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On mim-spaces

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Abstract The notion of idempotent measure is a counterpart of that of probability measure in the idempotent mathematics. In this note, we consider a metric on the set of compact, idempotent measure spaces (mim-spaces) and prove that this space is separable and non-complete.

Keywords idempotent measure, probability measure, mm-space

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1 Introduction

The notion of metric measure space (i.e., a space endowed with a measure; briefly, mm-space) plays an important role in different parts of mathematics. This notion also has numerous applications in computer science, in particular, in computer vision.

The notion of probability measure has its counterpart in the idempotent mathematics; the latter is a part of mathematics in which the usual arithmetic operations are replaced by idempotent ones (e.g., max). Namely, in [3] there were defined the idempotent measures (called also Maslov measures).

In this note, we introduce the notion of metric, idempotent measure space (briefly, mim-space). Recently, there were defined the so-called idempotent fractals as (ultrametric) spaces endowed with idempotent measures [4]; they can be considered as natural examples of mim-spaces.

We define a metric on the set of all compact mim-spaces and prove that the obtained space of mim-spaces is a separable noncomplete space.

2 Preliminaries

We begin with the notion of idempotent measure and space of idempotent measures (see [6] for details).

Let (M, d) be a compact metric space. As usual, by $C(M)$ we denote the Banach space of continuous functions on M (with the sup-norm). Given $\lambda \in \mathbb{R}$, by λ_M we denote the constant function in $C(M)$ equal to λ. Consider the following operations:

$$\begin{aligned} &\odot\colon \mathbb{R} \times C(M) \to C(M): \quad (\lambda, \varphi) \mapsto \lambda_M + \varphi; \\ &\oplus\colon C(M) \times C(M) \to C(M): \quad (\psi, \varphi) \mapsto \max\{\psi, \varphi\}. \end{aligned}$$

A functional $\mu\colon C(M) \mapsto \mathbb{R}$ is called an *idempotent measure* if it satisfies the following properties:

1. $\mu(c_M) = c$;
2. $\mu(c \odot \varphi) = c \odot \mu(\varphi)$;
3. $\mu(\psi \oplus \varphi) = \mu(\psi) \oplus \mu(\varphi)$.

Consider some examples of idempotent measures. For any $x \in M$, we denote by δ_x the Dirac measure concentrated at x, i.e. $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(M)$. Clearly, $\delta_x \in I(M)$. More generally, given $x_1, \ldots, x_n \in M$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\max\{\lambda_1, \ldots, \lambda_n\} = 0$, one can define $\mu = \oplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I(M)$.

Denote by $I(M)$ the set of all idempotent measures on M. We consider the weak*-topology on $I(M)$; the base of this topology consists of the sets

$$\langle \mu; \varphi_1, \ldots, \varphi_n; \varepsilon \rangle = \{\nu \in I(M) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon,\ i = 1, \ldots, n\},$$

where $\mu \in I(M)$, $\varphi_i \in C(M)$, $i = 1, \ldots, n$, $\varepsilon > 0$.

Let $\mu \in I(M)$. The support of μ is a minimal (with respect to inclusion) closed set A in M such that $\mu(\varphi) = \mu(\psi)$ whenever $\varphi, \psi \in M(X)$ and $\varphi|A = \psi|A$. We denote the support of μ by $\mathrm{supp}(\mu')$.

Given a map $f\colon M \to M'$ of compact metric spaces, we define a map $I(f)\colon I(M) \to I(M')$ by the formula $I(f)(\mu)(\varphi) = \mu(\varphi f)$, $\mu \in I(M)$, $\varphi \in C(M')$. In particular, if $\mu = \oplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I(M)$, then

$$I(f)(\mu) = \oplus_{i=1}^n \lambda_i \odot \delta_{f(x_i)} \in I(M').$$

We thus obtain a functor I on the category of compact metrizable spaces and continuous maps [6].

