THE RADIAL MOTIONS OF CHARGED PARTICLES IN THE FIELD OF THE CHARGED OBJECT IN GENERAL RELATIVITY AND THEIR CLASSIFICATION

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ABSTRACT. The radial motions of the charged test particles in the field of a spherically symmetric charged object in general relativity are considered and their classification is built. The conditions of equilibrium for these particles are studied and equilibrium stability conditions are received. It is shown, that stable states are only possible for the bound states of the weakly charged particle in the field of the abnormally charged central source.

Key words: classification of motions, effective potential; stability conditions.

1. Introduction

The consideration of motion of the charged particles is an important part of the general research of the behavior for the charged configurations in general relativity. The motion of the charged test particles in Reissner-Nordström field was considered by Cohen (1979), the radial motions was studied by Finley (1974). One of the main reasons of the researches is the receiving of the conditions under which the falling on the center takes no place, and the finding of the stability conditions for the charged particles in the field of the charged object. Thus, the paper Bonnor (1993) was dedicated the study of the equilibrium stability conditions for the charged particles. Note that in his paper this problem was considered without taking into account the conservation law and has been solved incompletely.

In this paper we carry out the classification of the radial motions for the charged particles in the spherically symmetric gravitational and electrical fields of the central source by using the energy conservation law. The research of this problem leads to the equilibrium conditions of stable states of the charged particles in the Reissner-Nordström field.

2. The equation of motion for the charged particle

The gravitational field of a spherically symmetric source with mass M and charge Q is described by Reissner-Notdström metric

$$ds^2 = Fc^2dT^2 - F^{-1}dR^2 - R^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad (1)$$

where

$$F = 1 - \frac{2\gamma M}{c^2 R} + \frac{\gamma Q^2}{c^4 R^2}\,, \quad (2)$$

γ and c — are the gravitation constant and velocity of light respectively. In dependence on the relation between mass and charge it is possible to separate following types of the charged relativistic objects: the charged black hole $(\sqrt{\gamma}M > |Q|)$, the extremely charged black hole $(\sqrt{\gamma}M = |Q|)$, the abnormally charged object $(\sqrt{\gamma}M < |Q|)$.

Lagrangian of a test particle with mass m and charge q is given by:

$$L(R,\dot{R}) = -mc\sqrt{Fc^2 - F^{-1}\dot{R}^2} - qQ/R\,, \quad (3)$$

where a dot denotes differentiation with respect to T. It is easy to see, that the total energy of the charged particle is conserved and equals to

$$E = \frac{qQ}{R} + \frac{mc^2F}{\sqrt{F - F^{-1}\dot{R}^2/c^2}}\,. \quad (4)$$

From here we have the equation of radial motion:

$$\left(mc^2\frac{dR}{ds}\right)^2 = \left(E - \frac{qQ}{R}\right)^2 - m^2c^4\left(1 - \frac{2\gamma M}{c^2R} + \frac{\gamma Q^2}{c^4R^2}\right)\,. \quad (5)$$

The similar equation for Reissner-Nordström-de Sitter metric has been received by Gonçalves (2001) by using Killing vector. Further, for the particle acceleration we find:

$$\frac{d^2R}{ds^2} = \frac{1}{m^2c^4}\left[\left(EqQ - m^2c^2\gamma M\right)\frac{1}{R^2} + \right.$$
$$\left. + \left(\gamma m^2 - q^2\right)\frac{Q^2}{R^3}\right]\,. \quad (6)$$

3. Classification of the radial motions

The standard method of the qualitative analysis of the particle trajectories is based on the study of effective "velocity potential" behaviour $\left(mc^2dR/ds\right)^2 \equiv -U$ (see, for example, Wilkins (1972), Cohen and Gautreau (1979), Dymnikova (1986), Gonçalves (2001)). Let us rewrite the potential (5) in the form

$$U(Q,M,R) = m^2c^4 - E^2 - \left(\gamma M m^2 c^2 - EqQ\right)\frac{2}{R} + \\ + \left(\gamma m^2 - q^2\right)\frac{Q^2}{R^2}\,.$$

The admissible motions are defined by an inequality $U(Q,M,R) \le 0$ where equality specifies the turning points. The presence of parameter of classification M in the potential U is not suitable for particles dynamics analysis. Therefore, let us introduce the "mass potential" U_m which is the solution of the equation $U = 0$ with respect to M

$$U_m(Q,R) = \frac{1}{2\gamma m^2c^2}\left[(m^2c^4 - E^2)R + 2EqQ + \\ + \left(\gamma m^2 - q^2\right)\frac{Q^2}{R}\right]. \tag{7}$$

In this case the regions of admissible motions are specified by an inequality $U_m(Q,R) \le M$. Besides the potential U_m let us introduce one more function U_g which can be find from the equation $F = 0$ defining horizons

$$M = \frac{1}{2}\left(\frac{Rc^2}{\gamma} + \frac{Q^2}{Rc^2}\right) \equiv U_g(Q,R)\,. \tag{8}$$

The function U_g determines a value of the mass of black hole having charge Q and radius of horizon R.

