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Homeotopy groups of one-dimensional foliations on surfaces

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Abstract. Let Z be a non-compact two-dimensional manifold obtained from a family of open strips $\mathbb{R} \times (0, 1)$ with boundary intervals by gluing those strips along their boundary intervals. Every such strip has a foliation into parallel lines $\mathbb{R} \times t$, $t \in (0, 1)$, and boundary intervals, whence we get a foliation Δ on all of Z. Many types of foliations on surfaces with leaves homeomorphic to the real line have such "striped" structure. That fact was discovered by W. Kaplan (1940-41) for foliations on the plane $\mathbb{R}^2$ by level-set of pseudo-harmonic functions $\mathbb{R}^2 \to \mathbb{R}$ without singularities.

Previously, the first two authors studied the homotopy type of the group $\mathcal{H}(\Delta)$ of homeomorphisms of Z sending leaves of Δ onto leaves, and shown that except for two cases the identity path component $\mathcal{H}_0(\Delta)$ of $\mathcal{H}(\Delta)$ is contractible. The aim of the present paper is to show that the quotient $\mathcal{H}(\Delta)/\mathcal{H}_0(\Delta)$ can be identified with the group of automorphisms of a certain graph with additional structure encoding the "combinatorics" of gluing.

1. Introduction

Let Z be a non-compact two-dimensional manifold and Δ be a one-dimensional foliation on Z such that each leaf ω of Δ is homeomorphic to $\mathbb{R}$ and is a closed subset of Z. These foliations on the plane often appear as level-sets of pseudoharmonic functions and from that point of view they were studied by W. Kaplan [6], [7], W. Boothby [1], [2], M. Morse and J. Jenkins [5], M. Morse [11] and others.

In particular, Kaplan proved that for every such a foliation there exists at most countably many leaves $\{\omega_i\}_{i \in J}$ such that for every connected component S of $\mathbb{R}^2 \setminus \{\omega_i\}_{i \in J}$ one can find a homeomorphism $\phi : S \to \mathbb{R}^2 \times (0, 1)$ sending the leaves in S onto horizontal lines $\mathbb{R} \times \{t\}$, $i \in (0, 1)$. However

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his construction was not *canonical*, as he tried to minimize the total number of strips, and for that reason the closure $\overline{S}$ may have very complicated structure. For instance the above homeomorphism ϕ not always extends to an embedding of $\overline{S}$ into $\mathbb{R} \times [0,1]$.

In [9, Theorem 1.8] the first two authors gave sufficient conditions for a one-dimensional foliation on a non-compact surface to have a similar *striped* structure, and proposed a certain *canonical* decomposition into strips whose closures homeomorphic to open subsets of $\mathbb{R} \times [0,1]$.

Also, in [8] the same authors considered arbitrary foliated surfaces (Z, Δ) glued from strips in the above way and studied the homotopy properties of the group of homeomorphisms $\mathcal{H}(\Delta)$ of Z mapping leaves of the foliation Δ into leaves. They proved that except for few cases the identity path component $\mathcal{H}_0(\Delta)$ of $\mathcal{H}(\Delta)$ is contractible. The principal technical assumption in [8] was that the gluing maps between boundary intervals of strips must be *affine*.

The quotient $\pi_0\mathcal{H}(\Delta) = \mathcal{H}(\Delta)/\mathcal{H}_0(\Delta)$ is an analogue of a mapping class group for foliated homeomorphisms and we call it the *homeotopy* group of the foliation Δ. In [12] and [13] the third author studied a special class of so-called "rooted tree like" striped surfaces, completely described algebraic structure of homeotopy groups of their foliations, and also related those groups with the homeotopy groups of the space of leaves Z/Δ.

The aim of the present paper is to extend the results of [8] to arbitrary "striped" surfaces and compute the corresponding homeotopy groups. Namely, we show that $\pi_0\mathcal{H}(\Delta)$ is isomorphic to a group of automorphism of a certain graph with additional structure, see Theorem 8.1. In particular, these results hold for all foliations considered in [9], [12], [13].