The following is proved in [1].

Proposition 1 *If $f\colon X \mapsto Y$ is a non-expanding map, then $I(f)\colon I(X) \mapsto I(Y)$ is also a non-expanding map.*

By $\exp X$ we denote the set of nonempty compact subsets in a metric space (X, d) (the hyperspace of X). The Hausdorff metric d_H on $\exp X$ is defined by the formula:

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\},\ A, B \in \exp X.$$

The Gromov-Hausdorff distance between compact metric spaces X_1 and X_2 is defined as follows:

$$d_{GH}(X_1, X_2) = \inf\{d_H(f_1(X_1), f_2(X_2)) \mid f_i\colon X_i \to Z,\ i = 1, 2,\ \text{is an isometric embedding into a metric space } Z\}$$

(see, e.g., [2]).

3 mim-spaces

Here we introduce the notion of mim-space.

Definition 1 A mim-space is a triple (M, d, μ), where

1. (M, d) is a metric space;
2. μ is an idempotent measure on M;
3. $\mathrm{supp}(\mu) = M$.

We say that mim-spaces (M, d, μ) are (M', d', μ') *isomorphic*, if there exists an isometry $f\colon M_0 \mapsto M'_0$ such that

$$\psi * \mu = \mu'.$$

By $[(M, d, \mu)]$ we denote the class of all mim-spaces isomorphic to (M, d, μ) mim-spaces. Denote $\mathbb{M} = \{[(M, d, \mu)] \mid (M, d, \mu) \text{ is an mim-space}\}$.

In order to simplify notation we will identify every mim-space (M, d, μ) and the class $[(M, d, \mu)]$. This allows us to interpret $\mathbb{M}$ as a set.

Let us define a metric on $\mathbb{M}$. First, we recall the definition of the metric on $M(X)$ (see [1]). A function $\varphi\colon M \mapsto \mathbb{R}$ is called *n-Lipschitz*, if

$$|\varphi(x) - \varphi(y)| \le nd(x, y),\ x, y \in X.$$

Let $n \in \mathbb{N}$. It is known (see [1]) that the function $\hat{d}_n\colon I(M) \times I(M) \to \mathbb{R}$ defined by the formula

$$\hat{d}_n(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi \text{ is n-Lipschitz}\}$$

is a continuous pseudometric on the space $I(M)$.

The metric $\tilde{d}$ on $\mathbb{M}$ is defined by

$$\tilde{d}(\mu,\nu)=\sum_{n=1}^{\infty}\frac{\hat{d}_n(\mu,\nu)}{n2^n},\ \mu,\nu\in I(M).$$

Let

$$I_\omega(M)=\{\oplus_{i=1}^n\lambda_i\odot\delta_{x_i}\mid \lambda_1,\dots,\lambda_n\in\mathbb{R},\ \max\{\lambda_1,\dots,\lambda_n\}=0,x_1,\dots,x_n\in M,\ n\in\mathbb{N}\}.$$

In other words, $I_\omega(M)$ consists of elements of finite support in $I(M)$. It is known (see [6]) that the set $I_\omega(M)$ is dense in the space $I(M)$.

4 Metric on the set of mim-spaces

Let $(M_i,d_i,\mu_i),\ i=1,2$, be mim-spaces. Consider the function

$$D((M_1,d_1,\mu_1)(M_2,d_2,\mu_2))=\inf\Big\{\hat{d}(I(f_1)(\mu_1),I(f_2)(\mu_2))\mid$$
$$f_i\colon M_i\to Z \text{ is an isometric embedding}\}$$

on the set $\mathbb{M}$.

We first remark that D is a well-defined function on $\mathbb{M}\times\mathbb{M}$. To this end, we have to show that the set from the right side of the formula defining D is nonempty.

