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THE ANALYSIS OF IMAGES OF A CIRCULAR SOURCE IN N-POINT GRAVITATIONAL LENSES

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ABSTRACT. Currently, in the field of computer processing, there are difficulties. When processing astrophysical experimental data, the difficulties are connected, on the one hand, with a large amount of information that requires processing, and, on the other hand, with the technical capabilities of computers. The use of analytical methods can significantly reduce information processing time. The information that is entered is recorded in an analytical form, more convenient for processing. Information is processed analytically (Kotvytskiy & Bronza, 2016), and not by methods of the Ray Tracing type. Some results were obtained strictly analytically, and they do not need further computer processing (Kotvytskiy et al., 2016). The goal of this paper is to analyze images in *N*-point gravitational lenses by:

– reduction of research of an arbitrary source to the study of a circular source;

– reduction of research of a circular source in an *N*-point gravitational line to its study in the Schwarzschild lens and other few-point lenses;

– analytical study of a circular source in the Schwarzschild lens and other few-point gravity lenses;

– research of the place of the Schwarzschild lens, in the set of *N* - point gravitational lenses.

Using the methods of algebraic geometry, algebraic topology and the theory of functions, we have prepared a problem for computer modeling: finding images of arbitrary sources in an *N*-point gravitational lens. We researched the images of a circular source in the Schwarzschild lens and proved that the study of images of any other sources is combinatorial reduced to this case.

АБСТРАКТ. В теперішній час в області обробки комп'ютерної інформації, спостерігаються труднощі. При обробці астрофізичних експериментальних даних, труднощі пов'язані з одного боку з великою кількістю інформації, яка потребує обробки, а з іншого боку з технічними можливостями комп'ютерів. Застосування аналітичних методів дозволяє значно скоротити час обробки інформації. Інформація, яка вводиться, записується в аналітичному вигляді, більш зручному для обробки. Інформація обробляється аналітично (Kotvytskiy & Bronza, 2016), а не методами типу Ray Tracing. Деякі результати вдалося отримати строго аналітично, і вони не потребують подальшої комп'ютерної обробки (Kotvytskiy et al., 2016).

Метою даної роботи є аналітичне дослідження зображень в *N*-точкових гравітаційних лінзах шляхом:

– редукції досліджень довільного джерела до вивчення кругового джерела;

– редукції досліджень кругового джерела в *N*- точковій гравітаційній лінзі до його дослідження в лінзі Шварцшильда та інших мало-точкових лінзах;

– аналітичного дослідження кругового джерела в лінзі Шварцшильда і інших мало - точкових гравітаційних лінзах;

– дослідження місця лінзи Шварцшильда, в множині N - точкових гравітаційних лінз.

Методами алгебраїчної геометрії, алгебраїчної топології та геометричної теорії функцій ми підготували задачу знаходження зображень довільних джерел в *N*-точкових гравітаційних лінзах до комп'ютерного моделювання. Аналітично досліджені зображення кругового джерела в лінзі Шварцшильда і деяких інших мало-точкових гравітаційних лінзах. Показано, що дослідження зображень будь-яких інших джерел комбінаторно зводиться до цього випадку.

Keywords: Gravitational lens, algebraic geometry, phases of images.

1. Introduction

Let R_X^2 and R_Y^2 be vector spaces. It is known, (Kotvytskiy & Bronza, 2017), (Kotvytskiy et al., 2017) that N - point gravitational lens sets a single-valued mapping

$$L:(R_X^2 \setminus \Lambda) \to R_Y^2 , \qquad (1)$$

were $\Lambda = \{ l_i \mid i = 1, 2, ..., N \}$ - set of radius - vectors $\vec{l}_i$ point masses. The mapping L can be written in the coordinate form (Weinberg, 2008):

$$\begin{cases} y_1 = x_1 - \sum_{i=1}^{N} m_i \frac{x_1 - a_i}{(x_1-a_i)^2+(x_2-b_i)^2} \\ y_2 = x_2 - \sum_{i=1}^{N} m_i \frac{x_2 - b_i}{(x_1-a_i)^2+(x_2-b_i)^2} \end{cases}, \quad (2)$$

were $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$, $\vec{l}_i = (a_i, b_i)$, and m_i normalized, dimensionless point masses satisfying the relation $\sum m_i = 1$. We add the vector spaces R_X^2 and R_Y^2 to affine ones. We will define orthonormal bases in them. For the unit of rationing, we take the Einstein-Chvolson radius. The resulting affine spaces R_X^2 and R_Y^2 are called the lens plane and the source plane, respectively. For some researches, they are combined and called the picture plane. The mapping L^{-1} inverse to (1) is, in general, multivalued (Kotvytskiy, et al., 2017), (Kotvytskiy, et al., 2017). It can, naturally, be continued from $\left(R_X^2 \setminus \Lambda\right)$ to all R_X^2 (we will leave the same notation behind the continued mapping), i.e.