The behavior of function $U_m(Q,R)$ depends on the relation between particle parameters defining its asymptotics. As a result we obtain the following cases:

1. when $R \to 0$ we have
- **1.a** if $\gamma m^2 > q^2$, then $U_m(Q,R) \to +\infty$, for the weakly charged particle;
- **1.b** if $\gamma m^2 = q^2$ then $U_m(Q,R) \to EqQ/\gamma m^2c^2$, for the extremely charged particle;
- **1.c** if $\gamma m^2 < q^2$ then $U_m(Q,R) \to -\infty$, for the abnormally charged particle.

2. when $R \to \infty$ we have
- **2.a** if $E^2 < m^2c^4$ then $U_m(Q,R) \to +\infty$, for the bound states of the particle;
- **2.b** if $E^2 = m^2c^4$ then $U_m(Q,R) \to EqQ/\gamma m^2c^2$, for the particle with a critical mass;
- **2.c** if $E^2 > m^2c^4$ then $U_m(Q,R) \to -\infty$, for the unbound states of the particle.

For certainty we suppose $Q > 0$ and $M > 0$. Thus, the introduced conditions define 9 types of behavior for potential U_m. Taking into account the sign of a charge q and type of the central object we obtain 54 types of behavior.

To plot and analysis functions U_m and U_g suitable to use the dimensionless variables

$$\widetilde{\mathcal{U}}_m(\varepsilon,\beta,x) = \frac{1}{2}\left[(1-\varepsilon^2)x + 2\varepsilon\beta + \left(1-\beta^2\right)\frac{1}{x}\right], \\ \widetilde{\mathcal{U}}_m(\varepsilon,\beta,x) \le \mathcal{M}\,, \tag{9}$$

$$\widetilde{\mathcal{U}}_g(x) = \frac{1}{2}\left(x + \frac{1}{x}\right), \tag{10}$$

where $\widetilde{\mathcal{U}}_m = \sqrt{\gamma}U_m/Q$, $\widetilde{\mathcal{U}}_g = \sqrt{\gamma}U_g/Q$, $x = Rc^2/\sqrt{\gamma}Q$, $\varepsilon = E/mc^2$, $\beta = q/\sqrt{\gamma}m$, $\mathcal{M} = \sqrt{\gamma}M/Q$.

Comparing $\widetilde{\mathcal{U}}_m$ and $\widetilde{\mathcal{U}}_g$, we come to the relation

$$\widetilde{\mathcal{U}}_g(x) = \widetilde{\mathcal{U}}_m(\varepsilon,\beta,x) + \left(\varepsilon\sqrt{x} - \frac{\beta}{\sqrt{x}}\right)^2 \tag{11}$$

from which it follows that $\widetilde{\mathcal{U}}_g(x) \ge \widetilde{\mathcal{U}}_m(\varepsilon,\beta,x)$. From this it follows that the curve $\widetilde{\mathcal{U}}_g(x)$ lies always below then the "mass potential" curve. The reason is that the turning radiuses do not exist in T-region where the radial coordinate becomes time-like. The equality $\widetilde{\mathcal{U}}_g(x) = \widetilde{\mathcal{U}}_m(\varepsilon,\beta,x)$ defines a point of a contact for these curves $x_{tang} = \beta/\varepsilon$ whence it follows that $R_{tang} = qQ/E$. Let us note, that a particle with the energy $E = qQ/R_+$, where $R_+ = \left(\gamma M + \sqrt{\gamma(\gamma M^2 - Q^2)}\right)$ — is the exterior horizon, has a turning point on the events horizon exactly (Chandrasekhar, 1986).

The plots for potentials $\widetilde{\mathcal{U}}_m(\varepsilon,\beta,x)$ and $\widetilde{\mathcal{U}}_g(x)$ for all cases of classification are represented at fig. 1-9.

