Conformally Flat Twisted Product Manifolds

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Abstract On 1995, T. Ikawa and J. B. Jun proved the following: If a doubly warped product manifold $M = M_1 \times M_2$ is conformally flat, then the manifolds M_1 and M_2 are conformally flat, too. As the corollary of this theorem, we can get the same properties in the case of product and warped product manifolds. But, about the case of a twisted product manifold, we can not see the similar theorem, yet. So, in this paper, we prove the same theorem in a twisted product manifold, that is, Let $M = M_1 \times_f M_2$ be a conformally flat twisted product manifold with the associated function f. Then the manifolds M_1 and M_2 are conformally flat, too.

Keywords Conformally flat, twisted product manifold

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1 Twisted product manifolds

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and M be a topological product manifold of M_1 and M_2. We define a Riemannian metric g of M as

$$g(U, V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V) \tag{1.1}$$

for any $U, V \in TM$, where f denotes a positive differentiable function on M . Then the manifold M is called a *twisted product manifold with an associated function* f and we write it $M = M_1 \times_f M_2$ ([2],[3]).

Remark. In a twisted product manifold $M = M_1 \times_f M_2$, if the associated function f is in M_2, then the the manifold M is a warped product Riemannian manifold ([4]) and f is constant (=1), then the manifold M is a product manifold.

Let $M = M_1 \times_f M_2$ be a twisted product manifold with the associated function f and let $\dim M_1 = n_1$, $\dim M_2 = n_2$ and $\dim M = n = n_1 + n_2$. Moreover, let $(x^1, x^2, ..., x^{n_1})$, $(x^{n_1+1}, ..., x^{n_1+n_2})$ be local coordinate systems of M_1 and M_2, respectively. Then we consider that $(x^1, x^2, ..., x^n)$ is a local coordinate system of M.

Using the above local coordinate systems, we can write

$$(g_{\mu\lambda}) = \begin{pmatrix} e^{f^2} g_{1ji} & 0 \\ 0 & g_{2ba} \end{pmatrix}, \qquad (g^{\mu\lambda}) = \begin{pmatrix} e^{-f^2} g_1{}^{ji} & 0 \\ 0 & g_2{}^{ba} \end{pmatrix}, \tag{1.2}$$

where the indices $(j, i, ..., h)$, $(d, c, ..., a)$ and $(\nu, \mu, ..., \lambda)$ run over the ranges $(1, 2, ..., n_1)$, $(n_1 + 1, n_1 + 2, ..., n_1 + n_2)$ and $(1, 2, ..., n_1 + n_2 = n)$, respectively.

By virtue of (1.2), the Christoffel symbols $\{_\nu{}^\lambda{}_\mu\}$ with respect to $g_{\mu\lambda}$ are given by

$$\begin{aligned} \{_j{}^h{}_i\} &= \{_j{}^h{}_i\}_1 + f^2\{(\partial_j \log f)\delta_i{}^h + (\partial_i \log f)\delta_j{}^h - (\partial_1{}^h \log f)g_{1ji}\}, \\ \{_b{}^h{}_i\} &= f^2(\partial_b \log f)\delta_i{}^h, \quad \{_b{}^h{}_a\} = 0, \\ \{_j{}^a{}_i\} &= -f^2 e^{f^2}(\partial_2{}^a \log f)g_{1ji}, \quad \{_j{}^a{}_b\} = 0, \quad \{_c{}^a{}_b\} = \{_c{}^a{}_b\}_2, \end{aligned} \tag{1.3}$$

where $\partial_1{}^h = g_1{}^{lh}\partial_l$ (resp. $\partial_2^a = g_2{}^{ea}\partial_e$) and $\{_j{}^h{}_i\}_1$ (resp. $\{_c{}^a{}_b\}_2$) denotes the Christoffel symbol of g_1 (resp. g_2).

