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Triples of infinite iterations of hyperspaces of max-plus compact convex sets

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Abstract. Geometry of the infinite iterated hyperspace of compact max-plus convex sets, their completions and compactifications is investigated.

1. Introduction

In [10] H. Toruńczyk and J. West investigated the construction of the iterated hyperspace functor. For a compact metric space X, this construction leads to the metric direct limit X' of the sequence

$$X \to \exp X \to \exp^2 X \to \dots,$$

where every map is the singleton embedding $x \mapsto \{x\}$. In particular, they proved that, for any Peano continuum X, the completion X^* of X' is homeomorphic to the separable Hilbert space ℓ^2.

The paper [14] is devoted to the construction of iterated superextension (the superextension functor was defined by J. de Groot [3]). It turned out that the completed infinite iterated superextension admits a natural compactification, which is the inverse limit of iterated superextensions. This result was considerably generalized by V. V. Fedorchuk [4]. He introduced the notion of perfectly metrizable functor and described the topology of obtained triples comprised of infinite iterations, their completions, and compactifications by means of inverse systems.

As a partial case, Fedorchuk considered the probability measure functor P. The direct and inverse sequences of iterated spaces of probability measures were also considered in [11], [12]. R. Mirzakhanyan [7], [8] investigated the case of the inclusion hyperspace functor.

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In [2] Ta Khac Cu proved counterparts of the results from [10] for the case of hyperspace of compact convex subsets in normed spaces.

The aim of this note is to extend results of [2] onto the case of the so-called max-plus convexity (see the definition below).

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2. Preliminaries

All spaces are assumed to be metrizable topological spaces. Let (X, d) be a metric space. By $\exp X$ we denote the hyperspace of a space X, i.e., the set of all nonempty compact subsets in X endowed with the Hausdorff metric d_{H}:

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

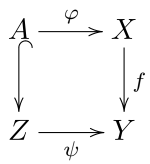
The Hausdorff metric on $\exp^2 X = \exp\exp X$ induced by the (Hausdorff) metric d_{H} will be denoted by d_{HH}.

By $Q = [0, 1]^\omega$ we denote the Hilbert cube. A closed set A in Q is called a *Z-set* in Q if the identity map of Q can be approximated by maps whose images miss A. A subset $A \subset Q$ is called a *Z-skeletoid* [1] if $A = \cup_{i=1}^\infty A_i$, where $A_1 \subset A_2 \subset \dots$ is a sequence of Z-sets satisfying the condition: for each $\varepsilon > 0$, $n \in \mathbb{N}$ and a Z-set $C \subset Q$ there exist $m > n$ and an autohomeomorphism $\psi_\varepsilon \colon Q \to Q$ such that

(1) $d(\psi_\varepsilon, \mathrm{id}) < \varepsilon$;
(2) $\psi_\varepsilon|_{C \cap A_n} = \mathrm{id}$;
(3) $\psi_\varepsilon(C) \subset A_m$.

(here d denotes a fixed compatible metric on Q). See [1] for the necessary properties of Z-skeletoids in Q.

Recall that a map $f \colon X \to Y$ is called *soft* [9] provided that for every commutative diagram



such that Z is a paracompact space and A is a closed subset of Z there exists a map $\Phi \colon Z \to X$ such that $f \circ \Phi = \psi$ and $\Phi|_A = \varphi$.

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and let τ be a cardinal number. Given $x, y \in \mathbb{R}^\tau$ and $\lambda \in \mathbb{R}$, we denote by $x \oplus y$ the coordinate wise maximum of x and y and by $\lambda \odot x$ the vector obtained from x by adding λ to every its coordinate. A subset A in $\mathbb{R}^\tau$ is said to be *max-plus convex* if $\alpha \odot a \oplus \beta \odot b \in A$ for all $a, b \in A$ and $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$. See, e.g., [6] for the history and applications of max-plus convexity.

A *max-plus convex body* in $\mathbb{R}^n$ is a max-plus convex set in $\mathbb{R}^n$ which is the closure of its interior.

The hyperspace of all compact max-plus convex subsets of $X \subset \mathbb{R}^\tau$ is denoted by $\mathrm{mpcc}(X)$.

Remark that there is a natural max-plus (respectively, max-min) convex structure on the hyperspace $\mathrm{mpcc}(X)$, where X is a max-plus (respectively max-min) convex compact subset of $\mathbb{R}^\alpha$, $1 \le \alpha \le \omega$.

