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A new curvature-like tensor in an almost contact Riemannian manifold

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Memories of Professor Victor Kuzakon at ONAFT

Abstract. In a M. Prvanović's paper [5], we can find a new curvature-like tensor in an almost Hermitian manifold.

In this paper, we define a new curvature-like tensor, named contact holomorphic Riemannian, briefly (CHR), curvature tensor in an almost contact Riemannian manifold. Then, using this tensor, we mainly research (CHR)-curvature tensor in a Kenmotsu and a Sasakian manifold. We introduce the flatness of a (CHR)-curvature tensor and show that a Kenmotsu and a Sasakian manifold with a flat (CHR)-curvature tensor is flat, see Theorems 3.1 and 4.1. Next, we introduce the notion of an (CHR)-η-Einstein in an almost contact Riemannian manifold. In particular, in a Sasakian or a Kenmotsu manifold, a (CHR)-η-Einstein manifold is η-Einstein, see Theorem 5.3. Finally, from this tensor, we introduce a notion of a (CHR)-space form in an almost contact Riemannian manifold. In particular, if a Kenmotsu and a Sasakian manifold are (CHR)-space form, then the (CHR)-curvature tensor satisfies a special equation, see Theorems 6.2 and 7.1.

1. Almost contact Riemannian manifolds

A real $(2n+1)$-dimensional differentiable Riemannian manifold (M, g) is said to be an almost contact Riemannian manifold if it has a $(1,1)$-tensor φ and a 1-form η which satisfy [6]

$$\begin{gathered} \varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\varphi X) = 0, \qquad \eta(\xi) = 1, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{gathered} \tag{1.1}$$

for any $Y, X \in TM$, where ξ is defined by $g(\xi, X) = \eta(X)$, and TM is the tangent bundle of M. From $(1.1)_2$ the vector field ξ is a unit vector field

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and we call it the *structure vector field* of the almost contact Riemannian manifold. Next, in an almost contact Riemannian manifold M, we define a 2-form F as $F(X,Y) = g(\varphi X, Y)$ for any $X, Y \in TM$. Then the 2-form F is skew-symmetric and we call it the *fundamental* 2-*form* of this almost contact Riemannian manifold.

In an almost contact Riemannian manifold a 2-dimensional distribution spanned by a unit vector field X and φX is called a φ-*section* of X. The sectional curvature $R(X,\varphi X,\varphi X,X)$ is said to be the φ-*holomorphic sectional curvature of* X, where R denotes the Riemannian curvature tensor with respect to g.

An almost contact Riemannian manifold (M,φ,g,ξ) is called a *Kenmotsu manifold*, [1], [2], [3], if the structure tensors satisfy

$$\begin{aligned}(\nabla_X\varphi)Y &= -\eta(Y)\varphi X - g(X,\varphi Y)\xi,\\ \nabla_X\xi &= X - \eta(X)\xi,\end{aligned}$$

for any $X, Y \in TM$, where ∇ means the covariant derivation with respect to g. In a Kenmotsu manifold (M,φ,g,ξ) we know

$$\begin{aligned}(\nabla_X\eta)Y &= g(\varphi X,\varphi Y),\\ \eta(R(X,Y)Z) &= \eta(Y)g(X,Z) - \eta(X)g(Y,Z),\\ R(X,Y)\xi &= \eta(X)Y - \eta(Y)X,\\ R(\xi,X)Y &= \eta(Y)X - g(X,Y)\xi,\\ R_1(\varphi X,\varphi Y) &= R_1(X,Y) + 2n\eta(X)\eta(Y),\\ R_1(X,\xi) &= -2n\eta(X),\end{aligned} \tag{1.2}$$

for any $X, Y, Z, W \in TM$, where R_1 denotes the Ricci tensor with respect to g.