Next, we calculate the Riemannian curvature tensor $R_{\omega\nu\mu}{}^\lambda$ with respect to $g_{\mu\lambda}$. Using (1.2) and (1.3) and by the straightforward calculation, we get

(1.4)

$$\begin{aligned} R_{kji}{}^h &= R^1{}_{kji}{}^h + f^2(2 - f^2)[(\partial_i \log f)\{(\partial_k \log f)\delta_j{}^h - \\ &-(\partial_j \log f)\delta_k{}^h\} - (\partial_1{}^h \log f)\{(\partial_k \log f)g_{1ji} - \\ &-(\partial_j \log f)g_{1ki}\}] + f^2\{(\nabla_{1k}\partial_i \log f)\delta_j{}^h - \\ &-(\nabla_{1j}\partial_i \log f)\delta_k{}^h - (\nabla_{1k}\partial_1{}^h \log f)g_{1ji} + \\ &+(\nabla_{1j}\partial_1{}^h \log f)g_{1ki}\} - (f^4(\|\log f_1\|^2 + 4e^{f^2}\|\log f_2\|^2)(g_{1ji}\delta_k{}^h - g_{1ki}\delta_j{}^h), \\ R_{kji}{}^a &= f^2 e^{f^2}[2(\partial_2{}^a \log f)\{(\partial_j \log f)g_{1hi} - (\partial_k \log f)g_{1ji}\} + \\ &+(\partial_j\partial_2{}^a \log f)g_{1ki} - (\partial_k\partial_2{}^a \log f)g_{1ji}], \\ R_{kjb}{}^h &= 2f^2(\partial_b \log f)\{(\partial_k \log f)\delta_j{}^h - (\partial_j \log f)\delta_k{}^h\} \\ &f^2\{(\partial_k\partial_b \log f)\delta_j{}^h - (\partial_j\partial_b \log f)\delta_k{}^h\}, \\ R_{kbi}{}^h &= -2f^2(\partial_b \log f)\{(\partial_i \log f)\delta_k{}^h - (\partial_1{}^h \log f)g_{1ki}\} - \\ &-f^2\{(\partial_c\partial_i \log f)\delta_k{}^h - (\partial_c\partial_1{}^h \log f)g_{1ki}\}, \\ R_{kci}{}^a &= f^2 e^{f^2}\{(2 + f^2)(\partial_c \log f)(\partial_2{}^a \log f) + \nabla_{2c}\partial_2{}^a \log f\}g_{1ki} \\ R_{kba}{}^h &= -f^2\{(2 + f^2)(\partial_b \log f)(\partial_a \log f) + (\nabla_{2b}\partial_a \log f)\}\delta_k{}^h, \\ R_{kjb}{}^a &= 0, \quad R_{kcb}{}^a = 0, \quad R_{dcb}{}^h = 0, \quad R_{dcb}{}^a = R^2{}_{dcb}{}^a, \end{aligned}$$

where ∇_1 (resp. ∇_2) is the covariant differentiation with respect to g_1 (resp. g_2) and $R^1{}_{kji}{}^h$ (resp. $R^2{}_{dcb}{}^a$) denotes the Riemannian curvature tensor with respect to g_{1ji} (resp. g_{2ba}).

Moreover, using (1.2) and (1.3), the calculation of the Ricci Tensor $R_{\mu\lambda}$ with respect to $g_{\mu\lambda}$ gives us

$$\begin{aligned} R_{ji} = R^1{}_{ji} + (n_1-2)f^2\{(2-f^2)(\partial_j \log f)(\partial_i \log f) + \nabla_{1j}\partial_i \log f\} - \\ -\{(2-2f^2+n_1f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f\} + \\ +e^{f^2}\{(2+n_1f^2)\|\nabla_2 \log f\|^2 + \nabla_{2e}\partial_2{}^e \log f\}g_{1ji}, \\ R_{ja} = -(n_1-1)f^2\{2(\partial_j \log f)(\partial_a \log f) + \partial_j\partial_a \log f\}, \\ R_{ba} = R^2{}_{ba} - n_1 f^2\{(2+f^2)(\partial_b \log f)(\partial_a \log f) + \nabla_{2b}\partial_a \log f\} \end{aligned} \tag{1.5}$$

where $R^1{}_{ji}$ (resp. $R^2{}_{ba}$) is the Ricci tensor with respect to g_{1ji} (resp. g_{2ba}).

Finally, the scalar curvature R with respect to $g_{\mu\lambda}$ is given by

$$\begin{aligned} R = e^{-f^2}R^1 + R^2 + (n_1-1)f^2e^{-f^2}\{(4-2f^2+n_1f^2)\|\nabla_1 \log f\|^2 + \\ +2\nabla_{1l}\partial_1{}^l \log f_1\} - n_1f^2\{(4+n_1f^2+f^2)\|\nabla_2 \log f\|^2 + \\ +2\nabla_{2e}\partial_2{}^e \log f\}, \end{aligned} \tag{1.6}$$

where R^1 (resp. R^2) denotes the scalar curvature with respect to g_{1ji} (resp. g_{2ba}).