Given a subset $\mathcal{A}$ of the hyperspace $\mathrm{mpcc}(X)$, we say that $\mathcal{A}$ is *max-plus convex* if, for every $A_1, \dots, A_n \in \mathcal{A}$ and every $\alpha_1, \dots, \alpha_n \in [-\infty, 0]$ with $\oplus_{i=1}^n \alpha_i = 0$, we have

$$\oplus_{i=1}^n \alpha_i \odot A_i = \{\oplus_{i=1}^n \alpha_i \odot a_i \mid a_i \in A_i,\ i = 1, \dots, n\} \in \mathcal{A}.$$

Remark that the set $\oplus_{i=1}^n \alpha_i \odot A_i$ is easily seen to be an element of the hyperspace $\mathrm{mpcc}(X)$. We denote by $\mathrm{mpcc}^2(X)$ the set of nonempty closed max-plus convex subsets in $\mathrm{mpcc}(X)$.

One can similarly define the iterations $\mathrm{mpcc}^m(X)$, $m \ge 3$.

3. Infinite iterated hyperspaces

Let $\mathrm{mpcc}^2(X)$ denote the set of all nonempty closed convex subsets in $\mathrm{mpcc}(X)$, where X is a compact max-plus convex subspace in $\mathbb{R}^n$, $n \ge 1$. We endow $\mathbb{R}^n$ with the ℓ_∞-metric: if

$$x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n,$$

then $d(x, y) = \max_i |x_i - y_i|$. Note that the union map

$$u_X \colon \mathrm{mpcc}^2(X) \to \mathrm{mpcc}(X)$$

is well-defined. Indeed, if $\mathcal{A} \in \mathrm{mpcc}(X)$ and for any $a, b \in u_X(\mathcal{A})$ there are $A, B \in \mathcal{A}$ such that $a \in A$ and $b \in B$. Since $\mathcal{A}$ is max-plus convex, for any $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$ we have $\alpha \odot A \oplus \beta \odot B \in \mathcal{A}$ and therefore $\alpha \odot a \oplus \beta \odot b \in u_X(\mathcal{A})$.

Lemma 3.1. *For every $a \in X$ and $\mathcal{A} \in \mathrm{mpcc}(X)$,*

$$d_{\mathrm{HH}}(\{\{a\}\}, \mathcal{A}) = d_{\mathrm{H}}(\{a\}, u_X(\mathcal{A})).$$

Proof. First,

$$d_{\mathrm{HH}}(\{\{a\}\}, \mathcal{A}) = \sup\{d_{\mathrm{H}}(\{a\}, A) \mid A \in u_X(\mathcal{A})\} \le d_{\mathrm{H}}(\{a\}, u_X(\mathcal{A})).$$

On the other hand, if $d_{\mathrm{HH}}(\{\{a\}\}, \mathcal{A}) < r$, then $O_r(a) \supset B$, for every $B \in \mathcal{A}$. Therefore, $O_r(a) \supset u_X(\mathcal{A})$ and thus $d_{\mathrm{H}}(\{a\}, u_X(\mathcal{A})) < r$. This proves the reverse inequality. □

Proposition 3.2. *The map u_X is soft.*

Proof. First we show that $u_X^{-1}(A)$ is max-plus convex for any $A \in \mathrm{mpcc}(X)$. If $\mathcal{B}, \mathcal{C} \in u_X^{-1}(A)$ and $\beta, \gamma \in \mathbb{R}_{\max}$ with $\beta \oplus \gamma = 0$, then we have

$$\beta \odot \mathcal{B} \ \oplus \ \gamma \odot \mathcal{C} \ = \ \{\beta \odot B \ \oplus \ \gamma \odot C \mid B \in \mathcal{B},\ C \in \mathcal{C}\}.$$

Given $x \ \in \ \beta \odot B \ \oplus \ \gamma \odot C \ \in \ \beta \odot \mathcal{B} \ \oplus \ \gamma \odot \mathcal{C}$, we see that there are $b \in B$ and $c \in C$ such that $x = \beta \odot b \oplus \gamma \odot c$. Since $b, c \in A$, we conclude that $u_X(\beta \odot \mathcal{B} \oplus \gamma \odot \mathcal{C}) \subset A$.