Moreover, in a Kenmotsu manifold we get

$$\begin{aligned}R(X,Y,\varphi Z,\varphi W) &= R(X,Y,Z,W)+\\ &\quad + g(\varphi X,Z)g(\varphi Y,W) - g(\varphi X,W)g(\varphi Y,Z)+\\ &\quad + g(X,W)g(Y,Z) - g(X,Z)g(Y,W),\\ R(X,Y,\varphi Z,W) &= R(X,Y,Z,\varphi W)+\\ &\quad + g(\varphi X,W)g(Y,Z) - g(\varphi Y,W)g(X,Z)+\\ &\quad + g(\varphi Y,Z)g(X,W) - g(\varphi X,Z)g(Y,W),\\ R(X,Y,\varphi Z,\varphi W) &= R(\varphi X,\varphi Y,Z,W),\\ R(\varphi X,\varphi Y,\varphi Z,\varphi W) &= R(X,Y,Z,W)+\\ &\quad + \eta(X)\eta(W)g(Y,Z) - \eta(X)\eta(Z)g(Y,W)+\\ &\quad + \eta(Y)\eta(Z)g(X,W) - \eta(Y)\eta(W)g(X,Z).\end{aligned} \tag{1.3}$$

A Kenmotsu manifold with constant φ-holomorphic sectional curvature c is called a *Kenmotsu space form* with c. Then, [1], its curvature tensor R is expressed by

$$\begin{aligned} R(X,Y,Z,W) = \frac{c-3}{4}\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big\} + \\ + \frac{c+1}{4}\Big[\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\}\eta(Z) + \\ + \{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\eta(W) + \\ + F(Y,Z)F(X,W) - F(X,Z)F(Y,W) - \\ - 2F(X,Y)F(Z,W)\Big]. \end{aligned} \tag{1.4}$$

A Kenmotsu manifold is said to be *flat* if its φ-sectional curvature is equal to zero on M.

An almost contact Riemannian manifold (M, φ, g, ξ) is called a *normal contact Riemannian* or *Sasakian* manifold if the structure vector field ξ and the fundamental 2-form F satisfy

$$\nabla_X \xi = \varphi X, \tag{1.5}$$

$$(\nabla_X F)(Y,Z) = \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \tag{1.6}$$

for any $X, Y, Z \in TM$.

In a Sasakian manifold the Riemannian curvature tensor R with respect to g satisfies

$$\begin{aligned} & R(X,Y,Z,\xi) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z), \\ & R(X,Y,\varphi Z,W) - R(X,Y,Z,\varphi W) = \\ & \qquad = g(\varphi X,Z)g(Y,W) - g(\varphi X,W)g(Y,Z) + \\ & \qquad + g(\varphi Y,W)g(X,Z) - g(\varphi Y,Z)g(X,W), \\ & R(X,Y,\varphi Z,\varphi W) = R(\varphi X,\varphi Y,Z,W) = R(X,Y,Z,W) + \\ & \qquad + g(X,Z)g(Y,W) - g(Y,Z)g(X,W) + \\ & \qquad + F(X,W)F(Y,Z) - F(Y,W)F(X,Z), \\ & R(\varphi X,\varphi Y,\varphi Z,\varphi W) = R(X,Y,Z,W) + \\ & \qquad + \eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z) + \\ & \qquad + \eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W), \\ & R_1(X,\xi) = 2n\eta(X) \end{aligned} \tag{1.7}$$

for any $X, Y, Z, W \in TM$.

A Sasakian manifold is said to be a *Sasakian space form* if it has constant φ-holomorpic sectional curvature. Then the curvature tensor field R

satisfies ([4])

$$\begin{aligned} R(X,Y,Z,W) = \frac{c+3}{4}\Big\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\Big\}+ \\ + \frac{c-1}{4}\Big[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)+ \\ + \eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W)+ \\ + F(Y,Z)F(X,W) - F(X,Z)F(Y,W)- \\ - 2F(X,Y)F(Z,W)\Big], \end{aligned} \tag{1.8}$$

where c is a constant holomorphic sectional curvature.