$$L^{-1} : R_Y^2 \to R_X^2 \quad (3)$$

Some authors call this mapping as a lens mapping, see, for example, (Schneider et al., 1999). A special case of the N – point gravitational lens is the Schwarzschild lens, (Weinberg, 2008), (Schneider et al., 1999), which is determined by the condition $N = 1,\ a_1 = 0,\ b_1 = 0$, and $m = 1$.

From the algebraic point of view, system (2) is a system of two rational equations, which are given over the field of real numbers. System (2) can also be considered over the field of complex numbers. If the coordinates of the source are fixed, the set of real solutions of the system is the set of its images in the lens.

In this note, system (2) is researched by methods of algebraic geometry, mathematical analysis and the theory of functions. This allows us to obtain analytical expressions for describing sources and images and to construct efficient algorithms for solving problems using computer algebra methods. In some cases, it is possible to accurately obtain an analytical solution.

2. Description of sources by the analytical method

In this article, we assume that the source is flat and located in the plane R_Y^2. Sources can be classified according to various criteria, for example: physical, homogeneous and non-homogeneous; topological, on:

– zero-dimensional, consisting of a finite number of points;

– one-dimensional, consisting of a finite number of lines;

– two-dimensional, consisting of a finite number of two-dimensional regions;

– combined.

Due to the physical nature of the problem, we restrict ourselves to considering sources with a finite number of connected components, and we assume that any component or:

– point;

– arc piecewise smooth curve;

– of course – a connected domain, the boundary of which consists of a finite number of arcs of smooth curves.

From the definition of mapping (1) by the system of equations (2), it follows that the source image is the union of the images of its connected components. It can also be assumed that the boundary of each connected component (a multiply connected domain) consists of a finite set of closed Jordan curves. From here follows: the image of a multiply-connected region can be represented as a finite combination of simply-connected regions. This combinatorial problem is effectively solvable in the framework of algebraic topology. The reduction of research of an arbitrary source to the study of a circular source provides

Riemann's theorem. For any simply connected domain G (the boundary of the region has at least two points), there is analytic function $\zeta = f(z)$ in the domain G, which conformally maps the G to the unit circle $|z| < 1$. In addition, if a is a certain point of the G domain, then the function $\zeta = f(z)$ is uniquely determined by the conditions: $f(a) = 0,\ f'(a) > 0$.

Thus, at the first stage, it suffices to study the image of a point, a Jordan curve and a circle.

Note also that the trajectory of a point source, also, can be considered as the union of Jordan curves. When considering inhomogeneous sources, the functions by which they are given can be approximated by step functions given by level lines. Due to the physical nature of the problem, each level line can also be considered as a Jordan curve, or their union.

To set:

– point source is enough to set the coordinates of the point;

– an extended one-dimensional source, it suffices to specify a Jordan curve (for our purposes, for example, as an irreducible polynomial in two y_1, y_2 variables, or as a parametric curve $y_1 = y_1(t),\ y_2 = y_2(t)$;

– an extended two-dimensional source, set, its boundaries, as Jordan curves, and the region itself, as a system of inequalities relative to its boundaries.

3. Asymptotic behaviour of a mapping

The direct task of the theory of gravitational lensing is the task of constructing images from a given source.

We make some assumptions regarding the source. We will assume that the source is homogeneous, flat and has a clear boundary. In topological terms, a source is a connected, finitely connected domain whose boundary is rectifiable, or in general, is an algebraic set (Lavrentiev & Shabat, 1973), (Gurvits & Kurant, 1968). The image of an S source in a lens, in topological terms, is an open set, generally speaking, consisting of several connected components.

We define some concepts necessary for further presentation.