Now, if $x \in A$, then there $B \in \mathcal{B}$ and $c \in \mathcal{C}$ such that $x \in B \cap C$. Then

$$x \ \in \ \beta \odot B \ \oplus \ \gamma \odot C \ \in \ \beta \odot \mathcal{B} \ \oplus \ \gamma \odot \mathcal{C}.$$

Thus, $u_X(\beta \odot \mathcal{B} \oplus \gamma \odot \mathcal{C}) \supset A$, i.e. finally $u_X(\beta \odot \mathcal{B} \oplus \gamma \odot \mathcal{C}) = A$.

We are going to prove that the map u_X is open. Since the spaces under consideration are compact and metrizable, it suffices to prove that for any $\mathcal{A} \in \mathrm{mpcc}^2(X)$ and any sequence (A_i) in $\mathrm{mpcc}(X)$ converging to

$$A = u_X(\mathcal{A})$$

there exists a sequence $(\mathcal{A}_i)$ in $\mathrm{mpcc}^2(X)$ converging to $\mathcal{A}$ and such that $u_X(\mathcal{A}_i) = A_i$, for every $i \in \mathbb{N}$ (see, e.g., [5]).

For any i, let $r_i = d_H(A, A_i)$ and let

$$\mathcal{A}_i = \overline{\mathrm{conv}}_{\mathrm{mp}}(\{A_i \cap \bar{O}_{r_i}(C) \mid C \in \mathcal{A}\})$$

(by $\overline{\mathrm{conv}}_{\mathrm{mp}}$ we denote the closed max-plus convex hull map). Since the map $K \mapsto \bar{O}_{r_i}(K)$ is continuous, we conclude that

$$\{A_i \cap \bar{O}_{r_i}(C) \mid C \in \mathcal{A}\} \in \exp \mathrm{mpcc}(X).$$

It is easy to see that $d_{\mathrm{HH}}(\{A_i \cap \bar{O}_{r_i}(C) \mid C \in \mathcal{A}\}, \mathcal{A}) \leq r_i$ and, since the closed max-plus convex hull map is nonexpanding, we obtain that $d_{\mathrm{HH}}(\mathcal{A}_i, \mathcal{A}) \leq r_i$.

Now, by [13, Theorem 3.3] the map u_X is soft as an open map with max-plus convex preimages. □

Given a compact convex set X consider the following sequence:

$$X \xrightarrow{\ s_X\ } \mathrm{mpcc}(X) \xrightarrow{\ s_{\mathrm{mpcc}(X)}\ } \mathrm{mpcc}^2(X) \xrightarrow{\ s_{\mathrm{mpcc}^2(X)}\ } \dots$$

Note that every map in this sequence is an isometric embedding. We denote the metric direct limit of this sequence by $\mathrm{mpcc}^+(X)$ and let $\mathrm{mpcc}^{++}(X)$ be the completion of $\mathrm{mpcc}^+(X)$. In the sequel, we identify the spaces $\mathrm{mpcc}^n(X)$ with the corresponding subspaces of $\mathrm{mpcc}^+(X)$ and $\mathrm{mpcc}^{++}(X)$.

Denote by $\mathrm{mpcc}^\omega(X)$ the inverse limit of the sequence

$$\mathrm{mpcc}(X) \xleftarrow{\ u_X\ } \mathrm{mpcc}^2(X) \xleftarrow{\ u_{\mathrm{mpcc}^2(X)}\ } \mathrm{mpcc}^3(X) \xleftarrow{\ u_{\mathrm{mpcc}^3(X)}\ } \dots$$

Let $\psi_n \colon \mathrm{mpcc}^\omega(X) \to \mathrm{mpcc}^n(X)$ denote the natural projection.

There exists a natural embedding $\theta\colon \mathrm{mpcc}^+(X)\to \mathrm{mpcc}^\omega(X)$. The restriction of this embedding onto the set $\mathrm{mpcc}^n(X)$ is uniquely determined by the maps

$$s_{nm}=s_{\mathrm{mpcc}^{m-1}(X)}\dots s_{\mathrm{mpcc}^{n}(X)}\colon \mathrm{mpcc}^n(X)\to \mathrm{mpcc}^m(X),\ n<m.$$

We write $\theta=(\theta_n)$, where $\theta_n=\psi_n\theta$.

