MOTION TYPES OF NEUTRAL TEST PARTICLES IN THE FIELD OF CHARGED OBJECT IN GENERAL RELATIVITY AND THEIR CLASSIFICATION

V.D. Gladush[1], D.A. Kulikov [2]

Theoretical Physics Department, Dniepropetrovsk National University
72 Gagarin av., Dniepropetrovsk 49010, Ukraine,
[1]*vgladush@gmail.com*, [2]*kulikov_d_a@yahoo.com*,

ABSTRACT. In the previous work of one of the authors the radial motions of charged test particles in the field of a spherically symmetric charged object in general relativity were considered and their classification was built. The present paper generalizes this approach to study the motion of particles with non-zero orbital momentum. We limit the consideration to neutral particles and stress on the peculiarities that emerge in the case of the Reissner-Nordström field with super-extremal charge.

Key words: effective potential, scale invariance, classification of motions, Reissner–Nordström metric.

1. Introduction

The analysis of test particles motion is a classical method for studying the space-time structure near a gravitating massive object. The electric charge of the gravitational field source, if present, affects substantially the space-time geometry. Search for particles motion peculiarities that emerge in this case receives considerable attention (see, for example, Chandrasekhar S. (1983) and other authors in the references) as it constitutes an important part of the investigation of relativistic configurations. In the present work we study the radial component of motion for neutral test particles with mass m and orbital moment L in the field of a central source with mass M and charge Q. This field is described by the Reissner-Nordström metric

$$ds^2 = Fc^2dt^2 - F^{-1}dR^2 - R^2d\sigma^2, \qquad (1)$$

where

$$F = 1 - \frac{2\kappa M}{c^2R} + \frac{\kappa Q^2}{c^4R^2}, \quad d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2. \quad (2)$$

Note that we consider all the possible types of spherically symmetric charged objects, namely: $Q^2 < \kappa M^2$ - black hole (BH), two horizons, $Q^2 = \kappa M^2$ - extreme BH, twofold horizon, $Q^2 > \kappa M^2$ - super-extremal (abnormally) charged object or naked singularity (NS).

In the case of static and spherically symmetric field we have the conservation of particle energy

$$E = c\sqrt{F}\sqrt{m^2c^2 + F\,p_R^2 + \frac{1}{R^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right)}, \quad (3)$$

and the square of the total orbital angular momentum

$$L^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} = const\,, \qquad (4)$$

where $P^\mu = mcdx^\mu/ds$ is the particle four-momentum. These formulae yield the expression for the radial component of velocity

$$\left(mc^2\frac{dR}{ds}\right)^2 = E^2 - F\left(m^2c^4 + \frac{c^2L^2}{R^2}\right). \qquad (5)$$

2. Scale invariance, new parameters and effective potential

The problem of how to investigate the motion can be simplified by using the potential method and scale invariance of the system. The dynamical system under consideration has five parameters: $\{M, Q, m, E, L\}$. However, the number of essential ones is less than five because the system admits the two-parametric group G^2 of scaling transformations:

$$\tilde{E} = E/\alpha,\ \tilde{m} = m/\alpha,\ \tilde{L} = L/\gamma,\ |\tilde{Q}| = \alpha|Q|/\gamma,$$
$$\tilde{M} = \alpha M/\gamma, \quad \tilde{R} = \alpha R/\gamma, \quad \tilde{s} = \alpha s/\gamma\,. \qquad (6)$$

As a new set of parameters and variables we choose the independent invariants of the G^2 transformations:

$$\epsilon = \frac{E}{mc^2}, \quad \eta = \frac{|Q|m\sqrt{\kappa}}{c|L|}, \quad \mu = \frac{Mm\kappa}{c|L|}\,,$$
$$z = \frac{Rmc}{|L|}, \quad \tau = s\frac{mc}{|L|}, \quad \varpi = \omega\frac{|L|}{mc^2}\,, \qquad (7)$$

where $\{\epsilon, \eta, \mu, z, \tau\}$ are the reduced energy, charge, mass, radius and proper time, respectively.