Structure of the paper. §2 contains a list of the principal results of the paper. First we give a formal definition of a strip and then show in Proposition 2.2 that up to a foliated homeomorphism it can be replaced by a *model* strip having better disposition of boundary intervals.

Next, we characterize homeomorphisms between boundaries of strips which extend to foliated homeomorphisms between strips, see Theorem 2.3. §3 and §4 are devoted to proofs of those results.

In §5 we introduce a notion of a *striped* atlas on a surface Z, being a decomposition into strips glued along boundary intervals, and prove that gluing homeomorphisms can be made affine, see Theorem 5.8.

Further, in §6, we associate to each striped atlas a certain *graph* G which encodes the "combinatorics" of gluing strips, and relate automorphisms of G with self-equivalences of the corresponding atlas, see Theorem 6.2.

§7 establishes relationships between distinct properties of foliated surfaces considered in [8], [9], and [10], see Theorem 7.4.

Finally, in §8 we consider the group $\mathcal{H}(\Delta)$ of homeomorphisms of the foliation Δ and deduce from [8] and results of previous sections that the homeotopy group $\pi_0\mathcal{H}(\Delta)$ is isomorphic with the group of automorphisms of the graph associated to some special striped atlas of Z.

2. Model strips

Let Z be a two-dimensional topological manifold. A *foliated chart of dimension* 1 on Z is a pair (U, φ), where $U \subset Z$ is an open subset and $\varphi : U \to (a, b) \times B$ is a homeomorphism with B being an open subset of $[0, +\infty)$. The set $P_y = \varphi^{-1}\big((a, b) \times \{y\}\big)$, $y \in B$, is then called a *plaque* of this foliated chart.

Suppose $\Delta = \{\omega_\alpha \mid \alpha \in A\}$ is a partition of Z into path connected subsets and there exists an atlas $\mathcal{A} = \{U_i, \varphi_i\}_{i\in\Lambda}$ of foliated charts of dimension 1 on Z such that for each $\alpha \in A$ and each $i \in \Lambda$ every path component of a set $\omega_\alpha \cap U_i$ is a plaque. Then Δ is said to be a *one-dimensional foliation* on Z and $\{U_i, \varphi_i\}_{i\in\Lambda}$ is called a *foliated atlas associated to* Δ. Every ω_α is then a *leaf* of the foliation Δ and the pair (Z, Δ) is a *foliated surface.*

Let (Z_1, Δ_1) and (Z_2, Δ_2) be two foliated surfaces. Then a homeomorphism $h : Z_1 \to Z_2$ is said to be *foliated* if for each leaf $\omega \in \Delta_1$ its image, $h(\omega)$, is a leaf of Δ_2.

Definition 2.1. A subset $S \subset \mathbb{R}^2$ will be called a *strip* if

(i) $\mathbb{R} \times (u, v) \ \subset \ S \ \subset \ \mathbb{R} \times [u, v]$;
(ii) S is open in the topology of $\mathbb{R} \times [u, v]$

for some $u < v \in \mathbb{R}$. Denote

$$\partial_-S := S \ \cap \ \mathbb{R} \times \{u\}, \qquad \partial_+S := S \ \cap \ \mathbb{R} \times \{v\},$$
$$\partial S := \partial_-S \ \cup \ \partial_+S, \qquad \mathrm{Int}S := \mathbb{R} \times (u, v).$$

We will call ∂S the *boundary* of S, while ∂_-S and ∂_+S will be the *sides* of S. It follows that ∂S is an open subset of $\mathbb{R} \times \{u, v\}$, and so it is a disjoint union of at most countably many open (possibly unbounded) intervals.