The following proposition is proved in [4] in general form; in turn, this is a generalization of a result from [14].

Proposition 3.3. *The (unique) extension* $\bar\theta\colon \mathrm{mpcc}^{++}(X)\to \mathrm{mpcc}^\omega(X)$ *of the map* θ *is an embedding.*

Proof. Similarly as in [10, Lemma 3], one can prove that the map $\bar\theta$ is injective. We are going to show that the map $\bar\theta^{-1}$ is continuous. To this end, for any $x\in \mathrm{mpcc}^{++}(X)$ and $\varepsilon>0$ one should find a neighborhood U of $\bar\theta(x)$ in $\mathrm{mpcc}^\omega(X)$ such that

$$\bar\theta^{-1}(U)\subset B_\varepsilon(x). \tag{3.1}$$

We write $\bar\theta=(\bar\theta_i)$, where $\bar\theta_i=\psi_i\circ\bar\theta$. Again, similarly as in [10, Lemma 3], the sequence $(\bar\theta_i(x))$ converges to x and therefore there exists n such that

$$d(\bar\theta_k(x),x)<\varepsilon/4 \ \text{ for all }\ k\ge n. \tag{3.2}$$

Put

$$V=O_{\varepsilon/4}(\bar\theta_n(x)))\subset \mathrm{mpcc}^{n+1}(X), \qquad U=\psi_{n+1}^{-1}(V).$$

Let us verify the inclusion

$$\mathrm{mpcc}^+(X)\cap\bar\theta^{-1}(U)\subset O_{3\varepsilon/4}(x). \tag{3.3}$$

Let $y\in \mathrm{mpcc}^+(X)\cap\bar\theta^{-1}(U)$. Then there exists $k\ge n+1$ such that $y\in \mathrm{mpcc}^k(X)\subset \mathrm{mpcc}^+(X)$. Since $y\in\bar\theta^{-1}(U)$, we have $\psi_{n+1}(y)\in V$. Since

$$\varepsilon/2>d(\psi_{n+1}(y),\bar\theta_n(x)),$$

from (3.2) and Lemma 3.1 it follows that

$$d(y,x)\le d(y,\bar\theta_n(x))+d(\bar\theta_n(x),x)<\varepsilon/2+\varepsilon/4+3\varepsilon/4.$$

Therefore, the inclusion in (3.3) is verified. Since $\mathrm{mpcc}^+(X)$ is dense in $\mathrm{mpcc}^{++}(X)$, from (3.3) we obtain that $\bar\theta^{-1}(U)\subset\overline{B_{3\varepsilon/4}(x)}\subset O_\varepsilon(x)$. □

Let $Q=[-1,1]^\omega$ be the Hilbert cube, $s=(-1,1)^\omega$ be its pseudointerior and $\operatorname{rint}Q=\{(x_i)\in Q\mid \sup_i|x_i|<1\}$ be its radial interior.

Theorem 3.4. *Let X be a compact max-plus convex body in $\mathbb{R}^n$. Then the triple* $(\mathrm{mpcc}^\omega(X),\mathrm{mpcc}^{++}(X),\mathrm{mpcc}^+(X))$ *is homeomorphic to the triple* $(Q,s,\operatorname{rint}Q)$.

Proof. Consider the following metric ϱ on $\mathrm{mpcc}^\omega(X)$,

$$\varrho((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{d(x_i,y_i)}{2^i},$$

where

$$(x_i),(y_i) \in \mathrm{mpcc}^\omega(X) \subset \prod_{i=1}^{\infty} \mathrm{mpcc}^i(X).$$

For every $i \in \mathbb{N}$ and $j > i$, the maps

$$q_{ij} = \mathrm{mpcc}(s_{i-1,j-1})\colon \mathrm{mpcc}^i(X) \to \mathrm{mpcc}^j(X)$$

determine the map $q_i\colon \mathrm{mpcc}^i(X) \to \mathrm{mpcc}^\omega(X)$.

