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ASSESSMENT OF ROBUSTNESS OF HINGED-BAR SYSTEMS

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Abstract

This paper focuses on methods for assessing the robustness of hinged bar systems, considering truss structures as an example. They are the simplest in terms of computation, but make it possible to fully illustrate the proposed approach.

The paper considers a well-known method of robustness assessment using a redundancy matrix determined by the forces that must be applied to assemble the system from elements with the length different from the design one. This method is opposed to the use of a projection matrix, the main diagonal elements of which indicate the degree of participation of the bars in ensuring robustness. The main properties of the idempotent projection matrix are considered. The paper illustrates the possibility of recalculating the projection matrix for the changed system with the help of the Jordan elimination step. A simple example demonstrates assembling and changing the projection matrix.

In addition to the failure of the bar, the case of its damage (partial failure) is also considered, it is shown how it affects the change in the projector and the redistribution of internal forces.

Keywords: Robustness, Progressive collapse, Robustness measure, Design matrix, Participation factors.

1. Introduction

Studying the response of a structure to possible catastrophic events and checking its robustness has now become an almost mandatory part of the design process. This made the engineers question some of the main ideas of the traditional approach to analysis, in particular, the orientation towards assessing the behavior of a structure under given actions, using the statistical properties of loads and materials to determine the failure probability, etc.

Catastrophic events that entail severe consequences are extremely rare and there is not enough statistical data for them. Therefore, the main approach is to shift attention from external actions to possible damage to the building structure. The relative novelty of this approach has created a certain confusion of such basic concepts as “progressive collapse”, “disproportionate collapse” and “robustness”. The phenomena they describe are very close, but do not coincide, and the mentioned concepts are not synonymous. Therefore, it is useful to consider their similarities and differences.

In this paper, the robustness of the system is considered as the main characteristic of the behavior of the system with initial damage. Quantitative assessment of robustness should be based on a comparison of the size and extent of the initial damage or the corresponding consequences [22, 8]. Various measures of robustness assessment have been proposed, based on some characteristic features that distinguish structural behavior in case of damage. Starossec U. and Haberland M. [17] introduced an energy-based metric based on the evaluation of the renewable energy required for damage propagation. In parallel, they suggested a stiffness-based measure that accounts for the ratio between the determinant of the stiffness matrix of the structure deprived of the damaged element (or connection) and the determinant of stiffness matrix of the undamaged structure. Baker et al. [2] proposed to measure the fraction of total system risk resulting from component failure. Biondini et al. [3] compared the effectiveness of various structural performance indicators and found that the ratios between either displacements or stored energies in the undamaged and damaged configurations are suitable for damage-tolerance analysis.

All of the above proposals were related to the assessment of the measure of system robustness and only to a small extent evaluated the contribution of elements, and this contribution was sequentially calculated for the elements of the systems and did not make it possible to simultaneously rank and compare the contributions of all elements.

An effective assessment of the role of elements in providing robustness is important at the conceptual design stage, especially for structures characterized by a high degree of static indeterminacy. And since the failure of an element is not necessarily related to an accidental action, and can be caused by human errors or poor workmanship, ranking allows you to indicate the places where you need to pay greater attention to the quality of work and justify the appropriate control procedures. And, finally, ranking helps to build a strategy for protection against accidental actions.

This is possible with the method for estimating contributions considered in this paper, which is based on using a projection matrix invariant with respect to the loading of the system and the stiffness properties of its elements.

2. Robustness and Progressive Collapse – Similarities and Differences

The very term “progressive collapse” is not very accurate, because it describes not the result, but rather the characteristic of the process, the nature of the collapse, which does not necessarily have the form of an avalanche-like expansion of the damage area up to a complete loss of system connectivity. But the term “disproportionate collapse” is more accurate, since it directly refers to the disproportionate scale of destruction compared to the primary structural damage, or in other words, to the discrepancy between the scale of accidental actions and their consequences.

Generally speaking, not every disproportionate collapse is progressive, and not every progressive collapse is disproportionate. That is, disproportionate collapse can occur with the sequential failure of elements one after another, in which case it will be progressive, or with an instant collapse of the entire structure. On the other hand, a large-scale progressive collapse may or may not become disproportionately large depending on whether the chain of failures stopped before a complete collapse.

The collapse of the World Trade Center towers in New York, which is often cited as an example, cannot reasonably be called disproportionate, since the initial impact was very large. In other words, disproportionate collapse describes the consequences of a local damage, while progressive collapse describes the mechanism of failure propagation. In addition, disproportionate collapse should not be confused with a general collapse due to a strong earthquake, wind, hurricane, etc. when a whole series of simultaneous failures is possible (Fig. 1).

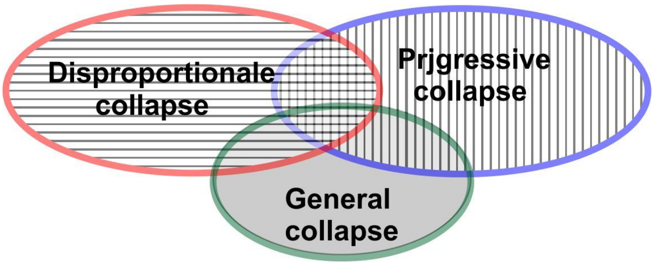


Fig. 1

Progressive (disproportionate) collapse stability analysis is usually identified with the robustness analysis [23, 9, 12]. This is how the situation is described, for example, in ISO 2394:2015, which provides the following definition: “Structural integrity (structural robustness): Ability of a structure not to be damaged by events like fire, explosions, impact or consequences of human errors, to an extent disproportionate to the original cause”.

