Matter from Toric Geometry and its Search at the LHC

T.V. Obikhod

Abstract Toric geometry is applied for construction the enhanced gauge groups in F-theory compactified on elliptic Calabi-Yau fourfolds. The Hodge numbers calculated from the polyhedra for the chain $H = SU(1), ..., SU(5), SO(10), E_6, E_7$ determine the number of tensor multiplets, vector multiplets and hypermultiplets of solitonic states that appear from singularities of elliptic fibration. Due to duality between the compactification of $E_8 \times E_8$ heterotic string and the type IIA string compactification on a Calabi-Yau manifold there is a natural sequence of E-group embeddings which gives the matter content of Minimal Supersymmetric Standard Model and the possibility of searching for supersymmetry at the LHC.

Keywords Toric geometry · Calabi-Yau manifold · Singularities of elliptic fibration · Matter content

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1 Introduction

At high energy, the three gauge interactions which define the electromagnetic, weak, and strong interactions, are merged into one single interaction characterized by one larger gauge symmetry Grand Unified Theory, (GUT) [1] and thus one unified coupling constant. This gauge coupling unification works quite well in the Minimal Supersymmetric Standard Model (MSSM) motivating not only grand unification (or some superstring theories with similar properties) but also that supersymmetry emerges. Supersymmetry refers to possible relations between the spectrum and interactions of fermions (half-integer spin particles) and bosons (integer spin particles). It can be viewed as a space-time extension of the Poincare (Lorentz plus translational invariance) group, involving new anti-commuting dimensions. Under reasonable assumptions it is the unique extension of the usual Poincare and internal symmetries of field theory. Supersymmetry provides a possible route to unify gravity with the other interactions through superstring theory [2]. For a consistent formulation superstring theories require additional dimensions and there are actually a large number of string theories, many of which include underlying grand unification symmetries (GUT). One of the branches of string theory is F-theory [3] which allows string theorists to construct new realistic vacua - in the form of F-theory compactified on elliptically fibered Calabi-Yau fourfolds.

2 Reflexive Polyhedra as Geometric Realization of Calabi-Yau Fourfolds

The elegant algorithm for obtaining the enhanced gauge groups in F-theory has been proposed by Candelas et al. [4,5]. We consider F-theory of the form $R^{3,1}\times X$ [6,7], where X is a Calabi-Yau fourfold. We consider superstring compactification on the Calabi-Yau fourfold X represented as the elliptic fibration over a complex threefold - the Hirzebruch surface F_n,

$$\begin{array}{ccc} \varepsilon & \rightarrow & X \\ & & \downarrow \\ & & F_n \end{array}$$

We also take X to be a local elliptic $K3$-fibration over S of the form

$$\begin{array}{ccc} Y & \rightarrow & X \\ & & \downarrow \\ & & S \end{array}$$

where we model the local elliptically-fibered $K3$-surface Y on a hypersurface in $\mathbb{C}^3$ with an isolated ADE singularity. A great many Calabi-Yau manifolds are known in terms of toric data [8] and there is the correspondence, pointed out by Batyrev, between Calabi-Yau manifolds and reflexive polyhedra $\triangle$ [9]. To a Calabi-Yau manifold (of any dimension) defined as a hypersurface in a weighted projective space one can associate its Newton polyhedron, which we denote by $\triangle$. The Newton polyhedron is often reflexive and when it is we may define the dual or polar polyhedron which we denote by ∇. Polyhedra $\triangle$ and ∇ are reflexive, when fourfold X is simultaneously $K3$-fibration and can also be viewed as an elliptic fibration over the Hirzebruch surface F_{2k}. By means of a computer program we have computed the dual polyhedra for the fourfolds

$$X_{18k+18}(1,1,1,3k,6k+6,9k+9) \tag{1}$$

$(k = 1,\ldots,6)$.