Indeed, let $M=M_1\times M_2$ and d be the metric on M defined by the formula $d((x_1,x_2),(y_1,y_2))=d_1(x_1,y_1)+d_2(x_2,y_2)$. Let $m_i^0\in M_i,\ i=1,2$. Define maps $f_i\colon M_i\to M,\ i=1,2$, by the formula $f_1(x)=(x,m_2^0),\ f_2(y)=(m_1^0,y)$. Clearly, f_1,f_2 are isometric embeddings.

Theorem 1 *The function D is a metric on $\mathbb{M}$.*

Proof Nonnegativity and symmetry of D are obvious.

We are going to prove nondegeneracy of D. Suppose that $D((M_1,d_1,\mu_1)(M_2,d_2,\mu_2))=0$. Then for every natural n there exists a compact metric space (Z_n,ϱ_n) and isometric embeddings $g_n\colon M_1\to Z_n$, $h_n\colon M_2\to Z_n$ such that

$$\lim_{n\to\infty}\hat{\varrho_n}(I(g_n)(\mu_1),I(h_n)(\mu_2))=0.$$

Without loss of generality, one may assume that $Z_n=M\sqcup M'_n$ and g_n is the inclusion map. Also, we assume that $M'_i\cap M'_j=\emptyset$ whenever $i\neq j$. Define $H=\cup_{n=1}^{\infty}Z_n$.

Define $\varrho\colon H\times H\to\mathbb{R}$ as follows:

$$\varrho(x,y)=\begin{cases}\varrho_i(x,y), & \text{if } x,y\in Z_i,\\ \inf\{\varrho_i(x,a)+\varrho_i(a,y)\mid a\in Z\}, & \text{if } x\in Z_i,\ y\in Z_j,\ i\neq j.\end{cases}$$

It is not difficult to show that ϱ is a metric on H and Z_n is a subspace of Z for every n.

We are going to prove that $Z_n\to Z$ in the hyperspace $\exp H$. Suppose the contrary. Without loss of generality, one may assume that there exists $\varepsilon>0$ and a nonempty open subset U of M_2 such that $h_n(U)$ lies in the complement of the ε-neighborhood of Z in H. Since $\operatorname{supp}(\mu_2)=M_2$, there exists a function $\varphi\in C(M_2)$ such that $\operatorname{supp}(\varphi)\subset U$ and $\mu_2(\varphi)=c\neq 0$.

Define $\psi\colon H\to\mathbb{R}$ as follows: $\psi(x)=\varphi h_n^{-1}(x)$ if $x\in Z_n$ and $\psi(x)=0$ otherwise. Then $I(h_n)(\mu_2)(\psi)=c$, for every n, and $\mu_1(\psi)=0$. We therefore obtain a contradiction.

Thus, $Z_n\to Z$ in the hyperspace $\exp H$ and therefore H is compact. Let $\{x_i\mid i\in\mathbb{N}\}$ be a dense set in M_2. By induction, we construct monotonically increasing subsequences $S_1\supset S_2\supset\dots$ such that the sequence $(h_n(x_i))_{n\in S_i}$ is convergent. Denote its limit by y_i. Clearly, the map $x_i\mapsto y_i$, $i\in\mathbb{N}$, is an isometry. It has a unique extension $u\colon M_2\to M_1$, which is also an isometry such that $I(u)(\mu_2)=\mu_1$.

Let us prove the triangle inequality. Suppose that (X_i,d_i,μ_i), $i=1,2,3$, are mim-spaces,

$$D((X_1,d_1,\mu_1),(X_2,d_2,\mu_2))=a,\ \ D((X_2,d_2,\mu_2),(X_3,d_3,\mu_3))=b.$$

Given $\varepsilon>0$, find metric spaces (Y_1,ϱ_1), (Y_2,ϱ_2) and isometric embeddings

$$f_1\colon X_1\to Y_1,\ \ f_2\colon X_2\to Y_1, f_3\colon X_2\to Y_2,\ \ f_4\colon X_3\to Y_2$$

such that

$$\hat\varrho(I(f_1)(\mu_1),I(f_2)(\mu_2))<a+\varepsilon,\ \ \hat\varrho(I(f_3)(\mu_2),I(f_4)(\mu_3))<b+\varepsilon.$$