As an example let us consider fig.1 which corresponds to the bound states of the weakly charged particles. The curves a and b correspond to the mass potentials $\widetilde{\mathcal{U}}+_m$ and $\widetilde{\mathcal{U}}-_m$ for the cases $q > 0$ and $q < 0$ respectively. The curve $\widetilde{\mathcal{U}}_g(x)$ describes a line of horizon. The segments of horizontal lines define the trajectories of the motions of particles. The intersections of curves $\widetilde{\mathcal{U}}+_m$ and $\widetilde{\mathcal{U}}-_m$ with a line $\widetilde{\mathcal{U}}_m(\varepsilon,\beta,x) = \mathcal{M}$ give the turning points. The line 1 describes the motion of a particle in the field of the weakly charged black hole, the line 2 — is for the case in the field of the extremely charged black hole, and the lines 3 and 4 — are for the motion of the particles with charges $q > 0$ (a curve a) and $q < 0$ (a curve b) in the field of the abnormally charged object. The character of motion of the particles for other cases can be consider in a similar way.

3. The stability conditions

The stationary positions R_{extr} of a particle are defined by conditions $dR/ds = 0$ and $d^2R/ds^2 = 0$. Moreover if $d^2R/ds^2 > 0$ when $R < R_{extr}$, and $d^2R/ds^2 < 0$ when $R > R_{extr}$ then the position R_{extr} will be steady. From definitions for the "velocity" and "mass" potentials it follows

$$U_m(Q,R) = \frac{R}{2\gamma m^2c^2}U(Q,M,R) + M\,. \tag{12}$$

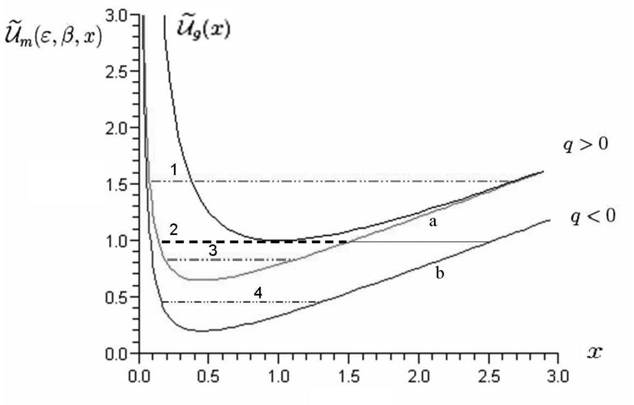

Figure 1: The bound states ($E^2 < m^2c^4$) of the weakly charged particle ($\gamma m^2 > q^2$).

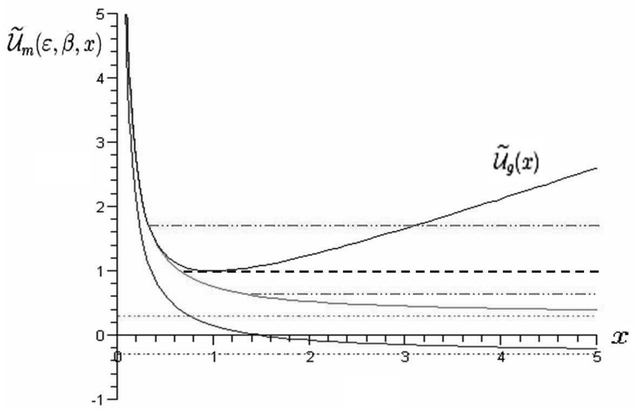

Figure 2: The weakly charged particle ($\gamma m^2 > q^2$) with a critical mass ($E^2 = m^2c^4$).

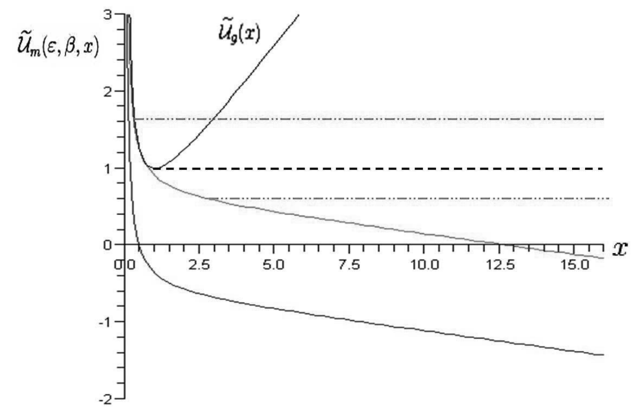

Figure 3: The unbound states ($E^2 > m^2c^4$) of the weakly charged particle ($\gamma m^2 > q^2$).