The following observations summarize the structure of the polyhedron $\bigtriangledown$:

1. Omitting the first two points and the last point of $\bigtriangledown$ leaves us with the dual polyhedron ${}^3\bigtriangledown$ of the $K3$ surface $X_{12}(1,1,4,6)$;
2. Omitting the first three points and the last two points of $\bigtriangledown$ leaves us with the dual polyhedron ${}^2\bigtriangledown$ of the torus $X_6(1,2,3)$;
3. The polyhedron ${}^3\bigtriangledown$ is divided into a ‘top’ and a ‘bottom’ by the polyhedron ${}^2\bigtriangledown$ and we may write

$${}^3\bigtriangledown = \bigtriangledown^H_{bot} \cup \bigtriangledown^{k=1}_{top},$$

where $\bigtriangledown^{k=1}_{top}$ depends only on $k = 1$ while $\bigtriangledown^H_{bot}$ depends only on the enhanced gauge group H.

Let us introduce special points

$$\begin{aligned}
pt_1^{(j)} &= (\ 0,\ 0,-j,\ 2,\ 3)\ ,\\
pt_2^{(j)} &= (\ 0,\ 0,-j,\ 1,\ 2)\ ,\\
pt_3^{(j)} &= (\ 0,\ 0,-j,\ 1,\ 1)\ ,\\
pt_4^{(j)} &= (\ 0,\ 0,-j,\ 0,\ 1)\ ,\\
pt_5^{(j)} &= (\ 0,\ 0,-j,\ 0,\ 0)\ ,\\
pt_6^{(j)} &= (\ 0,\ 0,-j,-1,\ 0)\ ,\\
pt_7^{(j)} &= (\ 0,\ 0,-j,\ 0,-1)\ ,
\end{aligned}$$

where j is a positive integer.

We consider now the possibility of adding to ${}^2\bigtriangledown$ combinations of the points $pt_r^{(j)}$ in all possible ways such that the bottom corresponds to a reflexive polyhedron. Tables 1 and 2 show the allowed bottoms leading to enhanced gauge groups H. Note that in each case the points of ${}^2\bigtriangledown$ are understood and $pt_r^{(j)}$

implies the presence of $pt_r^{(j-1)}, \ldots, pt_r^{(1)}$.

Applying the analogous consideration to tops we obtain the following gauge content for fourfolds (1) :

$$\begin{array}{lll} H \times SU(1) & \text{for} & k=1, \\ H \times SO(8) & \text{for} & k=2, \\ H \times E_6 & \text{for} & k=3, \\ H \times E_7 & \text{for} & k=4, \\ H \times E_8 & \text{for} & k=5, \\ H \times E_8 & \text{for} & k=6. \end{array}$$

The relations between the bottoms and the enhanced gauge groups are presented in Table 1.

H	Bottom
$SU(1)$	$\{pt_1^{(1)}\}$
$SU(2)$	$\{pt_1^{(1)}, pt_2^{(1)}\}$
$SU(3)$	$\{pt_1^{(1)}, pt_2^{(1)}, pt_3^{(1)}\}$
$SU(4)$	$\{pt_1^{(1)}, pt_2^{(1)}, pt_3^{(1)}, pt_4^{(1)}\}$
$SU(5)$	$\{pt_1^{(1)}, pt_2^{(1)}, pt_3^{(1)}, pt_4^{(1)}, pt_5^{(1)}\}$
$SU(6)$	$\{pt_1^{(1)}, pt_2^{(1)}, pt_3^{(1)}, pt_4^{(1)}, pt_5^{(1)}, pt_6^{(1)}\}$
$SO(10)$	$\{pt_1^{(2)}, pt_2^{(2)}, pt_3^{(1)}, pt_4^{(1)}, pt_5^{(1)}\}$
E_6	$\{pt_1^{(3)}, pt_2^{(2)}, pt_3^{(2)}, pt_4^{(1)}, pt_5^{(1)}\}$
E_7	$\{pt_1^{(4)}, pt_2^{(3)}, pt_3^{(2)}, pt_4^{(2)}, pt_5^{(1)}\}$
E_8	$\{pt_1^{(6)}, pt_2^{(4)}, pt_3^{(3)}, pt_4^{(2)}, pt_5^{(1)}\}$

Table 1: The relation between the bottoms and the enhanced gauge groups.