But the very concept of robustness can have a broader meaning; it makes it possible to describe the properties of damaged structures more adequately. From the general technical view robustness is defined as “the property of a structure to maintain limited operability under external actions leading to failures of its component parts”.

Considered in this way, robustness takes into account not only the possibility of progressive collapse, but also all other possible losses of the system functionality in case of failure of individual elements. For example, for structures whose functional purpose is to provide strength, the safety factor value can serve as a functional characteristic, for oscillatory systems – the natural frequencies of free vibrations, for precision objects – shape accuracy and stability.

The definition of robustness as the ability of a structure to perform its main functions, despite the damage received, seems more reasonable, since in many cases, the loss of functionality may not be due to the failure propagation, as evidenced by numerous examples. Thus, just the failure of one span of a multi-span bridge crossing can lead to the complete loss of its functionality, and just one wire break is enough to stop electric power transmission. Functioning of both of these systems depends on their connectivity. The same applies to the need to preserve continuity – it is necessary to assess the possibility of the tank operation, where a hole in the wall or bottom can lead to its complete inoperability.

Progressive (disproportionate) collapse is characterized by the following features:

- abnormal event causing accidental (identifiable and/or unidentifiable) actions;
- local failure of an individual structural member, causing the collapse of a part of the structural system;
- disproportionately large scale of negative consequences compared to the local failure that caused them.

Fig. 2 gives a good idea of the essence of the phenomenon. It shows a well-known formula for the probability of collapse $P[Collapse]$ as a product of probabilities, where $P[H]$ is the probability of occurrence of hazard H; $P[D|H]$ is the conditional probability of a local failure given the occurrence of hazard H; $P[Collapse|D]$ is the conditional probability of occurrence of a collapse given the occurrence of the local failure D.

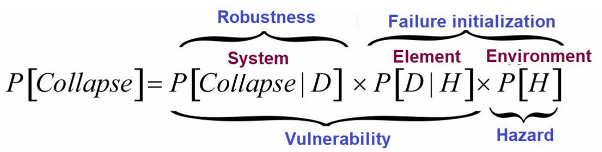


Fig. 2

Notes to the probabilistic formula clarify the role of the system, its element and the environment, and also indicate the relationship between the basic concepts and the individual elements of this formula. And it should be noted right away that, as shown in the figure, invulnerability is not synonymous with robustness. Vulnerability can characterize both an element and a system, and robustness is a quality inherent only in the load-bearing system as a whole [18]. The vulnerability of a structure may vary depending on various hazards, i.e. the structure may be vulnerable to vehicle impacts, but not to seismic loads, while robustness will be the same for both hazards.

Literature related to the problem of robustness often uses the concept of redundancy of a design model, which is defined as the ability to redistribute loads between its elements, creating an alternative load transfer path or using other methods of protection. Their redundancy is primarily related to static indeterminacy, but it can also be determined by additional safety margins (design of key elements), plastic structural behavior providing the ability to absorb energy, or special protection elements.

It is important to clarify the goals when we say that we want to protect the structure from progressive collapse. It is often implied (usually implicitly) that our goal is to *eliminate the possibility* of progressive collapse. However, there are also other reasonable goals that can be and sometimes even have to be achieved.

One of such goals can be for example *the limitation of the scale* of local damage. A protection strategy using partitioning is possible here, and typical examples are easily dropped partitions used for protection against explosions, when by limiting the scale of destruction only to the mentioned partitions, we interrupt the chain of failures. Firewalls or fire breaks in a fire protection system can also be mentioned as an example here.

Another goal is *the limitation of the rate of development* of the progressive collapse process. In fact, this is exactly what we are talking about when the fire resistance requirements (time to failure) are set taking into account the possibility of evacuating people from the fire zone (well, the building may collapse afterwards, although it is not desirable).

Finally, destruction can be limited by the *possibility of subsequent repair* of damaged structures. In fact, the whole theory of seismic protection is based on this idea.

3. Robustness Measure

According to the review [10] dozens of quantitative indicators of structural robustness are known. They can be classified into dependent on the stress state that preceded the failure of the element, and invariant with respect to the system load [1]. The former include all measures based on risk assessment, strength analysis, etc., the latter are based on the analysis of topological and stiffness characteristics, and in this sense are more general.

The analysis of the system properties invariant with respect to the stress state puts the focus on the structural analysis, the assessment of the role of individual parts of the structure and the ways of their interaction. It is related to the robustness study of systems whose local damage may occur due to unidentifiable accidental actions. In fact, the classical analysis of static geometric properties of bar systems designed for stable structures is developed here for the case of systems whose topology can change due to the failure of individual elements.

In this paper, we will consider only the analysis of hinged-bar systems, which will allow us to focus on the fundamental issues of the methodology.

The robustness of a bar system is directly related to static indeterminacy, and the degree of static indeterminacy is sometimes considered a measure of robustness. Despite the fact that the degree of static indeterminacy intuitively seems to be an ideal indicator of system safety [13], this metric provides only a necessary but not sufficient condition for robustness. A high degree of static indeterminacy does not mean a more robust structure (let's just take critical elements of the structure, for example, the removal of which leads to geometric instability).

In his works De Biagi [6, 5] related the robustness measure to the "complexity" of the system which is a metric based on the performance of the load paths through the structural scheme under an arbitrary loading and is estimated by the number of fundamental subsystems (geometrically stable main systems of the force method) that can be obtained for a given structure. As the degree of static indeterminacy increases, the number of fundamental subsystems increases at a faster rate, which requires special computational approaches.