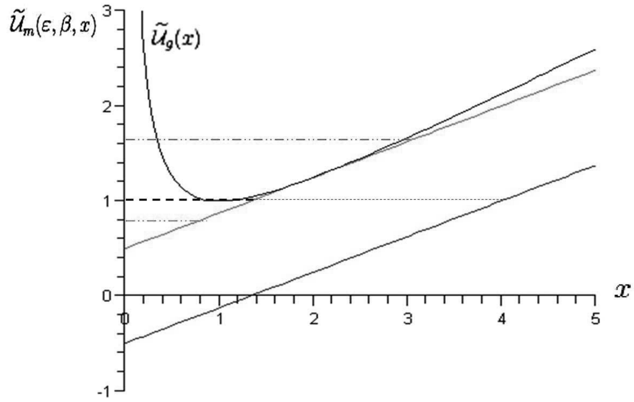

Figure 4: The bound states ($E^2 < m^2c^4$) of the extremely charged particle ($\gamma m^2 = q^2$).

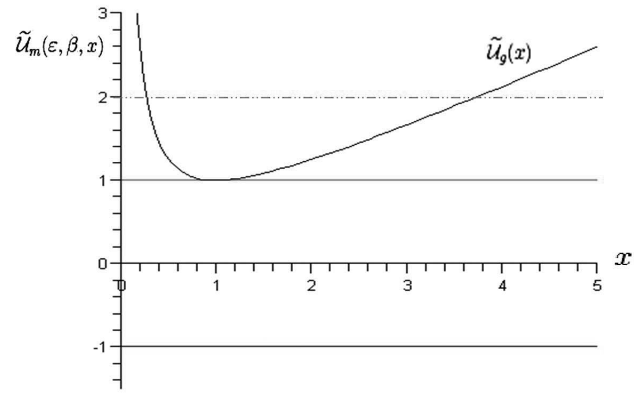

Figure 5: The extremely charged particle ($\gamma m^2 = q^2$) with a critical mass ($E^2 = m^2c^4$).

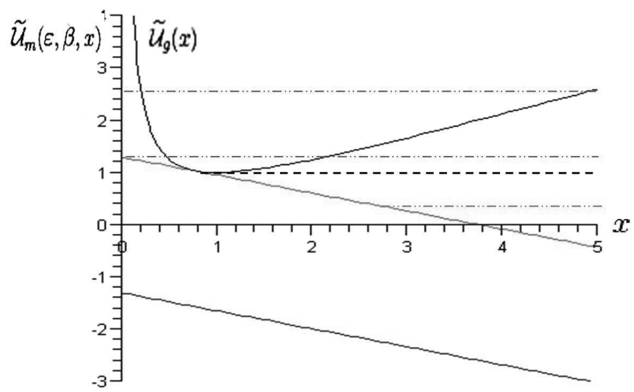

Figure 6: The unbound states ($E^2 > m^2c^4$) of the extremely charged particle ($\gamma m^2 = q^2$).

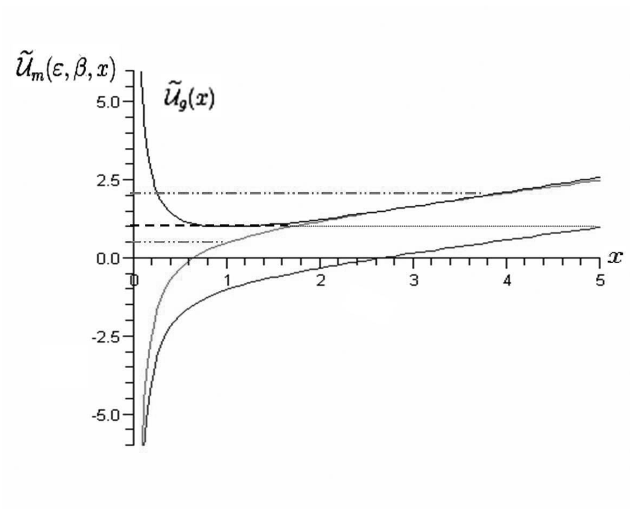

Figure 7: The bound states ($E^2 < m^2c^4$) of the the abnormally charged particle ($\gamma m^2 < q^2$).

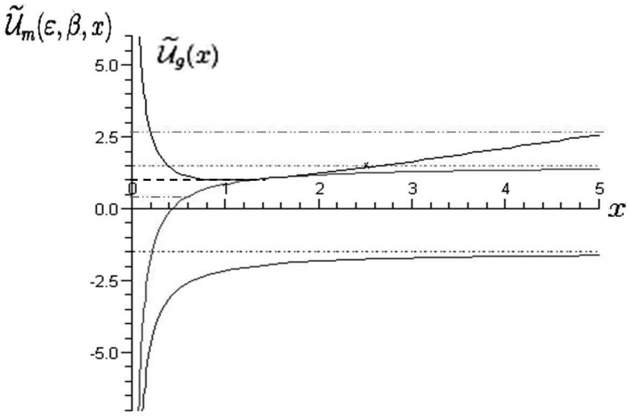

Figure 8: The abnormally charged particle ($\gamma m^2 < q^2$) with a critical mass ($E^2 = m^2c^4$).