3 Nesting of polyhedra

We consider now the possibility of adding combinations of the points in all possible ways such that the bottom corresponds to a reflexive polyhedron. All the groups do appear in Table 1 and, as a matter of consistency, we see the inclusions

$$E_7 \supset E_6 \supset SO(10) \supset SU(5) \supset SU(4) \supset SU(3) \supset SU(2) \supset SU(1) \ ,$$

as inclusions of the respective polyhedra. Thus we can see the duality between compactifications of the $E_8 \times E_8$ heterotic string and the type IIA string compactified on a Calabi-Yau manifold. The new contribution here is the observation that sequences of reflexive polyhedra associated to these spaces are nested in such a way as to reflect heterotic perturbative and non-perturbative processes. This qualitative observation is supported by quantitative agreement of the computed from the polyhedra Hodge numbers and the number of tensor, vector and hyper-multiplets in the heterotic side. It leads also to the observation that the Dynkin diagrams for the groups may be read off from the polyhedra. Table 2 for the chains of fourfolds demonstrates the nesting of polyhedra.

Group	Matter content
$SU(2)$	$(6n+2)\mathbf{2}$
$SU(3)$	$(6n+18)\mathbf{3}$
$SU(4)$	$(n+2)\mathbf{6}+(4n+16)\mathbf{4}$
$SU(5)$	$(3n+16)\mathbf{5}+(2+n)\mathbf{10}$
$SO(10)$	$(n+4)\mathbf{16}+(n+6)\mathbf{10}$
E_6	$(n+6)\mathbf{27}$
E_7	$(\frac{n}{2}+4)\mathbf{56}$

Table 2 : Matter content of models calculated from the polyhedra for the chain $H = SU(1), \ldots, SU(5), SO(10), E_6, E_7$.

For the chain $H = SU(1), \ldots, SU(5), SO(10), E_6, E_7$ we can determine the number of multiplets of solitonic states that appear from singularities of elliptic fibration. Nesting of polyhedra is interpreted as phase transition. This phase transition can be interpreted as a group chain of spontaneous symmetry breaking

$$SO(10) \to SU(5) \to SU(3) \times SU(2) \times U(1)$$

Group $SO(10)$ has group $SU(5)$ as its subgroup:

$$16 = 10 + \overline{5} + 1 \ .$$

Thus 16 multiplet of fermions consists of three $SU(5)$ multiplets. Particle content of Minimal Supersymmetric Standard Model (MSSM) can be presented in terms of $SU(5)$ multiplets

$$3 \times (24 + 5_H + \overline{5}_H + 5_M + \overline{5}_M + 10_M + \overline{10}_M) \ ,$$

where 5_H and $\overline{5}_H$ are Higgs multiplets, $\overline{5}_M$ and 10_M are multiplets of quark and lepton superpartners. The transition between $SU(5)$ and $SU(3) \times SU(2) \times U(1)$ can be realised with the help of partial widths of superpartners calculations. Using this restricted parameter set of MSSM model

$$\mathrm{M}_0 = 800 \text{ GeV} \ , \quad \mathrm{M}_{1/2} = 650 \text{ GeV} \ ,$$
$$A_0 = 0 \ , \quad \tan\beta = 10 \ , \quad \mathrm{sgn}(\mu) = +1 \ .$$

it is possible to calculate partial widths of superpartners by application of the computer program SDECAY [10]. These partial widths are shown in Tables 3, 4.

	channel	BR	channel	BR
$\tilde{u}_R$	$\tilde{\chi}_1^0 u$	0.997	$\tilde{\chi}_4^0 u$	0.002
$\tilde{d}_R$	$\tilde{\chi}_1^0 d$	0.997	$\tilde{\chi}_4^0 d$	0.002
$\tilde{c}_R$	$\tilde{\chi}_1^0 c$	0.997	$\tilde{\chi}_4^0 c$	0.002
$\tilde{s}_R$	$\tilde{\chi}_1^0 s$	0.997	$\tilde{\chi}_4^0 s$	0.002