A Sasakian space form with zero holomorphic sectional curvature is called *flat*.

An almost contact Riemannian manifold M^{2n+1} is said to be *η-Einstein* if the Ricci tensor R_1 with respect to g satisfies

$$R_1(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for certain differentiable functions a and b on M^{2n+1} which are called the *associated functions* of R_1. In particular, in an η-Einstein Kenmotsu and an η-Einstein Sasakian manifold, associated functions a and b respectively satisfy the following relations

$$a+b = -2n, \qquad \tau = 2n(a-1), \tag{1.9}$$

and

$$a+b = 2n, \qquad \tau = 2n(a+1), \tag{1.10}$$

where τ is the scalar curvature with respect to g.

2. A new curvature-like tensor field in an almost contact Riemannian manifold

In this section, we define a new curvature-like tensor field in an almost contact Riemannian manifold.

In a differentiable manifold M a $(0,4)$-type tensor field $T(X,Y,Z,W)$ is called *curvature-like* if it satisfies

$$\begin{gathered} T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z) = T(Z,W,X,Y), \\ T(X,Y,Z,W) + T(X,Z,W,Y) + T(X,W,Y,Z) = 0, \end{gathered}$$

for any $X,Y,Z,W \in TM$. Of course, a Riemannian curvature tensor, a conformal curvature tensor in a Riemannian manifold, a Bochner curvature tensor in a Kählerian manifold and a C-Bochner curvature tensor in a Sasakian manifold are examples of curvature-like tensor fields.

In an almost contact Riemannian manifold (M,φ,ξ,g), we define a $(0,4)$-type tensor field $(CHR)(X,Y,Z,W)$ as

$$
\begin{aligned}
(CHR)(X,Y,Z,W) = \frac{1}{16}\Big\{&3\big[R(X,Y,Z,W)+R(\varphi X,\varphi Y,Z,W)+\\
&+R(X,Y,\varphi Z,\varphi W)+R(\varphi X,\varphi Y,\varphi Z,\varphi W)\big]-\\
&-R(X,Z,\varphi W,\varphi Y)-R(\varphi X,\varphi Z,W,Y)-\\
&-R(X,W,\varphi Y,\varphi Z)-R(\varphi X,\varphi W,Y,Z)+\\
&+R(\varphi X,Z,\varphi W,Y)+R(X,\varphi Z,W,\varphi Y)+\\
&+R(\varphi X,W,Y,\varphi Z)+R(X,\varphi W,\varphi Y,Z)+\\
&+\eta(X)\big[2\{R(Z,W,Y,\xi)+R(\varphi Z,\varphi W,Y,\xi)\}+R(\varphi Z,\varphi Y,W,\xi)+\\
&\quad+R(\varphi Y,\varphi W,Z,\xi)-R(\varphi Y,W,\varphi Z,\xi)-R(Z,\varphi Y,\varphi W,\xi)\big]-\\
&-\eta(Y)\big[2\{R(Z,W,X,\xi)+R(\varphi Z,\varphi W,X,\xi)\}+R(\varphi Z,\varphi X,W,\xi)+\\
&\quad+R(\varphi X,\varphi W,Z,\xi)-R(Z,\varphi X,\varphi W,\xi)-R(\varphi X,W,\varphi Z,\xi)\big]+\\
&+\eta(Z)\big[2\{R(X,Y,W,\xi)+R(\varphi X,\varphi Y,W,\xi)\}+R(\varphi X,\varphi W,Y,\xi)+\\
&\quad+R(\varphi W,\varphi Y,X,\xi)-R(\varphi W,Y,\varphi X,\xi)-R(X,\varphi W,\varphi Y,\xi)\big]-\\
&-\eta(W)\big[2\{R(X,Y,Z,\xi)+R(\varphi X,\varphi Y,Z,\xi)\}+R(\varphi X,\varphi Z,Y,\xi)+\\
&\quad+R(\varphi Z,\varphi Y,X,\xi)-R(X,\varphi Z,\varphi Y,\xi)-R(\varphi Z,Y,\varphi X,\xi)\big]\Big\}
\end{aligned}
$$

for any $X,Y,Z,W\in TM$. Then, by the straightforward calculation, we can check the above tensor field is curvature-like. We call this tensor field a *contact holomorphic Riemannian curvature tensor* in an almost contact Riemannian manifold.