2 Conformal curvature tensors

In this section, we calculate the conformal curvature tensor $C_{\omega\nu\mu}{}^\lambda$ with respect to $g_{\mu\lambda}$. The conformal curvature tensor $C_{\omega\nu\mu}{}^\lambda$ is defined by

$$\begin{aligned} C_{\omega\nu\mu}{}^\lambda = R_{\omega\nu\mu}{}^\lambda + \frac{1}{n-2}(R_{\omega\mu}\delta_\nu{}^\lambda - R_{\nu\mu}\delta_\omega{}^\lambda + \\ +R_\nu{}^\lambda g_{\omega\mu} - R_\omega{}^\lambda g_{\nu\mu}) + \frac{R}{(n-1)(n-2)}(g_{\nu\mu}\delta_\omega{}^\lambda - g_{\omega\mu}\delta_\nu{}^\lambda). \end{aligned} \tag{2.1}$$

Using (1.2), (1.4), (1.5), (1.6) and (2.1), after the direct calculations, we obtain

$$
\begin{aligned}
C_{kji}{}^{h} &= R^{1}{}_{kji}{}^{h} + \frac{1}{n-2}T_{kji}{}^{h} + \frac{n_2(2-f^2)f^2}{n-2}S_{kji}{}^{h} + \\
&+ \frac{n_2 f^2}{n-2}\{(\nabla_{1k}\partial_i \log f)\delta_j{}^{i} - \nabla_{1j}\partial_i \log f)\delta_k{}^{h} - \\
&-(\nabla_{1k}\partial_1{}^{h} \log f)g_{1ji} + (\nabla_{1j}\partial_1{}^{h} \log f)g_{1ki} - \\
&-A(g_{1ji}\delta_k{}^{h} - g_{1ki}\delta_j{}^{h}) \\
C_{kjb}{}^{h} &= \frac{2(n_2-1)f^2}{n-2}(\partial_b \log f)\{(\partial_k \log f)\delta_j{}^{h} - (\partial_j \log f)\delta_k{}^{h}\} + \\
&+ \frac{(n_2-1)f^2}{n-2}\{(\partial_k\partial_b \log f)\delta_j{}^{h} - (\partial_j\partial_b \log f)\delta_k{}^{h}\}, \\
C_{kci}{}^{h} &= \frac{2(n_2-1)f^2}{n-2}[(\partial_c \log f)\{(\partial_1{}^{h} \log f)g_{ki} - (\partial_c \log f)\delta_k{}^{h}\} - \\
&-(\partial^{h}\partial_c \log f)g_{1ki} - (\partial_c\partial_i \log f)\delta_k{}^{h}], \\
&C_{dcb}{}^{h} = 0, \quad C_{kji}{}^{a} = 0, \quad C_{kjb}{}^{a} = 0, \quad C_{kcb}{}^{a} = 0, \\
C_{kcb}{}^{h} &= \frac{1}{n-2}(R^{2}{}_{cb}\delta_k{}^{h} + e^{-f^2}R^{1}{}_{k}{}^{h}g_{2cb}) - \\
&- \frac{(n_2-2)f^2}{n-2}\{(2+f^2)(\partial_c \log f)(\partial_b \log f) + \nabla_{2c}\partial_b \log f\}\delta_k{}^{h} + \\
&+ \frac{(n_1-2)f^2 e^{-f^2}}{n-2}\{(2-f^2)(\partial_k \log f)(\partial_1{}^{h} \log f) + \\
&+\nabla_{1k}\partial_1{}^{h} \log f\}g_{2cb} + Bg_{2cb}\delta_k{}^{h}, \\
C_{kci}{}^{a} &= \frac{1}{n-2}(R^{1}{}_{ki}\delta_c{}^{a} + e^{f^2}R^{2}{}_{c}{}^{a}g_{1ki}) + \\
&+ \frac{(n_2-2)f^2 e^{f^2}}{n-2}\{(2+f^2)(\partial_c \log f)(\partial_2{}^{a} \log f) + \\
&+\nabla_{2c}\partial_2{}^{a} \log f\}g_{1ki} - \frac{(n_1-1)f^2+}{n-2}\{(2-f^2)(\partial_k \log f)(\partial_i \log f) + \\
&+(\nabla_{1k}\partial_i \log f\}\delta_c{}^{a} + Cg_{1ki}\delta_c{}^{a}, \\
C_{kcb}{}^{a} &= -\frac{(n_1-1)f^2}{n-2}[2(\partial_k \log f)\{(\partial_b \log f)\delta_c{}^{a} - (\partial_2{}^{a} \log f)g_{2cb}\} \\
C_{dcb}{}^{a} &= R^{2}{}_{dcb}{}^{a} + \frac{1}{n-2}T_{dcb}{}^{a} + \frac{n_1 f^2(2+f^2)}{n-2}S_{dcb}{}^{a} - \\
&- \frac{n_1 f^2}{n-2}\{(\nabla_{2c}\partial_2{}^{a} \log f)g_{2db} - (\nabla_{2d}\partial_2{}^{a} \log f)g_{2cb} + \\
&+(\nabla_{2d}\partial_b \log f)\delta_c{}^{a} - (\nabla_{2c}\partial_b \log f)\delta_d{}^{a} + (\nabla_{2c}\partial_2{}^{a} \log f)g_{2db} - \\
&-(\nabla_{2c}\partial_2{}^{a} \log f)g_{2cb}\} + \frac{R}{(n-1)(n-2)}(g_{2cb}\delta_d{}^{a} - g_{2db}\delta_c{}^{a})
\end{aligned} \tag{2.2}
$$