The assembly of the system stiffness matrix involves both the topological and metric parameters of the system, therefore, its analysis was taken by many as the basis for developing the robustness measure.

It was proposed to use the condition number $k(\mathbf{K})\geq 1$ of the stiffness matrix in [14, 15]. Since the condition number serves as a measure of the matrix proximity to degeneracy, and we want the stiffness matrix to be "far" from the set of irreversible degenerate matrices, an inverse value is used

$$\delta_S = \frac{1}{k(\mathbf{K})} = \frac{n}{\|\mathbf{K}\| \cdot \|\mathbf{K}^{-1}\|}, \tag{1}$$

where $\|\square\|$ is the Euclidean matrix norm. δ_S is in the range from 0 to 1, with a higher value indicating a more robust system.

A measure of importance is proposed to represent the contribution of a structural member to the robustness of the system. It was defined as a ratio of the determinant of normalized stiffness matrices of the undamaged structure **K** and that of the damaged structure $\mathbf{K}_i$.

$$C_i = \frac{\det\mathbf{K}}{\det\mathbf{K}_i}. \tag{2}$$

This indicator varies from 1 to infinity, and the higher its value, the more critical the *i*-th element is for the robustness of the system. Starossec and Haberland [19, 20] proposed a similar measure to assess the system as a whole

$$R_S = \min_j \frac{\det\mathbf{K}}{\det\mathbf{K}_j}, \tag{3}$$

comparing the stiffness matrix of the undamaged structure and the structure damaged after the removal of *j*-th element.

The robustness of the system, its ability to function (perhaps with some loss of quality) without any failed element, indicates that it has a certain "redundancy", i.e. it has some safety margins that ensure the existence of alternative load transfer paths, and are based on its static indeterminacy.

The assessment using a redundancy measure, proposed in [11], is quantitative. In this case, it becomes possible to assess the contribution of individual elements to ensuring robustness. The distribution of these contributions significantly depends on the geometric and topological properties of the structural complex. Depending on the geometric and topological relationships between the components of the system, material and cross-section, each element has its own effect on robustness. It is worth knowing where there is redundancy in the system and where damage is unacceptable due to the lack of alternative load paths.

The authors of [11] propose to perform the measurement of redundancy on the basis of the problem of assembling a truss system from elements with inaccurate length values. To resolve geometric discrepancies, it is necessary to adjust the geometric length of the elements and/or their connections, and therefore we need to apply some forces. It is assumed that these forces are closely related to the distribution of redundancy components in the structure and can serve as an estimate of the role of elements in ensuring robustness. The sum of individual redundancy values of the elements, measured by a number from 0 to 1, is equal to the degree of static indeterminacy.

It should be noted that the redundancy matrix presented in [11] is, in fact, one of the variants introduced in [24] and developed in [16] of projection matrices that reflect load-independent static-kinematic properties of a multi-element statically indeterminate system. The use of projectors makes it possible to abandon the hypothesis that it is the assemblability that is the determining factor for assessing the contributions of individual elements to the robustness of the system.

4. Static-Kinematic Analysis. Projectors

In cases when the number of internal forces and displacements m exceeds the number of external unknowns n, the system of equilibrium equations, which has the following form in absence of external loads **p**

$$\mathbf{Qs} = \mathbf{0}, \tag{4}$$

allows a nontrivial solution

$$\mathbf{s}_0 = \mathbf{Ax} \tag{5}$$

with an arbitrary $(m–r)$-dimensional column **x**. Matrix **A** represents forces in the principal system of the force method caused by unit values of unknowns **x**.

Formula (5) defines a space of self-balanced forces $\mathbf{s}_0$, the dimension of which $k = (m - r)$ is equal to the degree of static indeterminacy of the system. Forces obtained from (5) define the self-balanced stressed state at an arbitrary vector **x**. These forces are initial ones (prestresses), and they usually arise during the erection of the structure, including the elimination of the mentioned discrepancy between the geometry of the system and the lengths of its elements, which form the vector of residuals **Δ**.

Forces that have to be applied to the bars for this are determined by the matrix formula of the displacement method

$$\mathbf{s} = (\mathbf{I} - \mathbf{FQ}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{Q})\mathbf{F\Delta} = (\mathbf{I} - \mathbf{FM})\mathbf{F\Delta}, \tag{6}$$

where $\mathbf{K} = \mathbf{QFQ}^{\mathrm{T}}$ is the stiffness matrix of the system, **F** is the matrix relating the bar elongations **Δ** with the forces **s** **(s=FΔ),** and **M** is the projection matrix

$$\mathbf{M} = \mathbf{Q}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{Q} = \mathbf{Q}^{\mathrm{T}}\left(\mathbf{QFQ}^{\mathrm{T}}\right)^{-1}\mathbf{Q}, \tag{7}$$

which was used in [11].

Since we are interested in the static and kinematic properties of the system, and more precisely, only in the conditions when it loses its variability property, it is more convenient to use another projector that does not contain the bar stiffness parameters. It is known [9], that the projection matrix **R** related to the full rank matrix **Q**

$$\mathbf{R} = \mathbf{I} - \mathbf{Q}^{\mathrm{T}}(\mathbf{QQ}^{\mathrm{T}})^{-1}\mathbf{Q} \tag{8}$$

transforms any vector $\mathbf{d}_0$ into a vector $\mathbf{s}_0 = \mathbf{Rd}_0$, belonging to the kernel of the matrix **Q**, i.e., satisfying homogeneous equations (1). But this means that the prestress force vector $\mathbf{s}_0$ is obtained using $\mathbf{d}_0$, which can be treated as a vector of arbitrary dislocation perturbations (such as bar elongations in a truss) causing prestress forces $\mathbf{s}_0$.