A contact Riemannian manifold is said to be (CHR)-*flat* if the (CHR)-curvature tensor vanishes, identically.

In an almost contact Riemannian manifold the (CHR)-curvature tensor satisfies

$$
\begin{aligned}
16(CHR)(\varphi X,\varphi Y,Z,W) = 16&(CHR)(X,Y,Z,W)+\\
&+\eta(X)\Big\{R(Z,W,Y,\xi)+R(\varphi Z,\varphi W,Y,\xi)\Big\}-\\
&-\eta(Y)\Big\{R(Z,W,X,\xi)+R(\varphi Z,\varphi W,X,\xi)\Big\}+\\
&+T(X,Y,Z,W),\\
16(CHR)(X,Y,\varphi Z,\varphi W) = 16&(CHR)(X,Y,Z,W)+\\
&+\eta(Z)\Big\{R(X,Y,W,\xi)+R(\varphi X,\varphi Y,W,\xi)\Big\}-\\
&-\eta(W)\Big\{R(X,Y,Z,\xi)+R(\varphi X,\varphi Y,Z,\xi)\Big\}+\\
&+T(X,Y,Z,W),
\end{aligned}
$$

$$\begin{aligned}
16(CHR)(\varphi X,\varphi Y,\varphi Z,\varphi W) = 16(CHR)(X,Y,Z,W)+\\
&+\eta(X)\Big\{R(Z,W,Y,\xi)+R(\varphi Z,\varphi W,Y,\xi)\Big\}-\\
&-\eta(Y)\Big\{R(Z,W,X,\xi)+R(\varphi Z,\varphi W,X,\xi)\Big\}+\\
&+\eta(Z)\Big\{R(X,Y,W,\xi)+R(\varphi X,\varphi Y,W,\xi)\Big\}-\\
&-\eta(W)\Big\{R(X,Y,Z,\xi)+R(\varphi X,\varphi Y,Z,\xi)\Big\}-\\
&-\eta(Z)\Big\{\eta(X)R(\xi,Y,W,\xi)+\eta(Y)R(X,\xi,W,\xi)\Big\}+\\
&+\eta(W)\Big\{\eta(X)R(\xi,Y,Z,\xi)+\eta(Y)R(X,\xi,Z,\xi)\Big\}+\\
&+T(X,Y,Z,W),\\
(CHR)(X,Y,Z,\xi) = \frac{1}{16}\Big[R(X,Y,Z,\xi)+R(\varphi X,\varphi Y,Z,\xi)+\\
&+\eta(X)\{2R(Y,\xi,Z,\xi)-R(\varphi Y,\xi,\varphi Z,\xi)\}-\\
&-\eta(Y)\{2R(X,\xi,Z,\xi)-R(\varphi X,\xi,\varphi Z,\xi)\}\Big],
\end{aligned}$$

where we put

$$\begin{aligned}
T(X,Y,Z,W) = \eta(Z)\Big\{&-2\eta(X)R(\xi,Y,W,\xi)-2\eta(Y)R(X,\xi,W,\xi)+\\
&+\eta(X)R(\xi\varphi W,\varphi Y,\xi)+\eta(Y)R(\varphi W,\xi,\varphi X,\xi)\Big\}-\\
-\eta(W)\Big\{&-2\eta(X)R(\xi,Y,Z,\xi)-2\eta(Y)R(X,\xi,Z,\xi)+\\
&+\eta(X)R(\xi,\varphi Z,\varphi Y,\xi)+\eta(Y)R(\varphi Z,\xi,\varphi X,\xi)\Big\}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&(CHR)(X,\varphi X,\varphi X,X) =\\
&\quad= \frac{1}{16}\Big[16R(X,\varphi X,\varphi X,X)+8\eta(X)R(X,\varphi X,\varphi X,\xi)+\\
&\qquad+\eta(X)^2\{11R(\xi,\varphi X,\varphi X,\xi)+4R(\xi,X,X,\xi)\}\Big].
\end{aligned} \tag{2.1}$$

