=1}^{4}a_{3i}=0$, $\sum_{i=1}^{4}a_{4i}=0$.

As a result, we have the following relationships:

$hy_n^4 f$: $a_{21}+a_{22}+\sum_{i=1}a_{3i}+\sum_{i=1}a_{4i}=0$,

$$h^2 y_n^4 Df: \ \frac{1}{2} - \{(a_{22}+a_{32}+a_{42})\alpha_2 + (a_{33}+a_{43})\alpha_3 + (a_{34}+a_{44})\alpha_4\} = 0,$$

$$h^3 y_n^4 \frac{\partial f}{\partial y} Df: \ \frac{1}{6} - \{(a_{33}+a_{43})\beta_{32}\alpha_2 + (a_{34}+a_{44})(\beta_{42}\alpha_2 + \beta_{43}\alpha_3)\} = 0,$$

$$h^3 y_n^4 D^2 f: \ \frac{1}{6} - \left\{(a_{22}+a_{32}+a_{42})\frac{\alpha_2^2}{2} + (a_{33}+a_{43})\frac{\alpha_3^2}{2} + (a_{34}+a_{44})\frac{\alpha_4^2}{2}\right\} = 0,$$

$$h^3 y_n^3 f \cdot Df: \ -\frac{1}{2} - \{(a_{22}+a_{32}+a_{42})\alpha_2 + (a_{33}+a_{43})\alpha_3 + (a_{34}+a_{44})\alpha_4\} +$$
$$+ 2\{(a_{22}+a_{32})\alpha_2 + a_{33}\alpha_3 + a_{34}\alpha_4\} = 0,$$

$$h^4 y_n^4 \cdot D^3 f: \frac{1}{24} - \frac{1}{6}\{(a_{22}+a_{32}+a_{42})\alpha_2^3 + (a_{33}+a_{43})\alpha_3^3 + (a_{34}+a_{44})\alpha_4^3\} = 0,$$

$$h^4 y_n^4 \frac{\partial f}{\partial y} D^2 f: \frac{1}{24} - \frac{1}{2}\{(a_{33}+a_{43})\beta_{32}\alpha_2^2 + (a_{34}+a_{44})(\beta_{42}\alpha_2^2 + \beta_{43}\alpha_3^2)\} = 0,$$

$$h^4 y_n^4 Df D \frac{\partial f}{\partial y}: \frac{3}{24} - \{(a_{33}+a_{43})\alpha_3\beta_{32}\alpha_2 + (a_{34}+a_{44})\alpha_4(\beta_{42}\alpha_2 + \beta_{43}\alpha_3)\} = 0,$$

$$h^4 y_n^4 \left(\frac{\partial f}{\partial y}\right)^2 Df: \frac{1}{24} - (a_{34}+a_{44})\beta_{43}\beta_{32}\alpha_2 = 0,$$

$$h^4 y_n^3 \cdot f \cdot D^2 f: -\frac{1}{6} + 2\left\{(a_{22}+a_{32})\frac{\alpha_2^2}{2} + a_{33}\frac{\alpha_3^2}{2} + a_{34}\frac{\alpha_4^2}{2}\right\} -$$
$$-\left\{(a_{22}+a_{32}+a_{42})\frac{\alpha_2^2}{2} + (a_{33}+a_{43})\frac{\alpha_3^2}{2} + (a_{34}+a_{44})\frac{\alpha_4^2}{2}\right\} = 0,$$

$$h^4 y_n^3 f \frac{\partial f}{\partial y} Df: -\frac{1}{6} + 2(a_{33}\beta_{32}\alpha_2 + a_{34}(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) -$$
$$-(a_{33}+a_{43})\beta_{32}\alpha_2 + (a_{34}+a_{44})(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) = 0,$$

$$h^4 y_n^3 (Df)^2: -\frac{1}{2}[(a_{22}+a_{32}+a_{42})\alpha_2 + (a_{33}+a_{43})\alpha_3 + (a_{34}+a_{44})\alpha_4] + (a_{22}\alpha_2)^2 = 0,$$