In their terms equation (5) becomes

$$\left(\frac{dz}{d\tau}\right)^2 = \epsilon^2 - 1 + \frac{2\mu}{z} - \frac{1+\eta^2}{z^2} + \frac{2\mu}{z^3} - \frac{\eta^2}{z^4} = -W_V \quad (8)$$

where W_V is the velocity potential. Allowed regions and turning points can be found from the conditions $W_V \leq 0$ and $W_V = 0$, whereas the circular orbits result from $W_V = 0$, $\partial W_V/\partial z = 0$.

Introduction of a suitable potential will further simplify the situation. In our case the energy potential (see Gladush V., Galadgyi M. (2011)) is the most convenient one, which is hereinafter called the effective potential (Pugliese B., Quevedo H., Ruffini R. (2011))

$$W_\epsilon^2 = 1 - 2\mu z^{-1} + (1+\eta^2)z^{-2} - 2\mu z^{-3} + \eta^2 z^{-4}. \quad (9)$$

Then the equation of motion becomes

$$(dz/d\tau)^2 + W_\epsilon^2 = \epsilon^2. \quad (10)$$

Now the allowed and forbidden regions and turning points are determined by the conditions $W_\epsilon^2 \leq \epsilon^2$ and $W_\epsilon^2 = \epsilon^2$. As a consequence, the classification of motions relies on distinguishing different types of the potential W_ϵ^2. Because of the scale invariance, it involves solely the reduced mass μ and charge η of the central body.

The circular orbits are determined by conditions $W_\epsilon^2 = \epsilon^2$, $\partial W_\epsilon^2/\partial z = 0$ that lead to algebraic equations

$$(1-\epsilon^2)z^4 - 2\mu z^3 + (1+\eta^2)z^2 - 2\mu z + \eta^2 = 0\,, \quad (11)$$

$$\mu z^3 - \left(1+\eta^2\right)z^2 + 3\mu z - 2\eta^2 = 0\,. \quad (12)$$

For a stable circular orbit, it is necessary to fulfill the condition of minimum $\partial^2 W_\epsilon^2/\partial z^2 > 0$. If there is an inflection point, then we have $\partial^2 W_\epsilon^2/\partial z^2 = 0$ that corresponds to the last circular orbit.

3. Circular orbit radii and their classification

Let us bring the cubic equation for circular orbits (12) to the standard form

$$z^3 + az^2 + bz + c = 0\,, \quad (13)$$

$$a = -(1+\eta^2)/\mu, \quad b = 3, \quad c = -2\eta^2/\mu\,. \quad (14)$$

Then the types of its solutions can be established from the sign of the discriminant

$$D = (-a^2/3 + b)^3/27 + (2a^3/27 - ab/3 + c)^2/4\,. \quad (15)$$

Equation (13) may have (i) one real root and a pair of complex conjugate roots ($D > 0$); (ii) three real roots from which at least two are equal ($D = 0$); (iii) three different real roots ($D < 0$). For identifying these cases, we rewrite the discriminant as

$$D = (1 - \mu_+^2\mu^{-2})(1 - \mu_-^2\mu^{-2})\,, \quad (16)$$

$$72\mu_\pm^2 = 3\left(1 + 14\eta^2 + \eta^4\right) \pm \pm\sqrt{3}\left(1-5\eta^2\right)\sqrt{(3+\eta^2)\left(1-5\eta^2\right)} \quad (17)$$

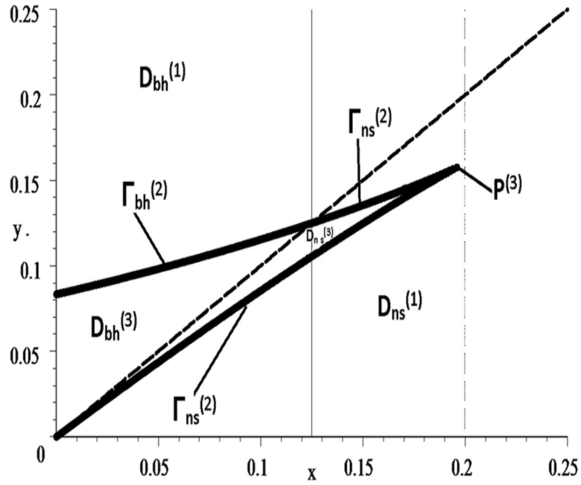

Figure 1: The curve $D = 0$ in the plane of parameters $(x = \eta^2, y = \mu^2)$ and the different regions of the roots.