By the way, besides formula (3), the following relation can be used to obtain **R**, as shown in [9]

$$\mathbf{R} = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}. \tag{9}$$

Matrix $\mathbf{R}=[\rho_{ij}]$ with elements ρ_{ij} has the following properties [10, 21]:

a. **R** is idempotent, i.e. $\mathbf{R}^2=\mathbf{R}$;
b. The trace of **R** is equal to the degree of static indeterminacy of the system, i.e. $tr\mathbf{R}=r$;
c. The eigenvalues of **R** are equal to 0 or 1, where $\lambda_1=1,\ldots,\lambda_r=1$, $\lambda_{r+1}=0,\ldots,$ $\lambda_m=0$, and the rank of **R** is equal to its trace, i.e. $rank\mathbf{R}=tr\mathbf{R}=r$;
d. If a diagonal element of the matrix **R** is equal to zero, then all elements of the corresponding column and corresponding row are equal to zero.

If the diagonal element is equal to 1, and the other elements of the corresponding column and the corresponding row are zero, then the corresponding element does not affect the behavior of other elements (their values r_{ij} do not change) and the removal of this element from the system, reducing the degree of static indeterminacy by one, does not affect the geometric stability.

The redundancy component matrix shows a geometrical property of the structure. Since the sum of the diagonal elements of the matrix **R** is equal to the degree of static indeterminacy, we can assume that the component ρ_{ii} indicates the degree of participation of the i-th element in the formation of the static indeterminacy of the system. Generally speaking, the smaller the value of component ρ_{ii}, the more important is the corresponding element in terms of ensuring stability. If the value equals 0, the corresponding element is essential. On the contrary, if its value is equal to 1, the corresponding element does not affect the behavior of other elements and its removal from the system reduces the degree of static indeterminacy by one but does not affect the stability in any way.

For structural elements that are *critical*, i.e. the removal of which leads to geometric instability [14], and the creation of prestressing forces with their help is impossible for any perturbation $\mathbf{d}_0$. Indeed, by definition, such an element is necessarily included in the principal system of the force method and, therefore, the force in it cannot be considered as a component of the vector of unknowns **x**. But this means that such an element must have corresponding zero-column and zero-row in the matrix **R** (and, accordingly, in **M**). It should be noted that it suffices to check whether the diagonal element of the projection matrix is equal to zero (see property “d”).

The reverse statement is true as well — *a conditionally critical* element (unlike a critical element, a conditionally critical element can be removed from the system without losing its geometric stability) has corresponding rows and columns with non-zero elements in the projection matrix, while the main diagonal has a nonzero element.

Thus, the values of the projection matrix components are quantitative estimates that demonstrate how and through which elements the static indeterminacy is embedded in the system.

Unlike the complexity estimate [6, 5], which is related to the sequential analysis of possible load transfer paths through all possible principal systems of the force method, the projection matrix, which applies to all possible principal systems simultaneously, gives such an estimate immediately.

The elements ρ_{ii} of the main diagonal of the matrix **R** show the importance of individual elements, and the non-diagonal elements evaluate the interaction between these elements. In this case, it is useful to normalize ρ_{ii}, and, given that they add up to *r*, it is reasonable to compare them using the values $c_i = (\rho_{ii}/r)$, the sum of which is equal to one.

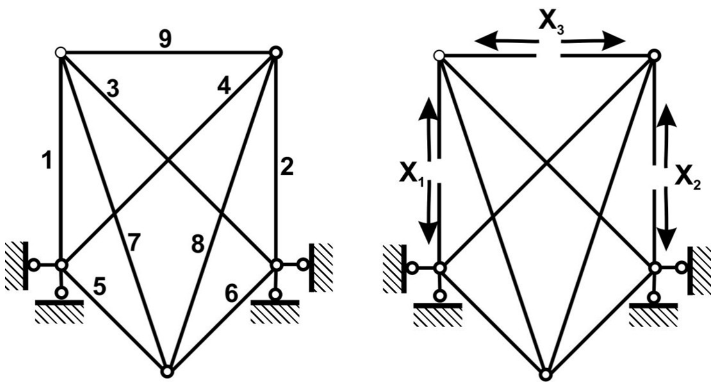
If we talk about assessing the robustness of the structure as a whole, then we can take the minimum value as a cautious measure of robustness:

$$\nu_R = \min_i (c_{ii}). \tag{10}$$

Then the presence of critical elements indicates zero robustness of the system, which can be destroyed by removing such an element.