If, in addition to (i) and (ii), the following conditions hold:

(iii) every connected component of ∂S is a *bounded* interval,
(iv) the closures of boundary intervals of ∂S in $\mathbb{R} \times [u, v]$ are mutually disjoint,

then S will be called a *model strip.*

Evidently, each strip S possesses an oriented one-dimensional foliation into horizontal lines $\mathbb{R} \times t$, $t \in (u, v)$ and boundary intervals of ∂S. We will call that foliation *canonical.*

The following statement allows to reduce any strip to a technically more convenient form. It will be proved in Section 3.

Proposition 2.2. *Each strip is foliated homeomorphic to a model strip.*

Monotone homeomorphisms of ∂S. Notice that the boundary of a strip can be regarded as a partially ordered set being a disjoint union of two linearly ordered sets $\partial_- S$ and $\partial_+ S$ that are incomparable with each other. In other words, for $(a, x), (b, y) \in \partial S$ we assume that $(a, x) < (b, y)$ if and only if $a < b$ and $x = y$.

More generally, let $A, B \subset \partial S$ be two subsets. Then we say that $A < B$ if and only if $a < b$ for all $a \in A$ and $b \in B$. In particular, this gives a linear order on the boundary intervals of $\partial_- S$ and $\partial_+ S$. Thus, if $I_\alpha = (a, b) \times \{x\}$ and $I_\beta = (c, d) \times \{y\}$ are boundary intervals of ∂S with $x, y \in \{u, v\}$, then $I_\alpha < I_\beta$ if and only if $x = y$ and $b < c$.

Now let S_1 and S_2 be two strips, $A \subset \partial S_1$ and $B \subset \partial S_2$ be subsets, and $h : A \to B$ be a bijection. We will say that h *preserves* (resp. *reverses*) order whenever for any $a, a' \in A$ we have that $a < a'$ if and only if $h(a) < h(a')$ (resp. $h(a) > h(a')$). In either of these cases h is said to be *monotone*.

Evidently, if $h : S_1 \to S_2$ is a foliated homeomorhism between two strips, then its restriction $h|_{\partial S_1} : \partial S_1 \to \partial S_2$ is monotone. The following statement is a converse to the latter observation. It will be proved in Section 4.

Theorem 2.3. *Every monotone homeomorphism $h : \partial S_1 \to \partial S_2$ between boundaries of two strips S_1 and S_2 extends to a foliated homeomorphism $\hat{h} : S_1 \to S_2$.*

3. Proof of Proposition 2.2

Lemma 3.1. *Let S be a half strip with $\mathbb{R} \times [0, 1) \subset S \subset \mathbb{R} \times [0, 1]$. Then there exists a half strip S' and foliated homeomorphism $h : S \to S'$ such that*

- $\mathbb{R} \times [0, 1) \subset S' \subset S \subset \mathbb{R} \times [0, 1]$;
- *the closures of boundary intervals of $\partial_+ S'$ are bounded in $\mathbb{R}^2$ and mutually disjoint;*
- *h is fixed on $\mathbb{R} \times 0$;*
- *h preserves the second coordinate, and therefore is foliated.*

Assuming lemma is true let us deduce Proposition 2.2. Let S be a strip with $\mathrm{Int}(S) = \mathbb{R} \times (-1, 1)$. Consider two half strips

$$A = S \cap \big(\mathbb{R} \times [-1, 0]\big), \qquad B = S \cap \big(\mathbb{R} \times [0, 1]\big).$$

Then by Lemma 3.1 one can find two half strips A' and B' and foliated homeomorphisms $f : A \to A'$ and $g : B \to B'$ such that

- $\mathbb{R} \times (-1, 0] \ \subset \ A' \ \subset \ A \ \subset \ \mathbb{R} \times [-1, 0]$;
- $\mathbb{R} \times [0, 1) \ \subset \ B' \ \subset \ B \ \subset \ \mathbb{R} \times [0, 1]$;
- the closures of boundary intervals of $\partial_- A'$ and $\partial_+ B'$ are bounded in $\mathbb{R}^2$ and mutually disjoint;
- f and g are fixed on $\mathbb{R} \times 0$ and preserve second coordinate.