In the (μ, η) plane the equality $D = 0$ is the boundary of the curvilinear angle $\Gamma^{(2)} = \Gamma_{bh}^{(2)} \bigcup \Sigma_{ns}^{(2)}$, which is depicted in Fig.1 by a heavy line and corresponds to the case of multiple roots ($z_1 = z_2, z_3$ or $z_1, z_2 = z_3$). The point $p^{(3)} \in \Gamma^{(2)}$, is associated with the triple root ($z_1 = z_2 = z_3$). The region $D^{(3)} = D_{bh}^{(3)} \bigcup D_{ns}^{(3)}$ that lies inside the angle $\Gamma^{(2)}$ refers to the case of $D < 0$, i.e., to three different real roots z_1, z_2, z_3. For points that lie outside the angle $\Gamma^{(2)}$, in the region $D^{(1)} = D_{bh}^{(1)} \bigcup D_{ns}^{(1)}$, we have $D > 0$. Here one root is real z_3 (a circular orbit) whereas two others are complex $z_1 = m_1 + im_2$, $z_2 = m_1 - im_2$.

Note that the dotted line $\eta^2 = \mu^2$ corresponds to the extreme BH. The regions above and below the line $\eta^2 = \mu^2$ relate to BH ($\eta^2 > \mu^2$) and NS ($\eta^2 < \mu^2$) respectively.

4. Parameterization in terms of circular orbits radii and classification of effective potentials

The roots of equation (13) and its coefficients (14) are interrelated as follows

$$z_1 + z_2 + z_3 = (1+\eta^2)/\mu\,, \quad z_1 z_2 z_3 = 2\eta^2/\mu\,, \quad z_1 z_2 + z_1 z_3 + z_2 z_3 = 3\,. \quad (18)$$

Taking into account these formulae, we obtain the parametrization for the effective potential (9) in terms of the roots of equation (13), i.e. in terms of the circular orbits radii,

$$W_\epsilon^2 = 1 - \frac{\mu}{z}\left(2 - \frac{z_1+z_2+z_3}{z} + \frac{2}{z^2} - \frac{z_1 z_2 z_3}{2z^3}\right)\,, \quad (19)$$

$$\mu = (z_3 + z_2 + z_1 - z_1 z_2 z_3/2)^{-1}, \eta^2 = \mu z_1 z_2 z_3/2\,. \quad (20)$$

This representation permits us to reduce the problem of classifying the trajectories to the task of identifying the type of the potential.

In the case of multiple roots, when $z_2 = z_3$, we get

$$W_\epsilon^2 = 1 - \frac{\mu}{z}\left(2 - \frac{z_1 + 2z_2}{z} + \frac{2}{z^2} - \frac{z_1 z_2^2}{2z^3}\right). \quad (21)$$

while $\mu = 2(2z_1 + 4z_2 - z_1 z_2^2)^{-1}$, $(2z_1 + z_2)\, z_2 = 3$. At the point $z = z_1$ the potential has an inflection. In the case of the triple root $z_1 = z_2 = z_3$ we have

$$W_\epsilon^2 = 1 - 0.4z^{-1}(2 - 3z^{-1} + 2z^{-2} - 0.5z^{-3}), \quad (22)$$

and $\mu = 0.4$, $\eta = 1/\sqrt{5}$, $z_1 = 1$.

In Fig. 1 the regions of parameters $\{\mu, \eta\}$ for the different types of the potential are shown. Choosing an arbitrary point inside each of these regions and calculating the roots of equation (13), we compose the potential according to (19). Some plots of $W(z)$ are given in Figs. 2 and 3. Here the allowed regions for particles with given energy ε are the horizontal line segments $W_\epsilon^2 = \varepsilon^2 = const$ bounded by the intersection points with the curve $W^2 = W^2(z)$ (turning points).

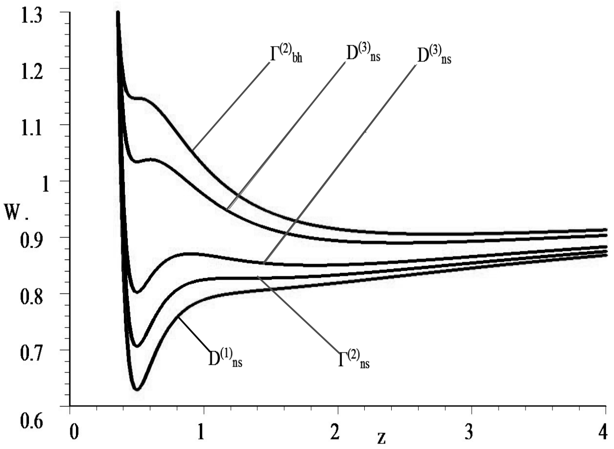

Figure 2: Effective potential versus radius z for different regions in the plane of parameters (η^2, μ^2).