The diameter of the S source is called the diameter of the minimum circle to which it belongs, and the center of the source S is the center O_S of this circle. We say that the source of S is small, if its diameter d_S is significantly less than the unit $d_S << 1$, and deleted (located far away) if the module $|\vec{O}_S|$ of the radius-vector $\vec{O}_S$ is significantly greater than one, i.e. $|\vec{O}_S| >> 1$. Similarly, we define the concept: the diameter and canter of the connected components of the image, and the remote image.

Occurs

Theorem 1. Let the source S and its images be viewed in the picture plane. If the source is small and deleted, then its image has:

– remote component of connectivity S_1, such that $|O_{S_1}| > |O_S|$;

– each point S_1, with $|O_S| \to \infty$, tends to its image in S_1;

– restriction of the mapping L to S_1 is a bijection of S_1 to S.

First we prove the lemma

Lemma 2. The remote image of a remote point source in N-point gravitational lens tends to its remote image in the Schwarzschild lens.

Proof. Let points $(a_i, b_i) \in R_X^2$ and $\delta = \sqrt{x_1^2 + x_2^2}$. Let $\delta \to \infty$. Where we have:

$$x_1 - \sum_{i=1}^{N} m_i \frac{x_1 - a_i}{(x_1 - a_i)^2 + (x_2 - b_i)^2} = x_1 - \frac{x_1}{x_1^2 + x_2^2} + o(\delta). \quad (4)$$

A similar relation holds for the right side of the second equation of system (2).

For system (2), with $\delta \to \infty$, we have:

$$\begin{cases} y_1 = x_1 - \sum_{i=1}^{N} m_i \frac{x_1 - a_i}{(x_1 - a_i)^2 + (x_2 - b_i)^2} \\ y_2 = x_2 - \sum_{i=1}^{N} m_i \frac{x_2 - b_i}{(x_1 - a_i)^2 + (x_2 - b_i)^2} \end{cases} \Rightarrow \begin{cases} y_1 = x_1 - \frac{x_1}{x_1^2 + x_2^2} \\ y_2 = x_2 - \frac{x_2}{x_1^2 + x_2^2} \end{cases} \quad (5)$$

Thus, when the image is removed from the origin of coordinates in an N-point lens, it differs little from the images in the Schwarzschild lens.

But for the Schwarzschild lens, it is known that the points of the remote source are close to the points of the remote component of the image. Indeed, if $y_1^2 + y_2^2 \to \infty$, then:

$$x = \frac{1}{2}\left(y_1 \pm y_1 \sqrt{1 + \frac{4}{y_1^2 + y_2^2}} \right) \to x_1 = \begin{cases} y_1 \\ 0 \end{cases}, \quad (6)$$

thus, the abscissa S_1 tends to abscissa S. The same is true for ordinates.

The proof is complete.

Proof of Theorem 1.

We show that the points of the S_1 (the remote component of the image of the source of the S in the Schwarzschild lens) tend to their images, if $|O_S| \to \infty$. For the Schwarzschild lens we have: $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 1/\sqrt{x_1^2 + x_2^2} = 1/\delta$. In addition, from $|O_S| \to \infty \Rightarrow r \to \infty$, and therefore $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 1/\delta \to 0$.

The restriction of the mapping L to S_1 is a bijection of S_1 to S. Indeed, a point from S has two pre-images, one of which belongs to the remote image S_1, and the second is in the unit circle. Considering the lemma, we have: the assertions of Theorem 1 are true.

4. Images of a circular source in the Schwarzschild lens

The Riemann theorem and Theorem 1 show the need to investigate images of a point and circular source in the Schwarzschild lens.

It is known that each point source located not at the origin of coordinates has two points (further conjugate) images in the Schwarzschild lens, one of which is in the unit circle and the other outside it.

Occurs

Theorem 2. If g_1, g_2 is the coordinates of one of the conjugate images in the Schwarzschild lens, then the coordinates of the second image are $-g_1\left(g_1^2 + g_2^2\right)^{-1}, -g_2\left(g_1^2 + g_2^2\right)^{-1}$.

Proof. Let's substitute the coordinates of each of the images into the lens equation. Both images have the same source.

The theorem is proved.

Corollary of Theorem 2. Let the source be small, simply connected, and not containing the origin. Then, in the Schwarzschild lens, it has two images. The points of the images are conjugate and their coordinates satisfy Theorem 2. We will call such images conjugate.