Without loss of generality, one may assume that

$$Y_1=f_1(X_1)\sqcup f_2(X_2),\ \ Y_2=f_3(X_2)\sqcup f_4(X_3).$$

Define $Y=(Y_1\sqcup Y_2)/\sim$, where the equivalence relation $\sim$ is defined by the condition: $Y_1\ni y\sim f_3(f_2^{-1}(y))\in Y_2$. Let $q\colon Y_1\sqcup Y_2\to Y$ be the quotient map.

Let a metric d on Y be defined by the conditions:

$$d(y,z)=\begin{cases} d_i(y,z), & \text{if } y,z\in q(f_i(X_i)),\ i=1,2,\\ \inf\{d_1(y,a)+d_2(a,z)\mid a\in q(Y_1)\cap q(Y_2)\}, & \text{if } y\in q(Y_1)\setminus q(Y_2),\\ & z\in q(Y_2)\setminus q(Y_1). \end{cases}$$

It is easy to see that d is a metric on Y. Then

$$\begin{aligned} D((X_1,d_1,\mu_1),(X_3,d_3,\mu_3)) &\le \hat{d}(I(qf_1)(\mu_1),I(qf_3)(\mu_3))\\ &\le \hat{d}(I(qf_1)(\mu_1),I(qf_2)(\mu_2))+\hat{d}(I(qf_2)(\mu_2),I(qf_4)(\mu_3))\\ &= \hat{d}(I(qf_1)(\mu_1),I(qf_2)(\mu_2))+\hat{d}(I(qf_3)(\mu_2),I(qf_4)(\mu_3))\\ &= \varrho_1(I(f_1)(\mu_1),I(f_2)(\mu_2))+\varrho_2(I(f_3)(\mu_3),I(f_4)(\mu_3))\\ &< a+b+2\varepsilon \end{aligned}$$

(here we used the fact that the functor I preserves isometries; this easily follows from Proposition 1). Letting $\varepsilon\to 0$, we are done.

The following statement is an immediate consequence of the definition.

Proposition 1 *Let X_1,X_2 be closed subspaces of a metric space (Y,d). If $\mu_1,\mu_2\in I(Y)$, then*

$$D((\mathrm{supp}(\mu_1),\mu_1,d|(\mathrm{supp}(\mu_1)\times\mathrm{supp}(\mu_1)),(\mathrm{supp}(\mu_2),\mu_2,d|(\mathrm{supp}(\mu_2)\times\mathrm{supp}(\mu_2))))\\ \le\tilde{d}(\mu_1,\mu_2).$$

We say that an idempotent measure $\mu=\oplus_{i=1}^k\alpha_i\odot\delta_{x_i}$ is rational if $\alpha_i\in\mathbb{Q}$, for every $i=1,\dots,k$.

Proposition 2 *The space $\mathbb{X}$ of all mim-spaces is separable.*

Proof We let

$$\mathbb{Y}=\{(X,\mu,d)\mid X \text{ is finite},\ d(X\times X)\subset\mathbb{Q},\ \mu \text{ is rational}\}.$$

Let $X=\{x_1,\dots,x_k)$ and let d be a metric on X. For any $\varepsilon>0$, one can find a metric space $Y=\{y_1,\dots,y_k\}$ (we will denote its metric by ϱ) with rational distances and such that $d_{GH}(X,Y)<\varepsilon$. Without loss of generality, one may assume that X and Y are subspaces of a common metric space (we will denote its metric by D) such that $D(x_i,y_i)<\varepsilon$, for every $i=1,\dots,k$.

