DOI: http://dx.doi.org/10.15673/2072-9812.4/2014.41441
Proc. Intern. Geom. Center 2014 **7**(4) 34–39 $d\omega$

Correspondences of probability measures with restricted marginals

Aleksandr Savchenko **Mykhailo Zarichnyi**

Abstract We derive the proof of continuity of the correspondence of probability measures with restricted marginals from the property of bicommutativity in the sense of E. Shchepin of probability measure functor.

Keywords probability measure, product, bicommutative functor

Mathematics Subject Classification (2010) 4B30, 54G60

1 Introduction

In [1] it is proved that the correspondence assigning to every probability measures on two coordinate spaces the set of probability measures on the product is continuous. Earlier, a similar result was proved by Eifler [2] and Schief [6].

In this note we develop a different approach to this problem and apply some known properties of probability measures in order to prove a more general result. Note that problems of this type arise in mathematical economy (see, e.g., introduction in [1]). Consider the income distributions at the time period k as probability measures μ_k on a space Y of possible incomes. Then any redistribution policy can be interpreted as a probability measure, τ, on the product $Y \times \cdots \times Y$ such that the marginal distributions of τ are μ_i and this leads to the problem of welfare maximization for prescribed sequence $\mu_1, \ldots, \mu_k$ and dependence of this maximum on $\mu_1, \ldots, \mu_k$.

A part of this text circulated as a preprint of the second-named author (see also the preprint [3]). In this note, we consider the problem not in full generality, our aim is rather unveiling the basic idea, which consists in reducing the situation

DOI: http://dx.doi.org/10.15673/2072-9812.4/2014.41441

to the case of finite spaces. Remark that the methods used in this note are based on general properties of functors in the category of compact Hausdorff spaces and Shchepin's theory of uncountable inverse spectra [7].

2 Preliminaries

By 1_X we denote the identity map of X. Given a product $\prod_i X_i$, we denote by π_i its projection onto the ith coordinate.

Given a topological space X, denote by $\exp X$ its hyperspace, i.e., the set of nonempty compact subsets in X endowed with the Vietoris topology. A base of this topology consists of the sets of the form

$$\langle U_1, \dots, U_n\rangle = \{A \in \exp X \mid A \subset \cup_{i=1}^n U_i,\ A \cap U_i \neq \emptyset \text{ for all } i\},$$

where $U_1, \dots, U_n$ are open subsets in X and $n \in \mathbb{N}$.

Given a compact-valued map (correspondence) $F\colon X \to Y$, we regard it as a (single-valued) map from X into $\exp Y$. The continuity of the correspondence F is equivalent to the continuity of f if we endow $\exp Y$ with the Vietoris topology.

Every continuous onto map $f\colon X \to Y$ determines the inverse map $f^{-1}\colon Y \to \exp X$, $y \mapsto f^{-1}(y)$. It is a well-known fact that f is open if and only if f^{-1} is continuous.

2.1 Inverse systems and bicommutative diagrams

A commutative diagram

$$X[r]^f[d]_g Y[d]^u Z[r]_v T \tag{1}$$

is called bicommutative [5] if its characteristic map

$$\chi = (f, g)\colon X \to Y \times_T Z = \{(y, z) \in Y \times Z \mid u(y) = v(z)\}$$

is onto. The following lemma is proved by Shchepin [7].

Lemma 1 *Suppose that in diagram (1) the spaces X, Y, Z, T are compact, the maps f, g, u, v are continuous and g, u are onto. If f is an open map, then so is v.*

The necessary definitions and results concerning σ-spectra (inverse systems) can be found in [7]. Here we only recall that a morphism $(f_\alpha)_{\alpha\in\mathcal{A}}$ of an inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ into an inverse system $\mathcal{S}' = \{X'_\alpha, p'_{\alpha\beta}; \mathcal{A}\}$ is called bicommutative if, for every $\alpha \geq \beta$, the diagram

$$X_\alpha[r]^{f_\alpha}[d]_{p_{\alpha\beta}} X'_\alpha[d]^{p'_{\alpha\beta}} X_\beta[r]_{f_\beta} X'_\beta$$

is bicommutative.

In [7], it is proved that for any bicommutative morphism of σ-spectra consisting of open maps the limit map $\varprojlim(f_\alpha)\colon \varprojlim \mathcal{S} \to \varprojlim \mathcal{S}'$.

2.2 Probability measures and bicommutative diagrams

By P we denote the probability measure functor in the category **Comp** of compact Hausdorff spaces and continuous maps.