Let us turn to the study of images of a circular source. Let the source be a disk $D_\varepsilon = D_\varepsilon(a)$ of radius ε with the canter at the point $(a, 0)$, and its boundary ∂D_ε:

$$\begin{cases} y_1 = t \\ y_2 = \pm\sqrt{\varepsilon^2 - (t_1 - a)^2} \end{cases}. \quad (7)$$

We substitute (7) into the system of equations that describes the Schwarzschild lens and exclude t. We have:

$$\left(x_1^2 + x_2^2 - 1\right)^2 - 2ax_1\left(x_1^2 + x_2^2 - 1\right) + (a^2 - \varepsilon^2)(x_1^2 + x_2^2) = 0 \quad (8)$$

We turn in (8) to the polar coordinates, we have:

$$\left(r^2 - 1\right)^2 - 2a\left(r^2 - 1\right) r\cos\phi + (a^2 - \varepsilon^2) r^2 = 0. \quad (9)$$

We research the equation (8) and (9), for which we consider special cases:

I) $a=0$; ***II)*** $|a|=\varepsilon$; ***III)*** $|a|<\varepsilon$; ***IV)*** $|a|>\varepsilon$.

Case I).

From (9) we have: the image of the disk D_ε under the mapping L^{-1} is the ring k_ε. The ring is formed by circles:

$$r_1=\left(\sqrt{\varepsilon^2+4}+\varepsilon\right)/2, \quad r_2=\left(\sqrt{\varepsilon^2+4}-\varepsilon\right)/2. \qquad (10)$$

The radii of the circles are reciprocal, i.e. $r_1=1/r_2$.

Case II).

If $|a|=\varepsilon \Rightarrow a=\pm\varepsilon$. Consider the case of $a=\varepsilon$. We substitute $a=\varepsilon$ into (8). We have: the equation is divided into two:

$$x_1^2+x_2^2=1,\ (x_1-\varepsilon)^2+x_2^2=1+\varepsilon^2 \qquad (11)$$

Equations (11) are equations of circles. Therefore, the image of the $D_\varepsilon=\left\{(y_1,y_2)\middle|y_1^2+y_2^2\le\varepsilon^2\right\}$ disk, when displaying $L^{-1}:R_Y^2\to R_X^2$, will be two circular wells. The wells are formed by circles (11), are conjugate and have the following areas:

$$S_I=\frac{\pi}{2}+\varepsilon-(1+\varepsilon^2)arcctg\varepsilon, \qquad (12)$$

$$S_{II}=\frac{\pi}{2}+\pi\varepsilon^2+\varepsilon-(1+\varepsilon^2)arcctg\varepsilon \qquad (13)$$

Similarly, we consider the case of $a=-\varepsilon$.

Case III). $|a|<\varepsilon$.

The value of the polar radius $r\ge 0$. From (9) we have: if $a<\varepsilon$ we have two ovals, which are one in the other and are the boundary of a doubly connected domain homeomorphisms to the K_1 ring.

Internal and external ovals:

$$r_\mp=\frac{1}{2}\left(a\cos\varphi\mp\sqrt{\varepsilon^2-a^2\sin^2\varphi}+\right.$$
$$\left.+\sqrt{\left(a\cos\varphi-\sqrt{\varepsilon^2-a^2\sin^2\varphi}\right)^2+4}\right), \qquad (14)$$

The area of this K_1 is

$$S=\int_0^\pi (r_+-r_-)(r_++r_-)\,d\varphi. \qquad (15)$$

Case IV). $|a|>\varepsilon$.

If $a>\varepsilon$, then we have: two closed, disjoint, conjugate curves (ovals) . The curves are one outside the other and are the boundaries of simply connected regions. Each of the areas is homeomorphism to a disk. Curves are defined if $\varepsilon^2-a^2\sin^2\varphi\ge 0$. Where do we get:

$$-\arcsin\frac{\varepsilon}{|a|}\le\varphi\le\arcsin\frac{\varepsilon}{|a|},\ \pi-\arcsin\frac{\varepsilon}{|a|}\le\varphi\le\pi+\arcsin\frac{\varepsilon}{|a|}. \qquad (16)$$

The ovals in the right and left half-planes are given for constraints (16) by the equation (14).

The far (near) arc of the right oval is associated with the far (near) arc of the left oval. The area of the area bounded by the right oval:

$$S=\int_0^{\arcsin\frac{\varepsilon}{a}}(r_+-r_-)(r_++r_-)d\varphi. \qquad (17)$$

5. Classification of circular images source in the Schwarzschild lens

Without loss of generality, we can assume that the center of the circular source is on the x-axis, and the radius of the source is small.