Given a rational $\mu=\oplus_{i=1}^k\alpha_i\odot\delta_{x_i}$, define $\nu=\oplus_{i=1}^k\alpha_i\odot\delta_{y_i}$. Let φ_n be an n-Lipschitz function on Z. Then it is easy to see that $|\mu(\varphi_n)-\nu(\varphi_n)|\le n\varepsilon$. Therefore, $d(\mu,\nu)\le\sum_{n=1}^\infty\frac{n\varepsilon}{n2^n}=\varepsilon$.

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Proposition 3 *The space $\mathbb{X}$ is not complete.*

Proof Consider a sequence of mim-spaces $((X_i, d_i, \mu_i))_{i=1}^{\infty}$, where:

1. $X_i = \{0, 1, \dots, i\} \subset \mathbb{R}$;
2. the metric d_i on X_i is inherited from $\mathbb{R}$;
3. $\mu_i = \oplus_{k=0}^{i} \alpha_i \oplus \delta_i$, where $\alpha_0 = 0$ and $\alpha_i \in (-\infty, 0]$ is such that $\hat{d}(\mu_{i-1}, \mu_i) \le 2^{-i}$; moreover, $\alpha_0 \ge \alpha_1 \ge \dots$.

In order to choose α_i, $i > 0$, by induction so that (3) is satisfied note that $\lim_{k\to\infty} \mu_{i-1} \oplus (k \odot \delta_i) = \mu_{i-1}$. Note also that (3) and Proposition 1 imply that

$$D((X_{i-1}, d_{i-1}, \mu_{i-1}), (X_i, d_i, \mu_i)) \le \hat{d}(\mu_{i-1}, \mu_i) \le 2^{-i}.$$

Now we are going to show that the sequence $((X_i, d_i, \mu_i))_{i=1}^{\infty}$ is not convergent. Suppose the contrary and denote the limit by (X, d, μ). Let C be an integer number with $C \ge \operatorname{diam}(X)$.

Without loss of generality, one may assume that $X \cup (\bigcup_{i=1}^{\infty} X_i \subset Y$, for some metric space (Y, ϱ), and the following are satisfied:

1. the metric d_i on X_i is inherited from Y;
2. $\lim_{i\to\infty} \mu_i = \mu$ (in the sense that $\lim_{i\to\infty} \hat{\varrho}(\mu_i, \mu) = 0$).

Let U denote the closed 1-neighborhood of X in Y. Clearly, the function $\psi_n \colon Y \to \mathbb{R}$, $\psi_n(y) = \varrho(y, X)$ is an n-Lipschitz function. For every $i \ge C+3$ find $j(i) \le C+3$ such that $x_{j(i)} \in X_i \setminus U$. Let $n > -\alpha_{C+3} + 1$ be a natural number. Then $\mu_i(\psi_n) \ge n + \alpha_{j(i)}$ and, since $\mu(\psi_n) = 0$, we see that

$$\hat{\varrho}(\mu_i, \mu) \ge |\mu(\psi_n)| \ge \left| \frac{n + \alpha_{j(i)}}{n2^n} \right| \ge \left| \frac{n + \alpha_{C+3}}{n2^n} \right| \ge \frac{1}{n2^n}.$$

This contradicts to the assumption that $\lim_{i\to\infty} \mu_i = \mu$.

Remarks

One can consider another metric on the space $I(M)$, for a compact metric space (M, d). Namely,

$$\check{d}(\mu, \nu) = \oplus_{n=1}^{\infty} \frac{\hat{d}_n(\mu, \nu)}{n2^n}, \ \mu, \nu \in I(M).$$

One can similarly prove that counterparts of the above results are also valid for this metric.

It is known that the space of mm-spaces is complete and separable (see, e.g., [5]). We do not know, however, what is a geometric model for this space. The same question is open also for the (completed) space of mim-spaces.

Another open problem is that of description of the elements of the completion of the space $\mathbb{M}$.

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