An alternative approach is possible, based on the hypothesis that in terms of robustness a system, where all elements are equally important and provide robustness to the same extent, will be the best. Since the trace of the matrix for a system of *m* bars is equal to the degree of static indeterminacy *r*, then such a situation will take place if all diagonal elements c_{ii} of the matrix **R** (importance indicators) have the value 1/*m*. Then, we can take the measure of the system robustness as the root-mean-square spread of its values:

$$C_R = \sqrt{(1/m)\sum_{i=1}^{m}\left(c_{ii} - (1/m)\right)^2}. \tag{11}$$



(a) (b)

Fig. 3

As an example, consider a three times statically indeterminate system with unit stiffness properties shown in Fig. 3(a). The principal system, which was used to assemble the matrix of equilibrium equations **A**, is shown in Fig. 3(b):

$$\mathbf{A}=\begin{bmatrix} 1,000 & 0,000 & -0,001 & 0,706 & 1,413 & 0,706 & -1,580 & 0,001 & 0,000 \\ 0,000 & 1,000 & 0,706 & -0,001 & 0,706 & 1,413 & 0,001 & -1,580 & 0,000 \\ 0,000 & 0,000 & -2,119 & -2,119 & -2,119 & -2,119 & 1,579 & 1,579 & 0,999 \end{bmatrix}.$$

Construct the projector **R** using the formula (4). We have

$$\mathbf{AA^T}=\begin{bmatrix} 6,490 & 1,990 & -8,477 \\ 1,990 & 6,490 & -8,477 \\ -8,477 & -8,477 & 23,945 \end{bmatrix},\quad \left(\mathbf{AA^T}\right)^{-1}=\begin{bmatrix} 0,313 & 0,091 & 0,143 \\ 0,091 & 0,313 & 0,143 \\ 0,143 & 0,143 & 0,143 \end{bmatrix}.$$

As a result we get:

$$\mathbf{R}=\begin{bmatrix} 0,313 & 0,091 & -0,239 & -0,082 & 0,203 & 0,046 & -0,269 & 0,083 & 0,143 \\ 0,091 & 0,313 & -0,082 & -0,239 & 0,046 & 0,203 & 0,083 & -0,269 & 0,143 \\ -0,239 & -0,082 & 0,371 & 0,260 & 0,033 & 0,144 & 0,058 & -0,190 & -0,202 \\ -0,082 & -0,239 & 0,260 & 0,371 & 0,144 & 0,033 & -0,190 & 0,058 & -0,202 \\ 0,203 & 0,046 & 0,033 & 0,144 & 0,320 & 0,209 & -0,321 & -0,073 & 0,000 \\ 0,046 & 0,203 & 0,144 & 0,033 & 0,209 & 0,320 & -0,073 & -0,321 & 0,000 \\ -0,269 & 0,083 & 0,058 & -0,190 & -0,321 & -0,073 & 0,425 & -0,131 & 0,000 \\ 0,083 & -0,269 & -0,190 & 0,058 & -0,073 & -0,321 & -0,131 & 0,425 & 0,000 \\ 0,143 & 0,143 & -0,202 & -0,202 & 0,000 & 0,000 & 0,000 & 0,000 & 0,143 \end{bmatrix}.$$

Having performed the control calculation of the trace of the matrix **R**, we obtain $tr\mathbf{R}$ = 3,000, which coincides with the degree of static indeterminacy of the system. The system robustness estimates turned out to be ν_R= 0,145 and μ_R=0,236. The deviation of μ_R from the ideal value r/m=0,333 was 30%.

The fact that the kinematic properties of the system can be analyzed with the help of the projector **R** can help to solve the issue of changing the static-kinematic class of the structure when removing an element of the system. If the k-th element is removed from the system, then the k-th component of the vector $\mathbf{s}_0$ must be equal to zero under any actions (including arbitrary dislocations $\mathbf{d}_0$), since there is no force in the missing element. To achieve this, it is necessary to exclude the variable $s_{0,k}$ from the system $\mathbf{s}_0 = \mathbf{Rd}_0$, by taking the Jordan elimination step [15] with the resolving element R_{kk}, after which the system $\mathbf{s}_0 = \mathbf{Rd}_0$ takes the following form

$$s_{0,i}=\sum_{j=1}^{k-1}\overset{*}{\rho}_{ij}d_{0,j}+\overset{*}{\rho}_{ik}s_{0,k}+\sum_{j=k+1}^{m}\overset{*}{\rho}_{ij}d_{0,j}\qquad (i=1,\ldots,k-1),$$

$$d_{0,k}=\sum_{j=1}^{k-1}\overset{*}{\rho}_{kj}d_{0,j}+\overset{*}{\rho}_{kk}s_{0,k}+\sum_{j=k+1}^{m}\overset{*}{\rho}_{kj}d_{0,j},$$

$$s_{0,i}=\sum_{j=1}^{k-1}\overset{*}{\rho}_{ij}d_{0,j}+\overset{*}{\rho}_{ik}s_{0,k}+\sum_{j=k+1}^{m}\overset{*}{\rho}_{ij}d_{0,j}\qquad (i=k+1,\ldots,m),\qquad (12)$$

where elements of the transformed matrix are marked with an asterisk. The condition $s_{0,k} = 0$ allows us to remove the k-th column, and the k-th row will contain an expression for determining the mutual approach of nodes connected by the removed element, and hence the k-th row can be removed as well.

The new matrix **R*** of order (m–1) is also a projector, but for the structure with the k-th element removed. If now **R*** contains new zero rows and columns, it means that the corresponding elements have now become critical after the removal of the k-th element (and because of this removal)!