where we put

$$
\begin{aligned}
T_{kji}{}^{h} &= R^{1}{}_{ki}\delta_{j}{}^{h} - R^{1}{}_{ji}\delta_{k}{}^{h} + R^{1}{}_{j}{}^{h}g_{1ki} - R^{1}{}_{k}{}^{h}g_{1ji},\\
T_{dcb}{}^{a} &= R^{2}{}_{db}\delta_{c}{}^{a} - R^{2}{}_{cb}\delta_{d}{}^{a} + R^{2}{}_{c}{}^{a}g_{2db} - R^{2}{}_{d}{}^{a}g_{2cb},\\
S_{kji}{}^{h} &= (\partial_1 \log f_1)\{(\partial_j \log f_1)\delta_k{}^h - (\partial_k \log f_1)\delta_j{}^h\}+\\
&\quad +(\partial_1{}^h \log f_1)\{\partial_k \log f_1)g_{1ji} - (\partial_j \log f_1)g_{1ki}\\
S_{dcb}{}^{a} &= (\partial_b \log f_2)\{(\partial_c \log f_2)\delta_d{}^a - (\partial_d \log f_2)\delta_c{}^a\}+\\
&\quad +(\partial_2{}^a \log f_2)\{\partial_d \log f_2)g_{2cb} - (\partial_c \log f_2)g_{2db},
\end{aligned}
\tag{2.3}
$$

and A, B and C are respectively the coefficient of $g_{1ji}\delta_k{}^h - g_{1k}\delta_j{}^h$, $g_{2db}\delta_k{}^h$ and $g_{1ki}\delta_c{}^a$.

3 Conformally flat twisted product manifolds

A Riemannian manifold (M, g) is called *conformally flat* if its conformal curvature vanishes. If $\dim M = 3$, the conforal curvature tensor vanishes, identically. So, in this case, the manifold M is conformally flat if anf only if

$$
\nabla_\nu R_{\mu\lambda} - \nabla_\mu R_{\nu\lambda} + \frac{1}{4}\{(\nabla_\nu R)g_{\mu\lambda} - (\nabla_\mu R)g_{\nu\lambda}\} = 0 \quad ([5]).
\tag{3.1}
$$

The following lemma is very useful to prove our theorem.

Lemma 3.1. *An n $(n > 3)$-dimensional Riemannian manifold M is conformally flat if and only if the Riemannian curvature tensor $R_{\omega\nu\mu}{}^{\lambda}$ satisfies*

$$
\begin{aligned}
R_{\omega\nu\mu}{}^{\lambda} + \frac{1}{n-2}(R_{\omega\mu}\delta_\nu{}^\lambda - R_{\nu\mu}\delta_\omega{}^\lambda + R_\nu{}^\lambda g_{\omega\mu} - R_\omega{}^\lambda g_{\nu\mu})+\\
+A(g_{\nu\mu}\delta_\omega{}^\lambda - g_{\omega\mu}\delta_\nu{}^\lambda) = 0
\end{aligned}
\tag{3.2}
$$

for a certain function A.