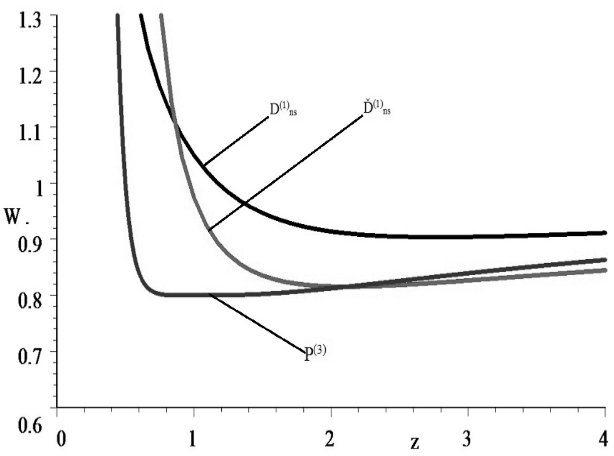

Figure 3: The same as Fig. 2 for other regions.

In Tables 1-3 we list the values of circular orbit radii z and particle energies ε calculated for parameters (η^2, μ^2) selected from the regions that correspond to different signs of D, assuming the super-extremely charged object $(\eta^2 > \mu^2)$.

Table 1: Circular orbit radii and energies of particles for selected values of parameters, $D > 0$.

	$D^{(3)}_{ns}$			$D^{(3)}_{ns}$		
μ	0.314			0.356		
η	0.334			0.382		
z	0.5	0.6	2.455	0.5	0.9	1.821
ε	1.017	1.019	0.944	0.896	0.933	0.538

Table 2: The same as Table 1, $D = 0$.

	$\Gamma^{(2)}_{ns}$		$\Gamma^{(2)}_{bh}$		$P^{(3)}$
μ	0.373		0.294		0.40
η	0.398		0.318		0.447
z	$z_1 = 0.5$	$z_{2,3} = 1.3$	$z_{1,2} = 0.5$	$z_3 = 2.75$	$z_{1,2,3} = 1$
ε	0.84	0.91	1.071	0.954	0.894

Table 3: The same as Table 1, $D < 0$.

	$D^{(1)}_{ns}$	$D^{(1)}_{ns}$	$D^{(1)}_{ns}$
μ	0.447	0.387	0.3
η	0.632	0.410	0.354
z	2.111	0.5	2.778
ε	0.903	0.793	0.951

A more detailed analysis of all the types of particles motion will be presented elsewhere.

5. Conclusion

Thus we have derived the characteristics of neutral particles motion in the field of the super-extremally charged object. The main peculiarity of this motion is that there may exist bound states with energy greater than the rest value ($\varepsilon > 1$). For example, this occurs for the effective potential from $D^{(3)}_{ns}$ (see Fig. 2) where the finite motions for the object with $\mu = 0.314, \eta = 0.334$ are allowed, so that the particles with energy $\varepsilon = 1.017$ move along the circular orbit with radius $z = 0.5$. As seen from Fig. 2, the curve $\Gamma^{(2)}_{bh}$ for the object with $\mu = 0.294, \eta = 0.318$ has an inflection point at the radius $z = 0.5$ that corresponds to the last stable circular object with energy $\varepsilon = 1.071$.

References

Chandrasekhar S. The mathematical theory of black holes. Oxford, 1983.

Dymnikova I.G.: 1986, *Uspekhi Fiz. Nauk,* **148**, 393.

Gladush V., Galadgyi M.: 2011, *General Relativity and Gravitation* **43**, 1347.

Pugliese B., Quevedo H., Ruffini R.: 2011, *Phys. Rev. D* **83**, 024021.

Pyragas A.K., Akeksandrov A.N., et al. Qualitative and analytic methods in relativistic dynamics. Minsk, 1995.