The Jordan elimination step with the resolving element r_{ks} transforms the matrix elements **R** that do not belong to the resolving row or to the resolving column according to the formula

$$\rho_{ij}^{*} = \rho_{ij} - \rho_{is}\rho_{kj} / \rho_{ks} \quad (i \neq k,\ j \neq r). \tag{13}$$

If we take the value r_{kk} as a resolving element, then the new values r_{ij}^{*} show how the role of the corresponding elements has changed after the removal of the k-th bar from the system. The Jordan elimination step with the resolving element r_{99} (the 9-th bar is removed) gives a new projector for the system in Fig. 3:

$$\mathbf{R}^{*} = \begin{bmatrix} 0,17 & -0,052 & -0,037 & 0,12 & 0,203 & 0,046 & -0,269 & 0,083 \\ -0,052 & 0,17 & 0,12 & -0,037 & 0,046 & 0,203 & 0,083 & -0,269 \\ -0,037 & 0,12 & 0,086 & -0,025 & 0,033 & 0,144 & 0,058 & -0,19 \\ 0,12 & -0,037 & -0,025 & 0,086 & 0,144 & 0,033 & -0,19 & 0,058 \\ 0,203 & 0,046 & 0,033 & 0,144 & 0,32 & 0,209 & -0,321 & -0,073 \\ 0,046 & 0,203 & 0,144 & 0,033 & 0,209 & 0,32 & -0,073 & -0,321 \\ -0,269 & 0,083 & 0,058 & -0,19 & -0,321 & -0,073 & 0,425 & -0,131 \\ 0,083 & -0,269 & -0,19 & 0,058 & -0,073 & -0,321 & -0,131 & 0,425 \end{bmatrix},$$

which shows how much the role of the 1st and 2nd bars has grown in the changed system. The calculation shows that $tr\mathbf{R}^{*}$=2,002.

The most interesting is the case when new critical elements can appear in a system that has changed due to the removal of the bar, as evidenced by the condition

$$\rho_{ss}\rho_{kk} = \rho_{rs}\rho_{sk}\,. \tag{12}$$

New critical elements indicate the lack of robustness of the analyzed system (their removal leads to a general collapse).

In conclusion, it should be noted that the possibility of using projectors and their transformations by Jordan elimination steps in the robustness analysis was pointed out in [25].

5. The Case of Partial Damage (Weakening) of Bars

Local failure, initiating further failure propagation through the elements of the system, does not necessarily have to be a complete collapse of the element of the system. In our case, we will assume that the initiating event is the damage to the bar, which reduces its cross-sectional area A_i by a value with a degree estimated by the factor $\xi \le 1$, and the initial value of the area is transformed into ξA_i.

To assess the effect of partial failure (weakening) of the bars, it is more convenient to use the projector **M**, which, as can be seen from (4), uses a diagonal matrix **F** with elements equal to the stiffness of the bars $F_{ii} = l_i / EA_i$. Like the projector **R**, the trace of the projector **M** is equal to the rank of the matrix of equilibrium equations **Q** [16].

$$tr\mathbf{M}=-r. \qquad (13)$$

Projector **M** for the system in Fig. 3:

$$\mathbf{M}=\begin{bmatrix} 0,382 & 0,053 & 0,058 & 0,101 & -0,098 & -0,054 & 0,111 & 0,009 & -0,062 \\ 0,053 & 0,312 & -0,024 & -0,113 & -0,024 & -0,113 & -0,052 & 0,160 & 0,027 \\ 0,058 & -0,024 & 0,266 & -0,151 & -0,054 & 0,028 & 0,108 & -0,087 & 0,112 \\ 0,101 & -0,113 & -0,151 & 0,334 & 0,004 & -0,011 & -0,014 & 0,020 & 0,094 \\ -0,098 & -0,024 & -0,054 & 0,004 & 0,340 & -0,102 & 0,172 & 0,035 & -0,011 \\ -0,054 & -0,113 & 0,028 & -0,011 & -0,102 & 0,359 & 0,050 & 0,142 & -0,029 \\ 0,111 & -0,052 & 0,108 & -0,014 & 0,172 & 0,050 & 0,268 & 0,057 & -0,006 \\ 0,009 & 0,160 & -0,087 & 0,020 & 0,035 & 0,142 & 0,057 & 0,304 & 0,038 \\ -0,062 & 0,027 & 0,112 & 0,094 & -0,011 & -0,029 & -0,006 & 0,038 & 0,435 \end{bmatrix}.$$

The weakening of the cross-section of the *k*-th bar, for example, changes the values of its stiffness, which becomes equal to $F_{kk} = l_k / \xi EA_k$. Naturally, this affects the magnitude of the main diagonal elements M_{ii} of the projector **M**, which, as in the case of the projector **R**, indicate the degree of participation of the *i*-th element in the formation of the static indeterminacy of the system. The role of the damaged bar decreases. In case of the complete collapse of the *k*–th bar ($\xi=\propto$), the *k*–th column and the *k*–th row are zeroed in the projector **M**.

Since the condition (11) must be satisfied, the role of the remaining undamaged elements increases, they take on the role of an alternative way of transferring the part of the load that the damaged element cannot take. The extent to which this role is transferred to undamaged elements can be seen from the value of the increase in the corresponding elements of **M**. For the example considered earlier, the value of **M** for the case ξ=0,5 will be as follows:

$$\mathbf{M}=\begin{bmatrix} 0,389 & 0,050 & 0,045 & 0,091 & -0,096 & -0,051 & 0,112 & 0,005 & -0,078 \\ 0,050 & 0,313 & -0,019 & -0,109 & -0,025 & -0,115 & -0,052 & 0,162 & 0,034 \\ 0,045 & -0,019 & 0,288 & -0,133 & -0,056 & 0,022 & 0,107 & -0,080 & 0,140 \\ 0,091 & -0,109 & -0,133 & 0,350 & 0,002 & -0,015 & -0,015 & 0,026 & 0,117 \\ -0,096 & -0,025 & -0,056 & 0,002 & 0,340 & -0,102 & 0,172 & 0,034 & -0,014 \\ -0,051 & -0,115 & 0,022 & -0,015 & -0,102 & 0,360 & 0,050 & 0,140 & -0,037 \\ 0,112 & -0,052 & 0,107 & -0,015 & 0,172 & 0,050 & 0,268 & 0,057 & -0,007 \\ 0,005 & 0,162 & -0,080 & 0,026 & 0,034 & 0,140 & 0,057 & 0,306 & 0,047 \\ -0,078 & 0,034 & 0,140 & 0,117 & -0,014 & -0,037 & -0,007 & 0,047 & 0,385 \end{bmatrix}.$$