A 3-dimensional Riemannian manifold M is conformally flat if and only if the Ricci tensor $R_{\mu\lambda}$ satsifies

$$
\nabla_\nu R_{\mu\lambda} - \nabla_\mu R_{\nu\lambda} + T_\nu g_{\mu\lambda} - T_\mu g_{\nu\lambda} = 0
\tag{3.3}
$$

for a certain vector field T_λ.

Proof. About (3.2), we can easily get

$$
A = \frac{R}{(n-1)(n-2)}
$$

which means $M(\dim M > 3)$ is conformally flat.

For about (3.3), using the Bianchi identity, we can easily see

$$T_\lambda = \frac{1}{4}\nabla_\lambda R$$

which means (3.1).

Let the twisted product manifold $M = M_1 \times_f M_2$ be conformally flat and we assume that $\dim M_1 > 3$. Since the conformal curvature tensor $C_{\omega\nu\mu}{}^\lambda = 0$, we have from $C_{kci}{}^a = 0$

$$(3.4)\quad (R^1{}_{ki}\delta_c{}^a + e^{f^2}R^2{}_c{}^a g_{1ki}) + (n_2-2)f^2e^{f^2}\{(2+f^2)(\partial_c \log f)(\partial_2{}^a \log f)+$$
$$+\nabla_{2c}\partial_2{}^a \log f\}g_{1ki} - (n_1-2)f^2\{(2-f^2)(\partial_k \log f)\partial_i \log f) + \nabla_{1k}\partial_i \log f\}\delta_c{}^a -$$
$$-f^2\{(2-2f^2+n_1f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f + (2+n_1f^2)e^{f^2}\|\nabla_2 \log f\|^2+$$
$$+e^{f^2}\nabla_{2e}\partial_2{}^e \log f\}g_{1ki}\delta_c{}^a - \frac{e^{f^2}R}{n-1}g_{1ki}\delta_c{}^a = 0.$$

Contracting (3.4) by c and a, we obtain

$$(3.5)\quad n_2R^1{}_{ki} + e^{f^2}R^2g_{1ki} + (n_2-2)f^2e^{f^2}\{(2+f^2)\|\nabla_2 \log f\|^2+$$
$$+\nabla_{2e}\partial_2{}^e \log f\}g_{1ki} - n_2(n_1-2)f^2\{(2-f^2)(\partial_k \log f)(\partial_i \log f)+$$
$$+\nabla_{1k}\partial_i \log f\} - n_2f^2\{(2-2f^2+n_1f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f+$$
$$+(2+n_1f^2)e^{f^2}\|\nabla_2 \log f\|^2 + e^{f^2}\nabla_{2e}\partial_2{}^e \log f\}g_{1ki} - \frac{n_2e^{f^2}R}{n-1}g_{1ki} = 0.$$

Moreover, a transvection of (3.5) by $g_1{}^{ki}$ gives us

$$(3.6)\quad e^{f^2}R^2 + (n_2-2)f^2e^{f^2}\{(2+f^2)\|\nabla_2 \log f\|^2 + \nabla_{2e}\partial_2{}^e \log f\}-$$
$$-n_2f^2\{(2-2f^2+n_1f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f+$$
$$+(2+n_1f^2)e^{f^2}\|\nabla_2 \log f\|^2 + e^{f^2}\nabla_{2e}\partial_2{}^e \log f\} - \frac{n_2e^{f^2}R}{n-1} =$$
$$= \frac{n_2}{n_1}R^1 + \frac{n_2(n_1-2)f^2}{n_1}\{(2-f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f\}.$$

Substituting the above equation into (3.5), we get

$$(3.7)\quad f^2\{(2-f^2)(\partial_k \log f)(\partial_i \log f) + \nabla_{1k}\partial_i \log f\} = \frac{1}{n_1-2}[R^1{}_{ki}-$$
$$-\frac{R^1}{n_1} + \frac{(n_2-2)f^2}{n_1}\{(2-f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f\}g_{1ki}].$$