$$h^4 y_n^2 f^2 Df: \ \frac{1}{2} - 3a_{22}\alpha_2 + 2\{(a_{22}+a_{32})\alpha_2 + a_{33}\alpha_3 + a_{34}\alpha_4\} = 0,$$

$$\alpha_1 = 0, \ \alpha_i = \sum_{j=1}^{i-1}\beta_{ij}, \ (i = 2,3,4), \ a_{11} = 1, \ \sum_{i=1}^{4} a_{4i} = 0, \ \sum_{i=1}^{4} a_{3i} = 0,$$
$$a_{21} + a_{22} = 0, \ a_{1i} = 0, \ (i = 3,4), \ a_{23} = a_{24} = 0 . \quad (9)$$

From the equations for the coefficients $h^2 y_n^4 \cdot Df$, $h^3 y_n^3 f \cdot Df$, $h^4 y_n^3 (Df)^2$, $h^4 y_n^2 f^2 Df$ it follows that

$$a_{21}\alpha_2 = \frac{1}{2}, \ \sum_{i=2}^{4} a_{3i}\alpha_i = 0, \ \sum_{i=2}^{4} a_{4i}\alpha_i = 0 .$$

From the equations for the coefficients $h^3 y_n^4 \cdot D^2 f$, $h^4 y_n^3 f \cdot D^2 f$ and $h^3 y_n^4 \cdot \frac{\partial f}{\partial y} \cdot Df$, $h^4 y_n^3 f \cdot Df$ we find that

$$(a_{22}+a_{32})\frac{\alpha_2^2}{2} + a_{33}\frac{\alpha_3^2}{2} + a_{34}\frac{\alpha_4^2}{2} = \frac{1}{6},$$

$$a_{22}\beta_{32}\alpha_2 + a_{34}(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) = \frac{1}{6},$$

and then

$$a_{42}\frac{\alpha_2^2}{2} + a_{43}\frac{\alpha_3^2}{2} + a_{44}\frac{\alpha_4^2}{2} = \frac{1}{6},$$

$$a_{43}\beta_{32}\alpha_2 + a_{44}(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) = 0 .$$

Thus we obtain a system of nonlinear equations:

$$(a_{22}+a_{32}+a_{42})\alpha_2+(a_{33}+a_{43})\alpha_3+(a_{34}+a_{44})\alpha_4=\frac{1}{2},$$

$$(a_{22}+a_{32}+a_{42})\frac{\alpha_2^2}{2}+(a_{33}+a_{43})\frac{\alpha_3^2}{2}+(a_{34}+a_{44})\frac{\alpha_4^2}{2}=\frac{1}{6},$$

$$(a_{22}+a_{32}+a_{42})\frac{\alpha_2^3}{6}+(a_{33}+a_{43})\frac{\alpha_3^3}{6}+(a_{34}+a_{44})\frac{\alpha_4^3}{6}=\frac{1}{24},$$

$$(a_{33}+a_{43})\beta_{32}\alpha_2+(a_{34}+a_{44})(\beta_{42}\alpha_2+\beta_{43}\alpha_3)=\frac{1}{6},$$

$$(a_{33}+a_{43})\beta_{32}\alpha_2^2+(a_{34}+a_{44})(\beta_{42}\alpha_2^2+\beta_{43}\alpha_3^2)=\frac{1}{12},$$

$$(a_{33}+a_{43})\alpha_3\beta_{32}\alpha_2+(a_{34}+a_{44})(\beta_{42}\alpha_2+\beta_{43}\alpha_3)=\frac{3}{24},$$

$$(a_{34}+a_{44})\beta_{43}\beta_{32}\alpha_2=\frac{1}{24},$$

$$\sum_{i=1}^{4}a_{4i}=0,\ \sum_{i=2}^{4}a_{4i}\alpha_i=0,\ \sum_{i=2}^{4}a_{4i}\frac{\alpha_i^2}{2}=0,$$