For images of a circular source, a classification theorem holds in the Schwarzschild lens.

Theorem 3. Images of a circular source in the Schwarzschild lens belong to only one of the sets defined below (we will call the sets phases, the circular source and the image of the circular source will be depicted in the picture plane).

Let the radius of the circular source ε, and the canter is at point $(a,0)$.

Phases and parameters by which they are defined: phase «-7»: $a<\varepsilon-\frac{1}{2\varepsilon}$; phase «-6»: $a=\varepsilon-\frac{1}{2\varepsilon}$; phase «-5»: $\varepsilon-\frac{1}{2\varepsilon}<a<-1+\varepsilon$; phase «-4»: $a=-1+\varepsilon$; phase «-3»: $-1+\varepsilon<a<-\varepsilon$; phase «-2»: $a=-\varepsilon$; phase «-1»: $-\varepsilon<a<0$; phase «0»: $a=0$; phase «1»: $0<a<\varepsilon$; phase «2»: $a=\varepsilon$; phase «3»: $\varepsilon<a<1-\varepsilon$; phase «4»: $a=1-\varepsilon$; phase «5»: $1-\varepsilon<a<\frac{1}{2\varepsilon}-\varepsilon$; phase «6»: $a=\frac{1}{2\varepsilon}-\varepsilon$; phase «7»: $a>\frac{1}{2\varepsilon}-\varepsilon$.

Corollary 1, Theorem 3. The classification of phases, in Theorem 3, is linear, has 15 phases, including 7 point and 8 intervals. The phases are symmetrical with respect to the central phase - the Einstein ring. Each of the phases is completely determined by the values of two parameters: the coordinates of the canter of the circular source, and its radius. Parameters are phase invariants.

Corollary 2, Theorem 3. Any small simply connected source containing the origin has a Schwarzschild lens, a single doubly connected image that contains a unit circle. The points of the image outside and inside the unit circle are conjugate and their coordinates satisfy Theorem 2.

6. Research of images in a N – point gravitational lens

The Schwarzschild Lens holds a special place in the set of point gravitational lenses:

– the point source has a long image - the Einstein ring;

– to it reduce other H-point gravitational lenses.

For a sufficiently complete study it is necessary to study the typical representatives of the set $N-$ point gravitational lenses: binary, 3-ary, etc. Also interesting are lenses with symmetries. The equation for specifying images of a circular source in an N-point gravitational lens, generally follows from (2) and (7).

Let the source be a disk $D_\varepsilon = D_\varepsilon(a)$ of radius ε centered at point (a,b) and its boundary ∂D_ε is given by parametric equations. Then the borders of the images are described by the equation:

$$\begin{gathered}\left(x_1-\sum_{i=1}^{N} m_i \frac{x_1-a_i}{(x_1-a_i)^2+(x_2-b_i)^2}-a\right)^2+\\ +\left(x_2-\sum_{i=1}^{N} m_i \frac{x_2-a_i b_i}{(x_1-a_i)^2+(x_2-b_i)^2}-b\right)^2=\varepsilon^2\end{gathered} . \quad (18)$$

Equation (18) is sufficient for computer modeling of images in small - point lenses. Thus, for the direct problem there is a constructive, quasi-analytical solution. For special cases, it is possible to solve the direct problem analytically.

Example.

Let the point source is on axis OY_1, that is $y_2 = 0$. Binary symmetric gravitational lens with masses $m_1 = 1/2$, $m_2 = 1/2$. For any real y_1, there are always three real solutions that are determined by the equation

$$x_1^3 - y_1 x_1^2 - \left(b^2+1\right)x_1 + b^2 y_1 = 0 . \quad (19)$$

In addition, with some ratios of parameters two more solutions can be implemented. For example, if $y_1 = 0$ and $b \le 1$, then we have the following, additional, pair of solutions $\vec{x} = \left(0, \pm\sqrt{1-b^2}\right)$.

7. Conclusion

In this paper, in the proof of the theorems, topological methods and the Riemann theorem on conformal equivalence of simply connected domains were used. It was proved that the study of images of any sources combinatorial reduces to the study of images of a circular source. It has been analytically proven that the Schwarzschild lens is a "limit" lens in the family of N-point gravitational lenses. The features of the Schwarzschild lens as a limiting lens are investigated, the classification of its images is proposed.

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