Using (3.7), we have

$$\frac{n_2f^2}{n-2}[\{(2-f^2)(\partial_k \log f)(\partial_i \log f) + \nabla_{1k}\partial_i \log f\}\delta_j{}^h -$$

$$-\{(2-f^2)(\partial_j \log f)(\partial_i \log f) + \nabla_{1j}\partial_i \log f\}\delta_k{}^h -$$
$$-\{(2-f^2)(\partial_k \log f)(\partial_1{}^h \log f) + \nabla_{1k}\partial^h \log f\}g_{ji} +$$
$$+\{(2-f^2)(\partial_j \log f)(\partial_1{}^h \log f) + \nabla_{1j}\partial^h \log f\}g_{ki}]$$
$$= \frac{n_2}{(n-2)(n_1-2)}\{R^1{}_{ki}\delta_j{}^h - R^1{}_{ji}\delta_k{}^h + R^1{}_j{}^h g_{1ki} - R^1{}_k{}^h g_{1ji}\} -$$
$$-\frac{2n_2}{(n-2)(n_1-2)}[\frac{R^1}{n_1} + \frac{(n_1-2)f^2}{n_1}\{(2-f^2)\|\nabla_1 \log f\|^2 +$$
$$+\nabla_{1l}\partial_1{}^l \log f\}](g_{1ki}\delta_j{}^h - g_{1ji}\delta_k{}^h).$$

Substituting the above equation into $C_{kji}{}^h = 0$, we have

$$R^1{}_{kji}{}^h + \frac{1}{n_1-2}(R^1{}_{ki}\delta_j{}^h - R^1{}_{ji}\delta_k{}^h + R^1{}_j{}^h g_{1ki} - \tag{3.8}$$
$$-R^1{}_k{}^h g_{1ji}) + A_1(g_{1ki}\delta_j{}^h - g_{1ji}\delta_k{}^h) = 0$$

for a certain function A_1.

For the manifold M_2, using $(2.2)_6$ and $(2.2)_8$, we can prove

$$R^2{}_{dcb}{}^a + \frac{1}{n_2-2}(R^2{}_{db}\delta_c{}^a - R^2{}_{cb}\delta_d{}^a + R^2{}_c{}^a g_{2db} - \tag{3.9}$$
$$-R^2{}_d{}^a g_{2cb}) + A_2(g_{2db}\delta_c{}^a - g_{2cb}\delta_d{}^a) = 0$$

for a certain function A_2, too.

Next, let M_1 be a 3-dimensional one, that is, $n_1 = 3$. Then, by virtue of $(1.5)_1$, the Ricci tensor R_{ji} satisfies

$$R_{ih} = R^1{}_{ih} - f^2\{(2-f^2)(\partial_i \log f)(\partial_h \log f) + \nabla_{1i}\partial_h \log f\} + \tag{3.10}$$
$$+\{(2+f^2)\|\nabla_1 \log f\|^2 + \nabla_{1l}\partial_1{}^l \log f + (2+3f^2)e^{f^2}\|\nabla_2 \log f\|^2 +$$
$$+e^{f^2}(\nabla_{2e}\partial_2{}^e \log f)\}f^2 g_{1ih}.$$

Since $\nabla_j R_{ih}$ is defined by

$$\nabla_j R_{ih} = \frac{\partial R_{ih}}{\partial x^j} - \{{}_j{}^l{}_i\}R_{lh} - \{{}_i{}^l{}_h\}R_{jl} - \{{}_j{}^e{}_i\}R_{ie} - \{{}_i{}^e{}_h\}R_{je},$$

we know

$$\nabla_j R_{ih} - \nabla_i R_{jh} = \frac{\partial R_{ih}}{\partial x^j} - \frac{\partial R_{jh}}{\partial x^i} - \tag{3.11}$$
$$-\{{}_j{}^l{}_h\}R_{il} + \{{}_i{}^l{}_h\}R_{jl} - \{{}_j{}^e{}_h\}R_{ie} + \{{}_i{}^e{}_h\}R_{je}.$$

Using (3.10), we obtain

$$\frac{\partial R_{ih}}{\partial x^j} - \frac{\partial R_{jh}}{\partial x^i} = \frac{\partial R^1{}_{ih}}{\partial x^j} - \frac{\partial R^1{}_{jh}}{\partial x^i} - 2f^2\{(\partial_j \log f)(\nabla_{1i}\partial_h \log f) \tag{3.12}$$

$$-(\partial_i \log f)(\nabla_{1j}\partial_h \log f)\} - (2-f^2)f^2\{(\partial_i \log f)(\partial_j\partial_h \log f)-$$

$$-(\partial_j \log f)(\partial_i\partial_h \log f)\} - f^2\{(\partial_j\nabla_{1i}\partial_h \log f) - (\partial_i\nabla_{1j}\partial_h \log f)\}-$$