For a unit vector field X the following tensor $(CHR)(X,\varphi X,\varphi X,X)$ is called a *ϕ-holomorphic (CHR)-curvature* of X, and an almost contact Riemannian manifold is called a *(CHR)-space form* if the holomorphic (CHR)-curvature constant on M.

3. (CHR)-curvature tensor in a Kenmotsu manifold

In this section we consider a (CHR)-curvature tensor in a Kenmotsu manifold. By virtue of (1.2) and (1.3) we have

$$\begin{aligned}(CHR)(X,Y,Z,W) &= R(X,Y,Z,W)+\\ &+\frac{1}{4}\Big\{g(\varphi X,Z)g(\varphi Y,W)-g(\varphi X,W)g(\varphi Y,Z)+2g(\varphi X,Y)g(\varphi Z,W)+\\ &\quad+\eta(X)\eta(W)g(Y,Z)-\eta(X)\eta(Z)g(Y,W)+\\ &\quad+\eta(Y)\eta(Z)g(X,W)-\eta(Y)\eta(W)g(X,Z)\Big\}+\\ &+\frac{3}{4}\Big\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\Big\}.\end{aligned} \tag{3.1}$$

Let the manifold be (CHR)-flat. Then we have from (3.1) that the curvature tensor R is written by

$$\begin{aligned}R(X,Y,Z,W) &= \frac{3}{4}\Big\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\Big\}+\\ &+\frac{1}{4}\Big\{g(\varphi X,W)g(\varphi Y,Z)-g(\varphi X,Z)g(\varphi Y,W)-2g(\varphi X,Y)g(\varphi Z,W)+\\ &\quad+\eta(X)\eta(Z)g(Y,W)-\eta(X)\eta(W)g(Y,Z)+\\ &\quad+\eta(Y)\eta(W)g(X,Z)-\eta(Y)\eta(Z)g(X,W)\Big\}.\end{aligned} \tag{3.2}$$

Comparing (3.2) and (1.4) we obtain

Theorem 3.1. *A (CHR)-flat Kenmotum manifold is flat.*

4. (CHR)-curvature tensor in a Sasakian manifold

In this section, we consider a (CHR)-flat Sasakian manifold and prove that this manifold is a Sasakian space form with zero holomorphic sectional curvature.

By virtue of (1.5), (1.6), (1.7) and Bianchi identity, [6], the (CHR)-curvature tensor (CHR) in a Sasakian manifold satisfies

$$\begin{aligned}(CHR)(X,Y,Z,W) &= R(X,Y,Z,W)+\\ &+\frac{3}{4}\Big\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\Big\}+\\ &+\frac{1}{4}\Big\{g(\varphi X,W)g(\varphi Y,Z)-g(\varphi X,Z)g(\varphi Y,W)-2g(\varphi X,Y)g(\varphi Z,W)+\\ &\quad+\eta(X)\eta(Z)g(Y,W)-\eta(X)\eta(W)g(Y,Z)+\\ &\quad+\eta(Y)\eta(W)g(X,Z)-\eta(Y)\eta(Z)g(X,W)\Big\}.\end{aligned} \tag{4.1}$$

Consider the (CHR)-flat case in a Sasakian manifold. Then we have from (3.1) that the curvature tensor R is written as

$$\begin{aligned}R(X,Y,Z,W) = \frac{3}{4}\Big\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\Big\}-\\ -\frac{1}{4}\Big\{g(\varphi X,W)g(\varphi Y,Z) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)+\\ +\eta(X)\eta(Z)g(Y,W) - \eta(X)\eta(W)g(Y,Z)+\\ +\eta(Y)\eta(W)g(X,Z) - \eta(Y)\eta(Z)g(X,W)\Big\}.\end{aligned}$$

Comparing the above equation and (1.3) we get

Theorem 4.1. *A (CHR)-flat Sasakian manifold is flat.*

5. The properties of the (CHR)-curvature tensor

In this section we consider other properties of the (CHR)-curvature tensor in Kenmotsu and Sasakian manifolds.