$$a_{43}\beta_{32}\alpha_2+a_{44}(\beta_{42}\alpha_2+\beta_{43}\alpha_3)=0,$$

$$\sum_{i=1}^{4}a_{3i}=0,\ \sum_{i=2}^{4}a_{3i}\alpha_i=0,$$

$$(a_{22}+a_{32})\frac{\alpha_2^2}{2}+a_{33}\frac{\alpha_3^2}{2}+a_{34}\frac{\alpha_4^2}{2}=\frac{1}{6},$$

$$a_{33}\beta_{32}\alpha_2+a_{34}(\beta_{42}\alpha_2+\beta_{43}\alpha_3)=\frac{1}{6},$$

$$a_{21}+a_{22}=0,\ a_{22}\alpha_2=\frac{1}{2},\ \alpha_i=\sum_{j=1}^{i-1}\beta_{ij}. \quad (10)$$

Let's describe the scheme of solving the system (10):

1). Let $\alpha_2=\alpha_3$. From the first two equations we define

$$a_{22}+a_{32}+a_{42}+a_{33}+a_{43}=\frac{3\alpha_4-2}{6\alpha_3(\alpha_4-\alpha_3)},$$

$$a_{34}+a_{44}=\frac{2-3\alpha_3}{6\alpha_4(\alpha_4-\alpha_3)}.$$

Substituting these values in the third equation of the system, we obtain a relationship that combines the parameters α_3 and α_4:

$$4(\alpha_3+\alpha_4)-6\alpha_3\alpha_4-3=0,$$

$$\alpha_4=\frac{3-4\alpha_3}{2(2-3\alpha_3)}. \quad (11)$$

From the fourth and fifth equations it follows that $\alpha_3=\frac{1}{2}$, then, in accordance with (11) $\alpha_4=1$. From the equations $a_{22}\alpha_2=\frac{1}{2}$, $a_{21}+a_{22}=0$, we get

$$a_{21}=-\frac{1}{2\alpha_2}=-1,\ a_{22}=\frac{1}{2\alpha_2}=1.$$

Now from the equations

$$\sum_{i=1}^{4}a_{3i}=0,\ \sum_{i=1}^{4}a_{3i}\alpha_i=0,$$

$$(a_{22}+a_{32})\frac{\alpha_2^2}{2}+a_{33}\frac{\alpha_3^2}{2}+a_{34}\frac{\alpha_4^2}{2}=\frac{1}{6},$$

we will get

$$a_{31}=\frac{1}{6},\ a_{32}+a_{33}=-\frac{1}{3},\ a_{34}=\frac{1}{6}.$$

From the system (9) from the eighth to eleventh equations follows that $a_{4i}=0$, ($i=1,4$).

Using the seventh equation we find β_{32} and β_{43}:

$$\beta_{32}=\frac{1}{24(a_{34}+a_{44})\beta_{43}\alpha_3}=\frac{1}{2\beta_{43}}.$$

From the fourth and sixth equations we will represent

a_{33} and β_{42} also through β_{43}:

$$a_{33}=\frac{1}{6\beta_{32}}=\frac{\beta_{43}}{3},\ \beta_{42}=1-\beta_{43}$$

and then $a_{32}=-\frac{1+\beta_{43}}{3}$.

From the relations $\alpha_i=\sum_{j=1}^{i-1}\beta_{ij}$ ($i=\overline{2,4}$) we will define $\beta_{21}=\frac{1}{2}$, $\beta_{31}=\frac{1}{2}-\frac{1}{2\beta_{43}}$, $\beta_{41}=0$.

Consequently, the solution of the system (10) has the form

$$\alpha_2=\alpha_3=\frac{1}{2},\ \alpha_4=1,\ \beta_{21}=\frac{1}{2},$$

$$\beta_{31}=\frac{1}{2}-\frac{1}{2\beta_{43}},\ \beta_{32}=\frac{1}{2\beta_{43}},\ \beta_{41}=0,$$

$$\beta_{42}=1-\beta_{43},\ a_{11}=1,\ a_{21}=-1,\ a_{22}=1,$$

$$a_{31}=\frac{1}{6},\ a_{32}=-\frac{1+\beta_{43}}{3},\ a_{33}=\frac{\beta_{43}}{3},$$