$$-2f^2\|\nabla_1 \log f\|^2\{(\partial_j \log f)g_{1ih} - (\partial_i \log f)g_{1jh}\}-$$

$$-(2+f^2)f^2\{(\partial_j\|\nabla_1 \log f\|^2)g_{1ih} - (\partial_i\|\nabla_1 \log f\|^2)g_{1jh}\}-$$

$$-f^2\{(\partial_j\nabla_{1l}\partial_1{}^l \log f)g_{1ih} - (\partial_i\nabla_{1l}\partial_1{}^l \log f)g_{1jh}\}-$$

$$-2(5+3f^2)f^4e^{f^2}\|\nabla_2 \log f\|^2\{(\partial_j \log f)g_{1ih} - (\partial_i \log f)g_{1jh}\}-$$

$$-(2+3f^2)f^2e^{f^2}\{(\partial_j\|\nabla_2 \log f\|^2)g_{1ih} - (\partial_i\|\nabla_2 \log f\|^2)g_{1jh}\}-$$

$$-2f^4e^{f^2}(\nabla_{2e}\partial_2{}^e \log f)\{(\partial_j \log f)g_{1ih} - (\partial_i \log f)g_{1jh}\}-$$

$$-f^2e^{f^2}\{(\partial_j\nabla_{2e}\partial_2{}^e \log f)g_{1ih} - (\partial_i\nabla_{2e}\partial_2{}^e \log f)g_{1jh}\}-$$

$$-2A_3\{(\partial_j \log f)g_{1ih} - (\partial_i \log f)g_{1jh}\} - A_3(\partial_j g_{1ih} - \partial_i g_{1jh}).$$

for a certain function A_3 on M. By virtue of (1.3) and (1.5), we have

$$-\{{}_j{}^l{}_h\}R_{il} + \{{}_i{}^l{}_h\}R_{jl} = -\{{}_j{}^l{}_h\}_1 R^1{}_{il} + \{{}_i{}^l{}_h\}_1 R^1{}_{jl}- \tag{3.13}$$

$$-f^2\{(\partial_j \log f)R^1{}_{ih} - (\partial_i \log f)R^1{}_{jh} - (\partial_1{}^l \log f)R^1{}_{il}g_{1jh} + (\partial_1{}^l \log f)R^1{}_{jl}g_{1ih}\}+$$

$$+f^2(2-f^2)\{(\partial_i \log f)\{{}_j{}^l{}_h\}_1(\partial_l \log f) - (\partial_j \log f)\{{}_i{}^l{}_h\}_1(\partial_l \log f)\}+$$

$$+f^2\{\{{}_j{}^l{}_h\}_1(\nabla_{1i}\partial_l \log f) - \{{}_i{}^l{}_h\}_1(\nabla_{1j}\partial_l \log f)\}+$$

$$+f^4\{(\partial_j \log f)(\nabla_{1i}\partial_h \log f) - (\partial_i \log f)(\nabla_{1j}\partial_h \log f)\}-$$

$$-f^4(2-f^2)\|\nabla_1 \log f\|^2\{(\partial_i \log f)g_{1jh} - (\partial_j \log f)g_{1ih}\}-$$

$$-f^4\{(\partial_1{}^l \log f)(\nabla_{1i}\partial_l \log f)g_{1jh} - (\partial_1{}^l \log f)(\nabla_{1j}\partial_l \log f)g_{1ih}\}+$$

$$+A_3\{\{{}_j{}^l{}_h\}_1 g_{1il} - \{{}_i{}^l{}_h\}_1 g_{1jl}\} + 2A_3f^2\{(\partial_j \log f)g_{1ih} - \partial_i \log f)g_{1jh}\}$$

and

$$-\{{}_j{}^e{}_h\}R_{ie} + \{{}_i{}^e{}_h\}R_{je} = 4f^4e^{f^2}\|\nabla_2 \log f\|^2\{(\partial_j \log f)g_{1ih}- \tag{3.14}$$

$$-(\partial_i \log f)g_{1jh}\} + f^4e^{f^2}\{(\nabla_{1j}\nabla_2 \log f\|^2)g_{1ih} - (\nabla_{1i}\nabla_2 \log f\|^2)g_{1jh}.\}$$