Since the (CHR)-curvature tensor in a Kenmotsu manifold satisfies (3.1), using (1.3) and (2.1), we obtain

Proposition 5.1. *In a Kenmotsu manifold we have*

$$\begin{aligned}&(CHR)(X,Y,Z,\xi) = 0,\\ &(CHR)(X,Y,\varphi Z,\varphi W) = (CHR)(\varphi X,\varphi Y,Z,W) =\\ &\qquad = (CHR)(\varphi X,\varphi Y,\varphi Z,\varphi W) = (CHR)(X,Y,Z,W), \qquad (5.1)\\ &(CHR)(X,\varphi X,\varphi X,X) = R(X,\varphi X,\varphi X,X)+\\ &\qquad +\eta(X)^2\{g(X,X) - \eta(X)^2\}\end{aligned}$$

Similarly, we have from (1.7) and (4.1)

Proposition 5.2. *In a Sasakian manifold, we have*

$$\begin{aligned}&(CHR)(X,Y,Z,\xi) = 0,\\ &(CHR)(X,Y,\varphi Z,\varphi W) = (CHR)(\varphi X,\varphi Y,Z,W) =\\ &\qquad = (CHR)(\varphi X,\varphi Y,\varphi Z,\varphi W) = (CHR)(X,Y,Z,W) \qquad (5.2)\\ &(CHR)(X,\varphi X,\varphi X,X) = R(X,\varphi X,\varphi X,X)-\\ &\qquad -\eta(X)^2\{g(X,X) - \eta(X)^2\}.\end{aligned}$$

Next, we put

$$\rho(CHR)(X,Y) = \sum_{i=1}^{2n+1}(CHR)(e_i,X,Y,e_i)$$

for a local orthonormal frame $(e_1, e_2, \ldots, e_{2n+1})$ of an almost contact Riemannian manifold M^{2n+1}. We say this $\rho(CHR)(X,Y)$ a (CHR)-*Ricci tensor.* By virtue of (3.1) and (4.1), the (CHR)-Ricci tensor in a Kenmotsu and a Sasakian manifold are respectively written by

$$\rho(CHR)(X,Y) = R_1(X,Y) + \\ + \frac{1}{2}\Big\{(3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y)\Big\}, \tag{5.3}$$

and

$$\rho(CHR)(X,Y) = R_1(X,Y) - \\ - \frac{1}{2}\Big\{(3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y)\Big\}. \tag{5.4}$$

We also put

$$\tau(CHR) = \sum_{i=1}^{2n+1} (CHR)(e_i, e_i)$$

which is called the (CHR)-*scalar curvature* of (CHR)-curvature tensor. Then, in a Kenmotsu and a Sasakian manifold, we respectively have from (5.3) and (5.4)

$$\tau(CHR) = \tau + n(3n+1),$$
$$\tau(CHR) = \tau - n(3n+1).$$

Now, a contact Riemannian manifold M^{2n+1} is called (CHR)-η-*Einstein* if its (CHR)-Ricci tensor $\rho(CHR)(X,Y)$ has the form

$$\rho(CHR)(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{5.5}$$

for certain functions α and β which are called *associated functions* of $\rho(CHR)$. In particular, if our manifold is Kenmotsu or Sasakian, then we have from (5.3) and (5.4) the Ricci tensors are respectively written as

$$R_1(X,Y) = \big(\alpha - \tfrac{3n-1}{2}\big)g(X,Y) + \big(\beta - \tfrac{n+1}{2}\big)\eta(X)\eta(Y), \tag{5.6}$$

$$R_1(X,Y) = \big(\alpha + \tfrac{3n-1}{2}\big)g(X,Y) + \big(\beta + \tfrac{n+1}{2}\big)\eta(X)\eta(Y). \tag{5.7}$$