Then $S' = A' \cup B'$ is a model strip with $\mathbb{R} \times [0, 1) \subset S' \subset S$, and a foliated homeomorphism $h : S \to S'$ can be given by the formula: $h|_{A'} = f$ and $h|_{B'} = g$. This proves Proposition 2.2 modulo Lemma 3.1.

Proof of Lemma 3.1. a) First we will show how to make closures of boundary intervals of $\partial_+ S'$ to be bounded though not necessarily disjoint. Fix any $a < b \in \mathbb{R}$ and consider the following half strip:

$$T \ = \ \mathbb{R} \times [0, 1) \ \cup \ (a, b) \times \{1\}.$$

Then by [8, Lemma 3.2] there exists a homeomorphism $h : \mathbb{R} \times [0, 1] \to T$ preserving second coordinate and fixed on $\mathbb{R} \times 0$. Hence $S' = h(S)$ is a half strip with $\partial S' \subset \partial T = (a, b) \times 1$.

b) To simplify the notation replace S with S' and assume that boundary intervals of $\partial_+ S'$ are bounded in $\mathbb{R}^2$. We should make their closures mutually disjoint.

Consider the following subset of $\mathbb{R}^7$:

$$A = \{ \, (a, u, b, c, v, d, t) \ | \ a < u < b, \ c < v < d, \ t \in [a, b] \, \}$$

and define the function $\gamma : A \to \mathbb{R}$ by

$$\gamma(a, u, b, c, v, d, t) = \begin{cases} \dfrac{t - a}{u - a}(v - c) + c, & t \in [a; u], \\ \dfrac{t - u}{b - u}(d - v) + v, & t \in [u; b]. \end{cases}$$

Then γ is continuous, and for any combination of the first six parameters a, u, b, c, v, d the map $t \mapsto \gamma(a, u, b, c, v, d, t)$ homeomorphically maps the segment $[a, b]$ onto $[c, d]$ so that u is sent to v.

Lemma 3.1.1. *Let T be a closed triangle in the plane $\mathbb{R}^2$ with vertices $A(x_a, y_a)$, $B(x_b, y_b)$, $O(x_o, y_o)$ such that $y_a = y_b > y_o$, and C be a point on the open interval (A, B). Let also*

$$T' = T \setminus \{A, B\}, \qquad\qquad T'' = T \setminus \{A \cup [B, C]\}.$$

Then there exists a homeomorphism $f : T' \to T''$ preserving second coordinate. In particular, f is fixed on the sides $(A, O]$, $[O, B)$ and maps the interval (A, B) onto (A, C).

Proof. Not loosing generality one can assume that $A(-1,0)$, $B(1,1)$, $O(0,0)$ and $C(0.5,1)$. Then f can be given by the formula:

$$f(x,y) = \begin{cases} \big(\gamma(-y, y^2, y, -y, 0, y, x), y\big), & 0 \leqslant y < 1, \\ \big(\frac{x+1}{2} - 1, 1\big), & y = 1. \end{cases}$$

Evidently, f maps the curve $x = y^2$ on the segment of the line $x = 0$. Moreover, we have that $f[A_y, Q_y] = [A_y, C_y]$, and $f[Q_y, B_y] = [C_y, B_y]$, see Figure 3.1, $\square$

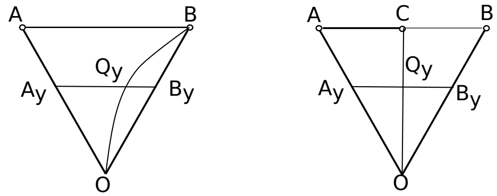

FIGURE 3.1. Triangles T' and T''