We first show that the pair $(\mathrm{mpcc}^\omega(X), \mathrm{mpcc}^{++}(X))$ is homeomorphic to the pair (Q, s). By [1, Theorems 2.3 and 3.3, Chapter V], it suffices to prove that the set

$$B = \mathrm{mpcc}^\omega(X) \setminus \mathrm{mpcc}^{++}(X)$$

is a Z-skeletoid in $\mathrm{mpcc}^\omega(X)$. Note first that the set B is σ-compact as the complement to the topologically complete set $\mathrm{mpcc}^{++}(X)$. Note also that every compact subset $K \subset B$ is a Z-set in $\mathrm{mpcc}^\omega(X)$. Indeed, the sequence of retractions $\psi_n\colon \mathrm{mpcc}^\omega(X) \to \mathrm{mpcc}^n(X)$ converges uniformly to the identity map of $\mathrm{mpcc}^\omega(X)$ and the image of every ψ_n misses K. By [1, Theorems 3.2, Chapter V], in order to show that B is a Z-skeletoid it suffices to find a Z-skeletoid in B. In turn, it suffices to find a sequence a sequence (L_i) of compact subspaces in B such that:

(1) every L_i is homeomorphic to Q;
(2) every L_i is a Z-set in Q_{i+1};
(3) for every i there is a retraction $r_i\colon \mathrm{mpcc}^\omega(X) \to L_i$ and the sequence (r_i) of retractions uniformly converges to the identity map.

The construction of such a sequence (L_i) is analogous to that in the proof of [4, Theorem 4], therefore we drop the details. We suppose that X is a max-plus convex body in $\mathbb{R}^n$, $n \geq 2$. Also, we suppose that $\operatorname{diam} X \leq 1$. Then $\operatorname{diam} \mathrm{mpcc}^\omega(X) \leq 1$.

By K_1 we denote the set

$$\{A \in \mathrm{mpcc}(X) \mid \text{ there is } x \in A \text{ such that } x + (\varepsilon,\dots,\varepsilon) \in A\},$$

where $\varepsilon > 0$. Clearly, K_1 is max-plus convex and if ε is small enough then K_1 is nonempty and can be made as close to $\mathrm{mpcc}(X)$ as we wish. We require that there is a retraction r_1 of $\mathrm{mpcc}(X)$ onto K_1 which is 1-close to the identity. Let $L_1 = q_1(K_1)$.

Assuming that K_i, $i \leq p$, are already constructed we let

$$K_{p+1} = \{A \in \mathrm{mpcc}^p(X) \mid \text{ there is } x \in A \text{ such that } x + (\varepsilon,\dots,\varepsilon) \in A\},$$

where $\varepsilon > 0$ is chosen small enough that

$$\mathrm{mpcc}(s_{\mathrm{mpcc}^{p-1}(X)})(K_p) \subset K_{p+1}$$

and there is a retraction

$$r_{p+1}\colon \mathrm{mpcc}^{p+2}(X) \to K_{p+1}$$

which is 2^{-p}-close to the identity. Let $L_{p+1} = q_{p+1}(K_{p+1})$.

Thus, $L = \cup_{i=1}^{\infty} L_i$ is a Z-skeletoid in $\mathrm{mpcc}^{\omega}(X)$. We conclude that the pair $(\mathrm{mpcc}^{\omega}(X), \mathrm{mpcc}^{++}(X))$ is homeomorphic to (Q, s).

Similarly, one can prove that $\mathrm{mpcc}^{+}(X)$ is a Z-skeletoid in $\mathrm{mpcc}^{\omega}(X)$. Therefore, the pair $(\mathrm{mpcc}^{\omega}(X), \mathrm{mpcc}^{+}(X))$ is homeomorphic to $(Q, \mathrm{rint}\, Q)$.

We now apply [4, Theorem 2] to finish the proof. □

4. Remarks and open questions

It is plausible that the main result can be extended to the case of all max-plus convex subsets of $\mathbb{R}^{\alpha}$, $\alpha \leq \omega$, of dimension ≤ 1.

We also conjecture that there is a counterpart of the main result for the hyperspace of max-min convex sets in $\mathbb{R}^{\tau}$. Given $\lambda \in \mathbb{R}_{\max} \cup \{\infty\}$ and $x = (x_{\alpha}) \in \mathbb{R}^{\tau}$, we define $\lambda \otimes x = (\min\{\lambda, x_{\alpha}\})$. A subset A in $\mathbb{R}^{\tau}$ is said to be *max-min convex* if $\alpha \otimes a \oplus b \in A$ for all $a, b \in A$ and $\alpha \in \mathbb{R}_{\max}$.

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