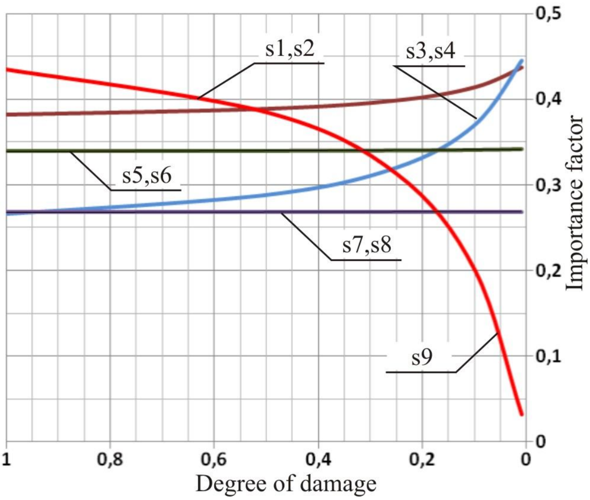


Fig. 4

Fig. 4 shows the graphs of elements of the main diagonal **M** versus the value of the parameter ξ, indicating the degree of damage to the 9th bar. It can be seen that the role of the alternative path is mainly played by the forces in bars 3 and 4.

If we consider the parameter ξ as a measure of accumulated damage, then from Fig. 4 it can be seen that at small ξ the system is characterized by low sensitivity to the failure of element 9 and such a weakening effect can be neglected. And only with a further increase in the degree of damage (ξ>0,5) the system begins to react intensively to the damage.

Sensitivity of internal forces in a system with the stress state characterized by the vector of internal forces

$$\mathbf{s}=\left[S_1^0\ S_2^0\ \cdots\ S_m^0\right]^{\mathrm{T}}, \tag{14}$$

is calculated by the formula [16]

$$\frac{\partial S_r}{\partial A_k}=\sum_{k=1}^{m}\mu_{rk}\frac{S_k^0}{A_k}=\sum_{k=1}^{m}\sigma_k^0\mu_{rk}\ , \tag{15}$$

where μ_{rk} are elements of the projection matrix **M**.

The change in the stress state of the bars caused by the weakening of the damaged element nonlinearly depends on the degree of damage ξ. This is evidenced by the graphs in Fig. 5, describing the change in the forces in the bars according to Fig. 3 depending on the degree of damage of the 9th bar.

It is known [26, 27], that the relationship between forces and the parameter ξ has a hyperbolic character and can be represented by the following expression:

$$S_i = A + \frac{B}{C+\xi}.$$

Three parameters *A*, *B* and *C* can be calculated for three force values obtained at different values of ξ, which then makes it possible to reject the numerical solution of the problem.

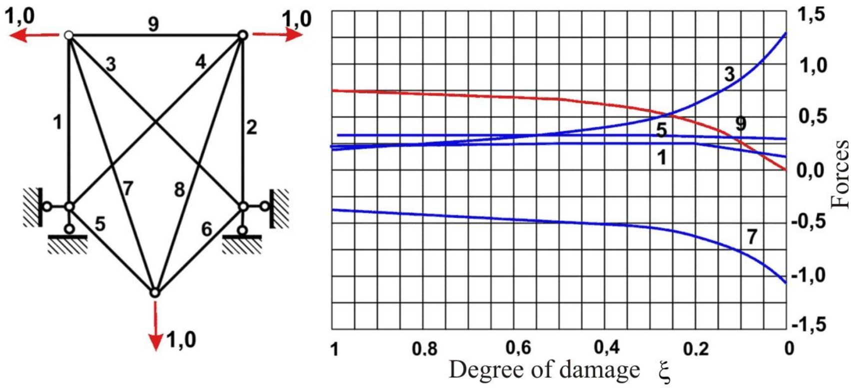


Fig. 5

And with relatively small changes in the cross-sectional area of the *k*-th bar, the forces in the truss bars will change by

$$\Delta S_r = \sum_{k=1}^{m} \sigma_k^0 \mu_{rk} \Delta A_k = (1-\xi) \sum_{k=1}^{m} S_k^0 \mu_{rk} \quad (r = 1, 2, ..., m). \qquad (16)$$

Linear prediction (16) is approximate, apparently, its use is justified for values of ξ not exceeding 0,5.

6. Conclusion

Representation of the static-kinematic properties of hinged-bar systems with the help of projection matrices of the system of equilibrium equations makes it possible to analyze the degree of participation of individual bars in the formation of static indeterminacy and geometric stability of the structure.

In addition, the matrix also indicates the rule for the redistribution of the action components in case of failure or weakening of the bar. The advantage of the considered approach, which distinguishes it from most other proposed methods for assessing robustness, is its invariance with respect to the load on the system. The latter is very important in the case of assessing robustness of structures under unidentifiable accidental actions.

Truss structures with a single-component vector of internal forces were analyzed as the simplest ones in terms of computation, but the principles of the approach to assessing the role of individual elements of the system using a projection matrix have a wider scope.