Returning back to the proof of Lemma 3.1 assume that

$$\partial_+ S = \bigsqcup_{i=1}^{N} (a_i, b_i) \times \{1\},$$

where N is either a finite number of $+\infty$. Fix a strictly monotone sequence $\{u_j\}_{j=1}^{\infty} \subset [-1, 1]$ such that $\lim\limits_{j\to\infty} = 1$ and for each i define the triangle T_i with vertices $A_i(a_i, 1)$, $B_i(b_i, 1)$, $O_i(\frac{a_i+b_i}{2}, u_i)$, and let $C(\frac{a_i+b_i}{2}, 1)$. Define also the following model half strips:

$$S_i = S \ \setminus \bigcup_{j\leq i} [C_j, B_j) \times \{1\},$$

and put

$$S_0 = S, \qquad S' = S \ \setminus \bigcup_{i=1}^{N} [C_i, B_i) \times \{1\} = \bigcap_{i=1}^{N} S_i.$$

Then the closures of the boundary intervals of S' are mutually disjoint.

Denote $T_i' = T_i \setminus \{A_i, B_i\}$ and $T_i'' = T_i \setminus \{A_i \cup [B_i, C_i]\}$. Then by Lemma 3.1.1 there exists a homeomorphism $f_i : T_i' \to T_i''$ preserving second coordinate and being identity on the sides $(A_i, O_i]$ and $[O_i, B_i)$. Therefore f_i extends by the identity to a homeomorphism $f_i : S_{i-1} \to S_i$.

Then a foliated homeomorphism $\varphi : S \to S^{'}$ can be defined as the composition of all f_i:

$$\varphi = \cdots \circ f_{i+1} \circ f_i \circ \cdots \circ f_2 \circ f_1 : S = S_0 \xrightarrow{f_1} S_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} S_i \xrightarrow{f_{i+1}} \cdots S'.$$

If N is finite, φ is well-defined.

For infinite N one should check that for each point $z \in S$ the sequence

$$\{\ g_j = f_j \circ \cdots \circ f_2 \circ f_1 : S \to S_j \subset S\ \}_{j\in\mathbb{N}}$$

of embeddings "stabilizes" on some neighborhood U_z of z, that is $g_j = g_{j+1}$ on U for all sufficiently large j.

So, let $z = (x, y) \in S$. If $y < 1$, then there exists i such that $y < u_i$. Let $U = \mathbb{R} \times (0, u_i)$. Then $f_j(U) = U$ for all j. Moreover, f_j is fixed on U for all $j > i$. Therefore $g_j = g_i$ on U for all $j > i$.

Suppose $y = 1$, so $(x, y) \in (a_i, b_i) \times \{1\}$ for some i, then each f_j with $j \neq i$ is fixed on the triangle T'_i. Then $U = T'_i \setminus \big((A_i, O_i] \cup [O_i, B_i)\big)$ is an open neighborhood of (x, y) in S, and $g_j = g_i = f_i$ on U for $j > i$. Thus φ is a homeomorphism. Lemma 3.1 is completed. □

4. Proof of Theorem 2.3

Let S_1 and S_2 be two strips and $h : \partial S_1 \to \partial S_2$ a monotone homeomorphism. We should prove that h extends to a foliated homeomorphism between S_1 and S_2.

If ∂S_1, and so ∂S_2, are empty, then any foliated homeomorphism between S_1 and S_2 can be regarded as an extension of h. Therefore we will suppose that $\partial S_1 \neq \varnothing$. Not loosing generality one can also assume that $h(\partial_- S_1) = \partial_- S_2$, $h(\partial_+ S_1) = \partial_+ S_2$, and the restrictions $h|_{\partial_- S_1}$ and $h|_{\partial_+ S_1}$ are increasing.