$$a_{34}=\frac{1}{6},\ a_{1i}=0,\ (i=\overline{2,4}),$$

$$a_{4i}=0,\ (i=\overline{1,4}),\ a_{23}=a_{24}=0,$$

$$\beta_{ij}=0,\ i\le j. \quad (12)$$

We give a specific set of parameter values at $\beta_{43}=1$,

0	0	0	0	1	0	0	0
1/2	1/2	0	0	−1	1	0	0
1/2	0	1/2	0	1/6	−2/3	1/3	1/6
1	0	0	1	0	0	0	0

2). If $\alpha_2\ne\alpha_3$ we obtain:

$$a_{21}=-\frac{1}{2\alpha_2},\ a_{22}=\frac{1}{2\alpha_2},\ \beta_{21}=\alpha_2,$$

$$a_{32}=\frac{3(1+2\alpha_3\alpha_4)-4(\alpha_3+\alpha_4)-6(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)}{12\alpha_2(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)},\ a_{33}=\frac{3(1+2\alpha_2\alpha_4)-4(\alpha_2+\alpha_4)}{12\alpha_3(\alpha_2-\alpha_3)(\alpha_4-\alpha_3)},$$

$$\beta_{32}=\frac{4\alpha_4-3}{24\alpha_2(\alpha_4-\alpha_3)\cdot a_{33}},\ a_{34}=\frac{3(1+2\alpha_3\alpha_3)-4(\alpha_2+\alpha_3)}{12\alpha_4(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)},\ \beta_{31}=\alpha_3-\beta_{32},$$

$$a_{31}=-(a_{32}+a_{33}+a_{34}),\ \beta_{41}=\alpha_4-\beta_{42}-\beta_{43},\ \beta_{42}=\frac{(3-4\alpha_3)(\alpha_3-\alpha_2)+2(2\alpha_2-1)(\alpha_4-\alpha_3)}{24\alpha_2(\alpha_4-\alpha_3)(\alpha_3-\alpha_2)\cdot a_{34}},$$

$$a_{12}=a_{13}=a_{14}=0,\ \beta_{43}=\frac{1-2\alpha_2}{12\alpha_3(\alpha_3-\alpha_2)\cdot a_{34}},\ a_{23}=a_{24}=0,\ a_{4i}=0,\ i=\overline{1,4}, \quad (13)$$

where $\alpha_4=1$, α_2 and α_3 are arbitrary numbers satisfying the relation

$$\alpha_2\cdot\alpha_3(\alpha_2-\alpha_3)(1-\alpha_2)(1-\alpha_3)\ne 0.$$

II. Construction of bilateral methods of the third order of accuracy

To build bilateral methods of Runge-Kutta [15–17] are looking approximate solution of problem (1) as a continued fraction (2), (3). Parameters a_{mi}, α_i, β_{ij} ($m,i,j=1,2,3$) are defined such that

$$y(x_{n+1})-y_{n+1}^{[k,l]}=$$
$$=\omega h^{(p)}KF^{[k,l]}(f)+O(h^{(k+l)}), \quad (14)$$

where $y(x_{n+1})$ and $y_{n+1}^{[k,l]}$ – respectively, exact and approximate solution of the problem (1), h – integration step, $F(f)$ – a differential operator, calculated at the point (x_n,y_n), K – constant, p – order accuracy, ω – parameter of bilateralism.

If in relations (9) we put

$$a_{42}\frac{\alpha_2^2}{2}+a_{43}\frac{\alpha_3^2}{2}+a_{44}\frac{\alpha_4^2}{2}=-\frac{\omega_1}{2},$$

$$a_{42}\beta_{32}\alpha_2+a_{44}(\beta_{42}\alpha_2+\beta_{43}\alpha_3)=-\frac{\omega_2}{2},$$

then

$$y(x_{n+1})-y_{n+1}^{[4,0]}=$$
$$=\omega h^4K\cdot F^{[4,0]}(f)+O(h^5). \quad (15)$$