Lemma 2 *For arbitrary maps $f_i\colon X_i \to X_i'$, $i = 1, \dots, k$, the diagram*

$$P(\prod X_i)[d]_{P(\prod f_i)}[rr]^{M_{X_1,\dots,X_k}} \prod P(X_i)[d]^{\prod P(f_i)} P(\prod X_i')[rr]_{M_{X_1',\dots,X_k'}} \prod P(X_i) \tag{2}$$

is bicommutative. We will use the fact that P is a bicommutative functor in the sense that it preserves the class of bicommutative diagrams (see [6]).

Proof Given $\tau' \in P(\prod X_i')$ and $(\mu_1, \dots, \mu_k) \in \prod P(X_i')$ such that

$$M_{X_1',\dots,X_k'}(\tau') = \prod P(f_i)(\mu_1, \dots, \mu_k) = (P(f_1)(\mu_1), \dots, P(f_k)(\mu_k))$$

we proceed as follows.

For every $j \le k$ denote by $\mathcal{D}_i$ the diagram

$$\prod_{i \le j} X_i \times \prod_{i > j} X_i'[d]_{\prod_{i\le j} f_i \times 1_{\prod_{i>j} X_i'}}[rr]^{\pi_i} X_j[d]^{f_j} \prod X_i'[rr]_{\pi_j} X_j',$$

which is obviously bicommutative.

Since $P(\pi_1)(\tau') = P(f_1)(\mu_1)$, applying the functor P to the diagram $\mathcal{D}_1$ we find $\tau_1 \in P(X_1 \times \prod_{i>1} X_i')$ such that

$$P(\pi_1)(\tau_1) = \mu_1,\ P(f_1 \times 1_{\prod_{i>1} X_i'})(\tau_1) = \tau'.$$

Consider natural l, $1 \le l \le k$, and suppose that, for every $j < l$, we have defined $\tau_i \in P\left(\prod_{i \le j} X_i \times \prod_{i > j} X_i'\right)$ such that $P(\pi_j)(\tau_j) = \mu_j$ and $P\left(f_i \times 1_{\prod_{i>j} X_i'}\right)(\tau_j) = \tau_{j-1}$. Note that

$$\begin{aligned}
P(f_l)(\mu_l) =& P(\pi_l)(\tau') = P(\pi_l)\left(P\left(f_1 \times 1_{\prod_{i>1} X_i'}\right)\right) \\
=& \dots \\
=& P(\pi_l)\left(P\left(f_1 \times 1_{\prod_{i>1} X_i'} \dots P\left(\prod_{i \le l-1} f_i \times 1_{\prod_{i>l-1} X_i'}\right)\right)\right) \\
=& P(\pi_l)(\tau_{l-1}).
\end{aligned}$$

Applying the functor P to the bicommutative diagram $\mathcal{D}_j$ we conclude that there exists $\tau_l \in P\left(\prod_{i\le l} X_i \times \prod_{i>l} X'_i\right)$ such that $P(\pi_l)(\tau_l) = \mu_l$ and $P\left(\prod_{i\le l} f_i \times 1_{\prod_{i>l} X'_i}\right)(\tau_l) = \tau_{l-1}$.

It is easy to see that $\tau = \tau_k$ has the following properties: $M_{X_1\dots X_k}(\tau) = (\mu_1,\dots,\mu_k)$ and $P\left(\prod f_i\right) = \tau'$. This proves the bicommutativity of diagram (2).

3 Result

The following is the main result of this note.

Let $X_1,\dots,X_k$ be a finite sequence of compact spaces. Then the multivalued map assigning to every $\mu_1,\dots,\mu_k$, where $\mu_i \in P(X_i)$, $i = 1,\dots,k$, the set

$$M(\mu_1,\dots,\mu_k) = M_{X_1,\dots,X_k}(\mu_1,\dots,\mu_k) = \{\nu \in P\left(\prod X_i\right) \mid P(\pi_i) = \mu_i,\ i = 1,\dots,k\}$$

is continuous.

Proof Our proof consists of three steps.