Case 1. Suppose that both S_1 and S_2 are half strips such that

$$\partial_+ S_1 = \bigsqcup_{i=1}^{K} X_i \times \{1\}, \qquad \partial_+ S_2 = \bigsqcup_{i=1}^{K} X'_i \times \{1\},$$
$$\partial_- S_1 = \partial_- S_2 = \mathbb{R} \times \{0\},$$

where $K \in \{0, 1, \dots, +\infty\}$, each $X_i \times \{1\}$ is a boundary interval of $\partial_+ S_1$, X'_i is a boundary interval of $\partial_+ S_2$, the closures $\overline{X_i}$ and $\overline{X'_i}$ are bounded,

$$\overline{X_i} \cap \overline{X_j} = \varnothing, \qquad\qquad \overline{X'_i} \cap \overline{X'_j} = \varnothing, \tag{4.1}$$

for all $i \neq j$, and $h(X_i \times \{1\}) = X'_i \times \{1\}$.

We will extend h to a homeomorphism $h : S_1 \to S_2$ preserving second coordinate and fixed on $\mathbb{R} \times 0$.

Fix an arbitrary strictly increasing sequence $\{u_j\}_{j=0}^{\infty} \subset [0,1)$ such that $u_0 = 0$ and $\lim\limits_{j\to\infty} u_j = 1$. For each $u_j = 0, 1, \dots, \infty$ we will now construct a homeomorphism $\psi_j : \mathbb{R} \to \mathbb{R}$ by the following rule.

Since $h(\mathbb{R} \times 0) = \mathbb{R} \times 0$ one can write $h(x,0) = (\psi_0(x), 0)$ for a unique homeomorphism $\psi_0 : \mathbb{R} \to \mathbb{R}$.

Further notice that there exists a unique homeomorphism

$$\overline{h} : \bigsqcup_{i=1}^{K} X_i \to \bigsqcup_{i=1}^{K} X_i'$$

such that $\overline{h}(X_i) = X_i'$. Then for $j \geq 1$ define $\psi_j : \bigsqcup\limits_{i=1}^{j} X_i \to \bigsqcup\limits_{i=1}^{j} X_i'$ by $\psi_j(x) = \overline{h}(x)$. The assumption that h preserves the order of boundary intervals means that $X_i < X_j$ if and only if $X_i^{'} < X_j^{'}$, $i, j \in 1, \dots, K$. Hence one can apply the following Lemma 4.1 to extend ψ_j to a homeomorphism $\psi_j : \mathbb{R} \to \mathbb{R}$.

Lemma 4.1. *Let $\alpha = \{X_i\}_{i=1}^{n}$ and $\beta = \{X_i'\}_{i=1}^{n}$ be two families of **open** segments in $\mathbb{R}$ having the following properties:*

(1) *the closures $\overline{X_i}$ and $\overline{X_i'}$ are bounded, and $\overline{X_i} \cap \overline{X_j} = \overline{X_i'} \cap \overline{X_j'} = \varnothing$ for all $i \neq j = 1, \dots, n$;*
(2) *α and β are "similarly ordered", that is $X_i < X_j$ if and only if $X_i' < X_j'$ for all $i \neq j = 1, \dots, n$.*

Suppose also that for each $i = 1, \dots, n$ we have an orientation preserving homeomorphism $\psi_i : X_i \to X_i'$. Then there is a homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi|_{X_i} = \psi_i$.

Proof. Due to assumptions on α and β one can renumber the elements in these families and assume that $X_i = (a_i, b_i)$ and $X_i^{'} = (c_i, d_i)$ for some $a_i, b_i, c_i, b_i \in \mathbb{R}$ such that

$$a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n, \qquad c_1 < d_1 < c_2 < d_2 < \dots < c_n < d_n.$$

Then the homeomorphism ψ can be given by the formula:

$$\psi(x) = \begin{cases} x - a_1 + c_1, & x \in (-\infty, a_1], \\ \psi_i(x), & x \in (a_i, b_i),\ i = 1, \dots, n, \\ \frac{c_{i+1} - d_i}{a_{i+1} - b_i}(x - b_i) + d_i, & x \in [b_i, a_{i+1}],\ i = 1, \dots, n-1, \\ x - b_n + d_n, & x \in [b_n, +\infty). \end{cases}$$

Lemma 4.1 is proved. □

Now define $h : S_1 \to S_2$ by the formula:

$$h(x,y) = \begin{cases} \bigl(\psi_j(x), y\bigr), & y = u_j, \\ & 0 \le j \le K, \\ \bigl(\varepsilon_j(y)\psi_j(x) + (1-\varepsilon_j(y))\psi_{j+1}(x),\ y\bigr), & y \in (u_j, u_{j+1}), \\ & 0 \le j \le K-1, \end{cases} \tag{4.2}$$

where $\varepsilon_j(y) = \frac{u_{j+1}-y}{u_{j+1}-u_j}$.