Having solved under these conditions a system of nonlinear equations (10), for $\alpha_2=\alpha_3$, we obtain

$$\alpha_2=\alpha_3=\frac{1}{2},\ \beta_{21}=\frac{1}{2},\ \beta_{31}=\frac{1}{2}-\frac{1}{2\beta_{43}},$$

$\beta_{32} = \dfrac{1}{2\beta_{43}}$, $\alpha_4 = 1$, $\beta_{41} = 0$, $\beta_{42} = 1 - \beta_{43}$,

$a_{11} = 1$, $a_{12} = a_{13} = a_{14} = 0$, $a_{21} = -1$,

$a_{22} = 1$, $a_{23} = a_{24} = 0$, $a_{31} = \dfrac{1}{6} + 2\omega_1$,

$a_{32} = 4\omega_1(\beta_{43} - 1) - 2\omega_2\beta_{43} - \dfrac{1}{3}(1 + \beta_{43})$,

$a_{33} = \dfrac{1}{3}\beta_{43} + 2\beta_{43}(\omega_2 - 2\omega_1)$, $a_{34} = \dfrac{1}{6} + 2\omega_1$,

$a_{41} = -2\omega_1$, $a_{42} = 4\omega_1(1 - \beta_{43}) + 2\omega_2\beta_{43}$,

$a_{43} = 2\beta_{43}(2\omega_1 - \omega_2)$, $a_{44} = -2\omega_1$, (16)

where β_{43} – nonzero parameter.

Putting $\omega_1 = \omega_2 = \omega$, we get the following set of parameters

0	0	0	0	1	0	0	0
1/2	1/2	0	0	-1	1	0	0
1/2	0	1/2	0	$1/6 + 2\omega$	$-2\omega - 2/3$	$1/3 - 2\omega$	$1/6 + 2\omega$
1	0	0	1	-2ω	2ω	2ω	-2ω

The estimate of the local error has the form

$$y_{n+1} - y_{n+1}^{[4,0]} = $$
$$= \omega h^4\left(D^2 f + f_y \cdot Df\right) + O\left(h^5\right) \cong$$
$$\cong h\sum_{i=1}^{4} \tilde{a}_{4i} k_i . \qquad (17)$$

With the help of the parameters ω and h is reached bilateral approximations and required accuracy on the whole interval of integration.

The proposed two-sided formulas at each step of integration use less number of calculations of the right-hand side of the differential equation than the known bilateral methods of Runge-Kutta type.

These formulas, using only four calculations of the right-hand side of the differential equation, allow at each step to obtain a method of fourth-order accuracy method and two bilateral formulas of the third order of accuracy.

CONCLUSIONS

Nonlinear formulas of the fourth order of accuracy for the solution of the Cauchy problem for ordinary differential equations based on continuous fractions have been constructed.

Two-sided methods of Runge-Kutta type of the third order of accuracy have been constructed. These formulas make it possible to obtain at each point the upper and lower approximations to the exact solution and determine the value of the leading (main) term of the local error without additional calculations of the right-hand side of the differential equation.

The proposed two-sided formulas at each step of integration use less number of calculations of the right-hand side of the differential equation than the known bilateral methods of Runge-Kutta type.

These formulas, using only four appeals to the right-hand side of the differential equation, allow us to construct a method of the fourth order of accuracy and two-sided method of the third order of accuracy**.**

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Пелех Ярослав Миколайович, Пахолок Богдан Богданович, Кунинець Андрій Володимирович, Сохан Петро Львович. **ЧИСЛОВІ МЕТОДИ РОЗВ'ЯЗУВАННЯ ПОЧАТКОВОЇ ЗАДАЧІ ДЛЯ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ОЦІНКОЮ ГОЛОВНОГО ЧЛЕНА ЛОКАЛЬНОЇ ПОХИБКИ**