Conversely, if our manifold M^{2n+1} is η-Einstein, then we can easily that a Kenmotsu and a Sasakian manifolds are (CHR)-η-Einstein. Moreover, we have from (1.9), (5.5) and (5.6)

$$\alpha + \beta = 4n \qquad : Kenmotsu, \tag{5.8}$$

$$\alpha + \beta = 0 \qquad : Sasakian. \tag{5.9}$$

By virtue of (5.3), (5.4), (5.8), and (5.9) the scalar curvature τ and the (CHR)-scalar curvature $\tau(CHR)$ are respectively written as

$$\tau = 2n\alpha - 3n(n-1), \quad \tau(CHR) = 2n(\alpha+2) \qquad : Kenmotsu, \tag{5.10}$$

$$\tau = 2n(\alpha + 3n + 1), \qquad \tau(CHR) = n(2\alpha + 3n + 1) \quad : \textit{Sasakian}. \tag{5.11}$$

Thus we have

Theorem 5.3. *A Kenmotsu or a Sasakian manifold M^{2n+1} is (CHR)-η-Einstein if and only if $M^{2n+1}$is η-Einstein and the scalar curvatures τ and $\tau(CHR)$ are respectively written by* (5.10) *and* (5.11) *which includes only the associated function α.*

Corollary 5.4. *In an (CHR)-η-Einstein Kenmotsu or Sasakian manifold M^{2n+1}, the scalar curvatures τ and $\tau(CHR)$ are constant if and only if one of the associated functions is constant.*

6. (CHR)-space form

In this section we define a (CHR)-space form in an almost contact Riemannian manifold.

Let M^{2n+1} be an $(2n+1)$-dimensional almost contact Riemannian manifold.

Definition 6.1. An almost contact Riemannian manifold M^{2n+1} is said to be a (CHR)-*space form* if its contact holomorphic (CHR)-sectional curvature is constant for any vector fields and any point on M^{2n+1}

$$(CHR)(X, \varphi X, \varphi X, X) = c\|X\|^2\{\|X\|^2 - \eta(X)^2\} \tag{6.1}$$

for a certain constant c and any $X \in TM$, where $\|X\|$ is the length of X with respect to g.

$$(CHR)(X, \varphi X, \varphi X, X) = c\|X\|^4$$

for any $X \in TM - \{\xi\}$.

First, we consider a Kenmotsu (CHR)-space form, that is a manifold M whose (CHR)-curvature tensor satisfies (6.1).

By virtue of (5.1)

$$c\|X\|^4 = R(X, \varphi X, \varphi X, X) + \eta(X)^2\{g(X, X) - \eta(X)^2\}$$

for any $X \in TM$. Moreover, from the above equation we know

$$c\|X\|^4 = R(X, \varphi X, \varphi X, X)$$

for any $X \in TM - \{\xi\}$. This means that the manifold is a Kenmotsu space form. So, using (3.1) and Proposition 5.1, the curvature tensor R is written as

$$
\begin{aligned}
(CHR)(X,Y,Z,W) = \frac{1}{4}\Big\{ & g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + \\
& + g(\varphi X,Z)g(\varphi Y,W) - g(\varphi X,W)g(\varphi Y,Z) + 2g(\varphi X,Y)g(\varphi Z,\varphi W) + \\
& + \eta(X)\eta(W)g(Y,Z) - \eta(X)\eta(Z)g(Y,W) + \eta(Y)\eta(Z)g(X,W) - \\
& - \eta(Y)\eta(W)g(X,Z)\Big\}.
\end{aligned} \tag{6.2}
$$