Table 3 : Partial widths of superpartners

	channel	BR	channel	BR
$\tilde{g}$	$\tilde{b}_1 b^*$	0.074	$\tilde{t}_1 t^*$	0.425
	$\tilde{b}_1^* b$	0.074	$\tilde{t}_1^* t$	0.425

Table 4 : Partial widths of superpartners

4 Conclusion

Thus we see the duality between the compactifications of the $E_8 \times E_8$ heterotic string and the type IIA string compactification on a Calabi-Yau manifold. The observations concerning this duality are the subject of the present article: we have a correspondence between reflexive polyhedra, $\triangle$, and Calabi-Yau manifolds. Combining this with the correspondence between vector bundles and Calabi-Yau manifolds gives a correspondence between vector bundles and reflexive polyhedra $\triangle$. It is known also that the moduli spaces of Calabi-Yau manifolds form a web in which continuous transitions between different Calabi-Yau manifolds (phases) occur [11]. Indeed it seems likely that all Calabi-Yau manifolds are connected by processes of this type [12]. In virtue of duality this web structure must exist also on the heterotic side, though the physical picture is different. Indeed, the space of heterotic vacua also forms a web in which different models are connected along branches parametrized by vacuum expectation values of scalars in vector and hypermultiplets. Subsequently, it was argued that the appearance of perturbative heterotic groups in the chains could be explained in the Calabi-Yau picture [4]. Indeed, there is a natural sequence of E-group embeddings which give the Standard Model gauge group and matter structure in an elegant manner:

$$E_3 \times U(1) \subset E_4 \subset E_5 \subset E_6 \subset \ldots$$

where $E_3 = SU(3) \times SU(2)$ denotes the non-abelian gauge group of the Standard Model, $E_4 = SU(5)$ and $E_5 = SO(10)$. Both gauge coupling unification as well as the matter content of the Standard Model hint at the presence of a unified gauge group structure at high energies. From this perspective, we interpret the unification of the gauge couplings in the minimal supersymmetric extension of the Standard Model as evidence that instead of the infinitely complex variety of classical groups and matter which can appear in string theory, the list of relevant simple gauge groups is limited to the finite number of exceptional gauge groups and their subgroups. This provides an economical way to break the exceptional gauge group to gauge symmetries closer to the MSSM. One of the nicest qualities of supersymmetry is that so much is known about it already, despite the present lack of direct experimental evidence. The interactions of the Standard Model particles and their superpartners are fixed by supersymmetry, up to mass mixing effects due to supersymmetry breaking. Even the terms and stakes of many of the important outstanding questions, especially the paramount issue "How is supersymmetry broken?", are already rather clear. Analyses of LHC searches corresponding to ~ 5 fb^{-1} at $\sqrt{s} = 7$ TeV have not found any superpartners, and have put strong lower bounds on the masses of squarks and gluinos in large classes of models. The most common templates used for reporting the results of experimental searches are the MSUGRA scenario with new parameters $m_0, m_{1/2}, A_0, \tan\beta, \mathrm{Arg}(\mu)$. However, one should not lose sight of the fact that the only indispensable idea of supersymmetry is simply that of a symmetry between fermions and bosons. If supersymmetry is experimentally verified, the discovery will not be an end, but rather a beginning in high energy physics. It seems likely to present us with questions and challenges that we can only guess at presently. The measurement of sparticle masses, production cross-sections, and decay modes will rule out some models for supersymmetry breaking and lend credence to others. We will be able to test the principle of R-parity conservation, the idea that supersymmetry has something to do with the dark matter, and possibly make connections to other aspects of cosmology including baryogenesis and inflation.

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T.V. Obikhod

Institute for Nuclear Research, NAS of Ukraine 03680 Kiev, Ukraine

E-mail: obikhod@kinr.kiev.ua