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ДО ОЦІНКИ ЖИВУЧОСТІ ШАРНІРНО-СТРИЖНЕВИХ СИСТЕМ

Як правило, проектування конструкцій враховує передбачувані навантаження і для таких варіантів роботи підбираються розміри поперечних перерізів. Однак конструкції можуть зазнавати і непередбачених подій, таких як інтенсивні явища навколишнього середовища, аварії, зловмисні дії, а також помилки планування або виконання. Ця обставина визначає інтерес до проблеми живучості конструкцій, якій останнім часом присвячується багато робіт.

Ця стаття присвячена методам оцінки живучості шарнірно-стрижневих систем. Об'єктом дослідження обрані фермові конструкції, найпростіші у обчислювальному відношенні, але що дають можливість повністю проілюструвати пропонований підхід.

Спочатку аналізуються відмінності прогресуючого обвалення (опис процесу) від непропорційного розвитку локальних руйнувань (опис стану). Вказується на узагальнюючий характер поняття живучості та її від поняття невразливості.

Розглядається проблема виміру живучості. Аналізуються відомі кількісні оцінки живучості, основна увага при цьому спрямована на оцінки, інваріантні по відношенню до напруженого стану як більш загальні. Розглядаються оцінки, що використовують такі властивості матриці жорсткості як число обумовленості, або засновані на зіставленні детермінантів початкової матриці жорсткості, що змінилася. Вказується те що, що ступінь статичної невизначеності може лише необхідним, але недостатнім вимірником живучості.

Відзначається відомий варіант оцінки живучості за допомогою матриці надмірностей, що визначається зусиллями, які необхідно докласти для складання системи з елементів, що мають довжину, відмінну від проектної. Цьому варіанту протиставляється використання матриці-проектора, елементи головної діагоналі якої вказують на ступінь стрижнів у забезпеченні живучості. Розглянуто основні властивості проектора, обумовлені тим, що він є матрицею нильпотентною. Показана можливість перерахунку початкової матриці-проектора до проектора системи, що змінилася, за допомогою кроку жорданових виключень. На найпростішому прикладі демонструються формування та зміни матриці-проектора.

Крім руйнування стрижня розглядається і випадок його ушкодження (часткового руйнування), показано як це позначається зміні проектора і перерозподіл внутрішніх зусиль.

Ключові слова: живучість, прогресуюче руйнування, межа живучості, проективна матриця, фактори участі.

Perelmuter A. V.

ASSESSMENT OF ROBUSTNESS OF HINGED-BAR SYSTEMS

Typically in structural design, foreseeable loads are assumed in a dimensioning exercise. Structures can, however, be exposed to largely unforeseeable events such as intense environmental phenomena, accidents, malicious acts, and planning or execution errors. This circumstance determines the interest in the problem of structural robustness, which has been the subject of many recent works.

This paper focuses on methods for assessing the robustness of hinged bar systems, considering truss structures as an example. They are the simplest in terms of computation, but make it possible to fully illustrate the proposed approach.

First, the differences between progressive collapse (description of the process) and the disproportionate propagation of local failures (description of the state) are analyzed. The generalizing nature of the concept of robustness and its differences from the concept of invulnerability are pointed out.

The paper considers the problem of measuring robustness. The known quantitative estimates of robustness are analyzed focusing on estimates that are invariant with respect to the stress state, as more general ones. The paper considers estimates that use such properties of the stiffness matrix as the condition number, or based on a comparison of the determinants of the original and changed stiffness matrices. It is pointed out that the degree of static indeterminacy can serve only as a necessary, but insufficient measure of robustness.

The paper considers a well-known method of robustness assessment using a redundancy matrix determined by the forces that must be applied to assemble the system from elements with the length different from the design one. This method is opposed to the use of a projection matrix, the main diagonal elements of which indicate the degree of participation of the bars in ensuring robustness. The main properties of the idempotent projection matrix are considered. The paper illustrates the possibility of recalculating the projection matrix for the changed system with the help of the Jordan elimination step. A simple example demonstrates assembling and changing the projection matrix.

In addition to the failure of the bar, the case of its damage (partial failure) is also considered, it is shown how it affects the change in the projector and the redistribution of internal forces.

Keywords: Robustness, Progressive collapse, Robustness measure, Design matrix, Participation factors.

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Перельмутер А. В. **До оцінки живучості шарнірно-стрижневих систем** // Опір матеріалів і теорія споруд: наук.-тех. збірн. – К.: КНУБА, 2022. – Вип. 109. – С. 3-19.
Стаття присвячена методам оцінювання живучості шарнірно-стрижневих систем. Пропонується використання матриці-проектора, елементи головної діагоналі якої вказують на ступінь участі стрижнів у забезпеченні живучості. Розглянуто основні властивості проектора, обумовлені тим, що він є нільпотентною матрицею.
Іл. 5. Табл. 0. Бібліог. 27 назв

УДК 624.024.046.5+006.036
Perelmuter A. V. **Assessment of robustness of hinged-bar systems** // Strength of Materials and Theory of Structures: Scientific-and-technical collected articles – Kyiv: KNUBA, 2022. – Issue 109. – P. 3-19.
The article is devoted to methods of assessing the survivability of hinge-rod systems. It is proposed to use a projector matrix, the elements of the main diagonal of which indicate the degree of participation of the rods in ensuring survivability. The main properties of the projector due to the fact that it is a nilpotent matrix are considered.
Figs. 5. Tabs. 0. Refs. 27.

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