1) Suppose that the spaces $X_1,\dots,X_k$ are finite. Then the map $M_{X_1,\dots,X_k}$ is an affine surjective map of compact convex polyhedra. In order to prove that every such map, say, $f\colon A \to B$ is open, it suffices to show that any point a of A lies in the image of a selection of f. Denote by C the union of simplices of the geometric boundary of B that do not contain the point $f(a)$. For every vertex c of a simplex in C let $g(c)$ be an arbitrary point of $f^{-1}(c)$. Extend the so-defined map g onto C affinely onto every simplex of C. Now, every point b in B can be uniquely represented in the form $tf(a) + (1-t)c$, where $c \in C$. Define $g(b) = ta + (1-t)g(c)$. We see that $fg = 1_B$ and $a \in g(B)$.

2) Suppose now that the spaces $X_1,\dots,X_k$ are zero-dimensional. Then, for each i, there exists an inverse σ-system $\mathcal{S}_i = \{X_{i\alpha}, p_{i\alpha\beta}; \mathcal{A}\}$ consisting of finite spaces and surjective maps such that $X_i = \varprojlim \mathcal{S}_i$, $i = 1,\dots,k$.

By Lemma 2, the maps $(M_{X_{1\alpha},\dots,X_{k\alpha}})_\alpha$ form a bicommutative morphism of the systems $\{P(\prod_i X_{i\alpha}), P(\prod_i p_{i\alpha\beta}); \mathcal{A}\}$ and $\{\prod_i P(X_{i\alpha}), \prod_i P(p_{i\alpha\beta}); \mathcal{A}\}$. The result of Shchepin mentioned above together with was proved in case 1) show that the limit map of the morphism, namely, the map $M_{X_1,\dots,X_k}$ is continuous.

3) $X_1,\dots,X_k$ are arbitrary compact Hausdorff spaces. Then there exist maps $f_i\colon Y_i \to X_i$, where Y_i are compact Hausdorff zero-dimensional spaces. Consequently applying Lemmas 1 and 2 we obtain the result.

ne can generalize the main result in different directions. First of all, the products in Theorem 3 need not be finite. The proof requires transfinite induction instead of finite one.

Second, one can replace the probability measure functor by another functors acting in the category **Comp** (see, e.g., the preprint [3]). Namely, consider the functor ccP defined as follows. The space $ccP(X)$ is the subspace in $\exp P(X)$ consisting of convex sets; for a map $f\colon X \to Y$, the map $ccP(f)\colon ccP(X) \to ccP(Y)$ acts by the formula $ccP(f)(A) = P(f)(A)$, for $A \in ccP(X)$. The proof that a counterpart of Theorem 3 holds also for the functor ccP consists in establishing a counterpart of Lemma 2 for the functor ccP and finite spaces $X_1, \dots, X_k$. Note that this approach leads to a proof which is simpler than that in [3].

The second-named author considered the functor of idempotent measures (Maslov measures) in the category **Comp** (see [8]). In [8], it is proved, in particular, that one cannot replace the probability measure functor by the idempotent measure functor in Theorem 3.

A functor in the category **Comp** is said to be open-bicommutative if this functor preserves the class of open-bicommutative diagrams, i.e., diagrams (1) for which the characteristic maps are onto and open. A more general notion of open multi-commutativity of functors is introduced in the preprint [4].

References

1. J. Bergin, On the continuity of correspondences on sets of measures with restricted marginals, Economic Theory, 13(1999), 471–481.
2. L.Q. Eifler, Some open mapping theorems for marginals, Trans. Amer. Math. Soc., 211 (1975), pp. 311–319.
3. R. Kozhan, Open-multicommutativity of some functors related to the functor of probability measures, preprint (arXiv:math/0409566).
4. R. Kozhan, M. Zarichnyi, Open-multicommutativity of the probability measure functor, preprint (arXiv:math/0409590).
5. K. Kuratowski, Topology, Vol. 1. - Academic Press, New York–London, 1996.
6. A. Schief, An open mapping theorem for marginals, J. Math. Analysis Appl., Volume 147, Issue 2, 1990, 506–511.
7. E.V. Shchepin, Functors and uncountable powers of compacta, Russian Mathematical Surveys, 1981, 36:3, 1–71.
8. M.M. Zarichnyi, Spaces and maps of idempotent measures, Izvestiya: Mathematics, 2010, 74:3, 481–499.

Aleksandr Savchenko
Kherson State Agrarian University, 23 Rozy Liuksemburg Str., 73006 Kherson, Ukraine
E-mail: savchenko1960@rambler.ru

Mykhailo Zarichnyi
Department of Mechanics and Mathematics, Lviv National University, Universytetska Str. 1, 79000 Lviv, Ukraine
E-mail: mzar@litech.lviv.ua