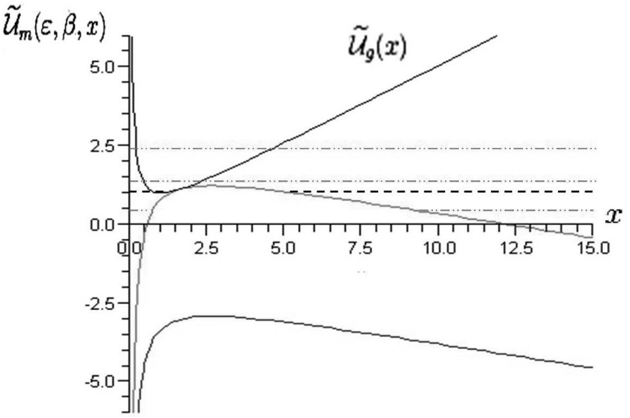

Figure 9: The unbound states ($E^2 > m^2c^4$) of the abnormaly charged particle ($\gamma m^2 < q^2$).

In a stable point R_{extr}, where $U(R_{extr}) = 0$ or $U_m(R_{extr}) = M$, the conditions $(\partial U/\partial R)|_{R_{extr}} = 0$ and $(\partial U_m/\partial R)|_{R_{extr}} = 0$ coincide and give

$$R_{extr} = Q\sqrt{\frac{\gamma m^2 - q^2}{m^2c^4 - E^2}}\,. \tag{13}$$

Excluding variable R from $dR/ds = 0$ and $d^2R/ds^2 = 0$ we receive the relation

$$(m^2c^4 - E^2)(\gamma m^2 - q^2)Q^2 = (m^2c^2\gamma M - EqQ)^2 \tag{14}$$

from which we find two systems of inequalities

$$\begin{aligned} |q| < m\sqrt{\gamma}, \quad |E| < mc^2\,, \\ |q| > m\sqrt{\gamma}, \quad |E| > mc^2\,. \end{aligned} \tag{15}$$

Rewriting (16) in the form

$$(Q^2 - \gamma M^2)(\gamma m^2 - q^2) = \gamma\left(\frac{EQ}{c^2} - qM\right)^2 \tag{16}$$

in a similar way we obtain

$$\begin{aligned} |q| < m\sqrt{\gamma}, \quad |Q| > M\sqrt{\gamma}\,, \\ |q| > m\sqrt{\gamma}, \quad |Q| < M\sqrt{\gamma}\,. \end{aligned} \tag{17}$$

From classification of motions and relations (15) and (17) we find the following conditions: $|E| < mc^2$, $|q| < m\sqrt{\gamma}$, $|Q| > M\sqrt{\gamma}$ is for stable equilibrium of the bound states of weakly charged particle in the field of the abnormally charged object; $|E| > mc^2$, $|q| > m\sqrt{\gamma}$, $|Q| < M\sqrt{\gamma}$ is for instable equilibrium of the unbound states of the abnormally charged particle in the field of the charged black hole. The stability conditions (17) were received by Bonnor (1993) without use of the conservation laws. However he did not indicate, which of the conditions of equilibrium is stably and did not receive stable point of a particle. Note that in the work of Markov and Frolov (1972) it is stated that in case of the abnormally charged object $|Q| > M\sqrt{\gamma}$ the stable system is impossible.

Thus, the stable stationary positions are only possible for the bound states of weakly charged particle in the field of the abnormally charged central object.

References

Wilkins D.: 1972, *Phys. Rev. D,* **5**, 814.
Dymnikova I.G.: 1986, *Usp. Phys. Nauk,* **148**, 393.
Bonnor W.B.: 1993, *Class. Quantum Grav.,* **10**, 2077.
Chandrasekhar S.: 1983, *The mathematical theory of black holes,* Oxford.
Cohen J.M., Gautreau R.: 1979, *Phys. Rev. D,* **19**, 2273.
Gonçalves S.M.: 2001, *Phys. Rev. D,* **63**, 124017.
Markov M.A., Frolov V.P.: 1972, *J. Theor. and Math. Phys.,* **13**, 41.
Finley J.D.: 1974, *J. of Mat. Phys.,* **15**, 1698.