From the above equation, we obtain

$$\rho(CHR)(X,Y) = -\frac{n+1}{2}\Big\{g(X,Y) - \eta(X)\eta(Y)\Big\}. \tag{6.3}$$

and

$$\tau(CHR) = -n(n+1). \tag{6.4}$$

Summing up, we have

Theorem 6.2. *If an $(2n+1)$-dimensional (CHR)-Kenmotsu space form, then the (CHR)-curvature tensor, the (CHR)-Ricci tensor and the (CHR)-scalar curvature are respectively given by* (6.2)*,* (6.3)*, and* (6.4)*.*

7. Sasakian (CHR)-space forms

In this section, we consider a Sasakian (CHR)-space form and we determine the Riemannian curvature tensor of this manifold.

Let M be a Sasakian manifold. Then the (CHR)-curvature tensor satisfies (4.1). From this we have

$$
\begin{aligned}
(CHR)(X,\varphi X,\varphi X,X) = R(X,\varphi X,\varphi X,X) - \\
- \eta(X)^2 g(X,X) + \eta(X)^4.
\end{aligned} \tag{7.1}
$$

Now suppose that M is a Sasakian (CHR)-space form. Then, we have from (6.1) and (7.1) that

$$R(X,\varphi X,\varphi X,X) = c\|X\|^4 + \eta(X)^2 g(X,X) - \eta(X)^4$$

for any $X \in TM$. In particular, for $X \in TM - \{\xi\}$ the above equation means

$$R(X,\varphi X,\varphi X,X) = c\|X\|^4,$$

that is, the manifold is a Sasakian space form with the constant φ-holomorphic sectional curvature c. So, its curvature tensor $R(X,Y,Z,W)$ satisfies (1.8). Substituting (1.8) into (4.1), we get

$$\begin{aligned}(CHR)(X,Y,Z,W) = \frac{c}{4}\Big\{&g(X,W)g(Y,Z) - g(X,Z)g(Y,W)+\\ &+\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z)-\\ &-\eta(X)\eta(W)g(Y,Z) - g(\varphi X,W)g(\varphi Y,Z)+\\ &+g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)\Big\}.\end{aligned} \tag{7.2}$$

From (7.2), we obtain

$$\rho(CHR)(X,Y) = \frac{(n+1)c}{2}\Big\{g(X,Y) - \eta(X)\eta(Y)\Big\} \tag{7.3}$$

and

$$\tau(CHR) = n(n+1)c. \tag{7.4}$$

As a result, we have

Theorem 7.1. *If M is an $(2n+1)$-dimensional Sasakian (CHR)-space form, then its (CHR)-curvature tensor, the Ricci (CHR)-tensor and the scalar (CHR)-curvature are respectively given by* (7.2), (7.3), *and* (7.4).

Rerefences

[1] K. Arslan, R. Ezentas, I. Mihai, C. Murathan, C. Ozgur. Ricci curvature of submanifolds in Kenmotsu space forms. *International Journal of Mathematics and Mathematical Sciences*, 29(12):719–726, 2002.

[2] Avik De. On Kenmotsu manifold. *Bulletin of Mathematical Analysis and Applications*, 2(3):1–6, 2010.

[3] Katsuei Kenmotsu. A class of almost contact Riemannian manifolds. *Tohoku Mathematical Journal*, 24(1):93–103, 1972.

[4] Koji Matsumoto, Ion Mihai. Ricci tensor of c-totally real submanifolds in Sasakian space forms. *Nihonkai Mathematical Journal*, 13(2):191–198, 2002.

[5] Mileva Prvanovic. Conformally invariant tensors of an almost Hermitian manifold associated with the holomorphic curvature tensor. *Journal of Geometry*, 103(89):89–101, 2012.

[6] Kentaro Yano, Masahiro Kon. *Structures on Manifolds*, volume 3 of *Series in pure mathematics*. World Scientific, 1984.

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