Substituting (3.12), (3.13) and (3.14) into (3.11), we obtain

$$\nabla_j R_{ih} - \nabla_i R_{jh} = \nabla_{1j}R^1{}_{ih} - \nabla_{1i}R^1{}_{jh}+ \tag{3.15}$$

$$+B_1\{(\partial_j \log f)g_{1ih} - (\partial_i \log f)g_{1jh}\} + f^2R^1{}_{jih}{}^l\partial_l \log f-$$

$$-\frac{1}{2}(4+f^2)f^2\{(\partial_j\|\nabla_1 \log f\|^2)g_{1ih} - (\partial_i\|\nabla_1 \log f\|^2)g_{1jh}\}-$$

$$-2(1+f^2)f^2\{(\partial_j\|\nabla_2 \log f\|^2)g_{1ih} - (\partial_i\|\nabla_2 \log f\|^2)g_{1jh}\}-$$

$$-f^2\{(\partial_j \nabla_{1l}\partial_1{}^l \log f)g_{1ih} - (\partial_i \nabla_{1l}\partial_1{}^l \log f)g_{1jh}\}-$$
$$-f^2 e^{f^2}\{(\partial_j \nabla_{2e}\partial_2{}^e \log f)g_{1ih} - (\partial_i \nabla_{2e}\partial_2{}^e \log f)g_{1jh}\}-$$
$$-f^2\{(\partial_j \log f)R^1{}_{ih} - (\partial_i \log f)R^1{}_{jh} - (\partial_1{}^l \log f)R^1{}_{il}g_{1jh} + (\partial_1{}^l \log f)R^1{}_{jl}g_{1ih}\}$$

for a certain function B_1 on M.

On the other hand, in any 3-dimensional Riemannian manifold, the conformal curvature tensor is zero, identically. So we have in the 3-dimensional manifold M_1

$$C^1{}_{kji}{}^h = R^1{}_{kji}{}^h + R^1{}_{ki}\delta_j{}^h - R^1{}_{ji}\delta_k{}^h + R^1{}_j{}^h g_{1ki} - R^1{}_k{}^h g_{1ji}+$$
$$+\frac{R^1}{2}(g_{1ji}\delta_k{}^h - g_{1ki}\delta_j{}^h) = 0,$$

where $C^1{}_{kji}{}^h$ denotes the conformal curvature tensor with respect to g_1.

Substituting the above equation into (3.15), we have

$$\nabla_j R_{ih} - \nabla_i R_{jh} = \nabla_{1j}R^1{}_{ih} - \nabla_{1i}R^1{}_{jh}+ \tag{3.16}$$
$$+B\{(\partial_j \log f)g_{1ih} - (\partial_i \log f)g_{1jh}\}-$$
$$-\frac{1}{2}(4+f^2)f^2\{(\partial_j\|\nabla_1 \log f\|^2)g_{1ih} - (\partial_i\|\nabla_1 \log f\|^2)g_{1jh}\}-$$
$$-2(1+f^2)f^2\{(\partial_j\|\nabla_2 \log f\|^2)g_{1ih} - (\partial_i\|\nabla_2 \log f\|^2)g_{1jh}\}-$$
$$-f^2\{(\partial_j \nabla_{1l}\partial_1{}^l \log f)g_{1ih} - (\partial_i \nabla_{1l}\partial_1{}^l \log f)g_{1jh}\}-$$
$$-f^2 e^{f^2}\{(\partial_j \nabla_{2e}\partial_2{}^e \log f)g_{1ih} - (\partial_i \nabla_{2e}\partial_2{}^e \log f)g_{1jh}\}+$$
$$+\frac{R^1}{2}f^2\{(\partial_j \log f)g_{1ih} - (\partial_i \log f)g_{1jh}\}.$$

Moreover, since M is conformally flat, we know

$$\nabla_j R_{ih} - \nabla_i R_{jh} = \frac{1}{n-1}\{(\partial_j R)g_{ih} - (\partial_i R)g_{jh}\}.$$

Thus we have from the above equation and (3.16)

$$\nabla_{1j}R^1{}_{ih} - \nabla_{1i}R^1{}_{jh} + T_j g_{1ih} - T_i g_{1jh} = 0 \tag{3.17}$$

for a certain vector field T_i on M_1.

When $\dim M_2 = 3$, we can get the similar equation with (3.17), that is,

$$\nabla_{2c}R^2{}_{ba} - \nabla_{2b}R^2{}_{ca} + T_c g_{2ba} - T_b g_{2ca} = 0 \tag{3.18}$$

for a certain vector field T_a on M_2.

Thus we have from (3.8), (3.9), (3.17), (3.18) and the lemma, we have our theorem.

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