Одним з сучасних наукових методів дослідження явищ та процесів є математичне моделювання. Математичне моделювання є ефективним методом вивчення економічних процесів, у багатьох важливих випадках дозволяє замінити реальний процес, а також дає можливість одержати як якісну, так і кількісну картину досліджуваних процесів. Оскільки точні розв'язки таких моделей можна отримати лише у дуже часткових випадках, то необхідно використовувати числові методи. При дослідженні економічних проблем виникає необхідність не тільки знайти наближені розв'язки, але і оцінити їх локальні та глобальні похибки. В обчислювальній математиці широкого застосування набули неперервні дроби. Вони дають можливість отримувати монотонні і двосторонні наближення, мають слабку чутливість до похибок заокруглень, а також вірно відображають основні властивості досліджуваних задач. Об'єктом дослідження є початкова задача для звичайних диференціальних рівнянь. Метою дослідження є розробка методів та алгоритмів чисельного розв'язування задачі Коші для звичайних диференціальних рівнянь. Виведені формули типу Рунге-Кутти четвертого порядку точності розв'язування початкової задачі для звичайних диференціальних рівнянь на основі неперервних дробів. Запропоновано нові двосторонні чисельні методи третього порядку точності, які в кожній вузловій точці дозволяють отримати не тільки верхні та нижні наближення до точного розв'язку, але і оцінювати значення головного члена локальної похибки без додаткових звертань до правої частини диференціального рівняння. Двосторонні формули використовують на кожному кроці інтегрування менше звертань до правої частини диференціального рівняння порівняно з відомими двосторонніми методами типу Рунге-Кутта. Запропоновані формули, використовуючи лише чотири звертання до правої частини диференціального рівняння, дозволяють отримувати не тільки односторонній метод четвертого порядку точності, але і двосторонні формули третього порядку точності.

Ключові слова: *початкова задача, неперервні дроби, методи типу Рунге-Кутта, двостороння апроксимація, нелінійні числові методи.*

Пелех Ярослав Николаевич, Пахолок Богдан Богданович, Кунынец Андрей Владимирович, Сохан Петр Львович. ЧИСЛЕННЫЕ МЕТОДЫ РЕШЕНИЯ НАЧАЛЬНОЙ ЗАДАЧИ ДЛЯ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОЦЕНКОЙ ГЛАВНОГО ЧЛЕНА ЛОКАЛЬНОЙ ПОГРЕШНОСТИ

Одним из современных научных методов исследования явлений и процессов является математическое моделирование. Математическое моделирование – эффективный метод изучения экономических процессов, во многих важных случаях позволяет заменить реальный процесс, а также дает возможность получить как качественную, так и количественную картину исследуемых процессов. Поскольку точные решения таких моделей можно получить только в очень частных случаях, то необходимо использовать численные методы. При исследовании экономических проблем возникает необходимость не только найти приближенные решения, но и оценить их локальные и глобальные погрешности. В вычислительной математике широкое применение получили непрерывные дроби. Они дают возможность получать монотонные и двусторонние приближения, имеют слабую чувствительность к погрешностям округления, а также верно отражают основные свойства изучаемых задач. Объектом исследования является начальная задача для обыкновенных дифференциальных уравнений. Целью исследования является разработка методов и алгоритмов численного решения задачи Коши для обыкновенных дифференциальных уравнений. Выведены формулы типа Рунге-Кутты четвертого порядка точности для решения начальной задачи для обыкновенных дифференциальных уравнений на основе непрерывных дробей. Предложены новые двусторонние численные методы третьего порядка точности, которые в каждой узловой точке позволяют получать не только верхние и нижние приближения к точному решению но и оценивать значение главного члена *локальной погрешности без дополнительных вычислений правой части дифференциального уравнения. Двусторонние формулы используют на каждом шагу интегрирования меньше обращений к правой части дифференциального уравнения по сравнению с известными двусторонними методами типа Рунге-Кутты. Предложенные формулы, используя только четыре обращения к правой части дифференциального уравнения, позволяют получать не только односторонний метод четвертого порядка точности, но и двусторонние формулы третьего порядка точности.*

***Ключевые слова**: начальная задача, непрерывные дроби, методы типа Рунге-Кутты, двусторонняя аппроксимация, нелинейные численные методы.*

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