Evidently, h is bijective, preserves the second coordinate and homeomorphically maps $S_1 \setminus \partial_+ S_1$ onto $S_2 \setminus \partial_+ S_2$.

It remains to check that h is a homeomorphism. Since $\psi_j = \psi_{j+1} = \overline{h}$ on X_i for $i \le j$, it follows from the second line in (4.2) that

$$h(x,y) = (\overline{h}(x), y).$$

for all $(x,y) \in X_i \times (u_i, 1]$. Therefore h homeomorphically maps the open set $X_i \times (u_i, 1]$ of S_1 onto the open set $X'_i \times (u_i, 1]$ of S_2. Since the family

$$\{X_i \times (u_i, 1]\}_{i=1}^{K} \ \cup\ \{S_1 \setminus \partial_+ S_1\}$$

constitutes an open covering of S_1, it follows that h is a homeomorphism.

Case 2. Suppose S_1 and S_2 are arbitrary half strips.

One can assume that $\mathbb{R} \times [0,1) \subset S_i \subset \mathbb{R} \times [0,1]$, $i = 1,2$. Then by Lemma 3.1 one can find a half strip S'_i and a homeomorphism $\phi_i : S_i \to S'_i$ such that

- $\mathbb{R} \times [0,1) \subset S'_i \subset S_i \subset \mathbb{R} \times [0,1]$,
- the closures of boundary intervals in $\partial_+ S'_i$ are bounded and mutually disjoint;
- ϕ_i is fixed on $\mathbb{R} \times 0$ and preserves the second coordinate.

Hence the composition $h' = \phi_2 \circ h \circ \phi_1^{-1} : \partial S'_1 \to \partial S'_2$ is a homeomorphism preserving order and orientations of boundary intervals and coincides with h on $\mathbb{R} \times 0$. Therefore, by Case 1, it extends to a foliated homeomorphism $h' : S'_1 \to S'_2$. Hence $\phi_2^{-1} \circ h' \circ \phi_1 : S_1 \to S_2$ is the required extension of h.

Case 3. Consider the general case when S_1 and S_2 are strips. Not loosing generality one can assume that

$$\mathbb{R} \times (-1,1) \subset S_i \subset \mathbb{R} \times [-1,1]$$

for $i = 1,2$. Similarly to the proof of Proposition 2.2 consider two half strips

$$A_i = S_i \cap \mathbb{R} \times [-1,0], \qquad B_i = S_i \cap \mathbb{R} \times [0,1].$$

Evidently,

$$\partial_- A_i = \partial_- S_i, \qquad \partial_+ A_i = \partial_- B_i = \mathbb{R} \times 0, \qquad \partial_+ B_i = \partial_+ S_i.$$

Define two homeomorphisms $f : \partial A_1 \to \partial A_2$ and $g : \partial B_1 \to \partial B_2$ by the rule:

$$f|_{\partial_- A_1} = h|_{\partial_- A_1}, \qquad f|_{\partial_+ A_1} = g|_{\partial_- B_1} = \mathrm{id}_{\mathbb{R}\times 0}, \qquad f|_{\partial_+ B_1} = h|_{\partial_+ S_1}.$$

Then, by Case 2, f and g extend to foliated homeomorphisms $f : A_1 \to A_2$ and $g : B_1 \to B_2$. Hence a required extension $h : S_1 \to S_2$ of h can be given by the formula: $h|_{A'} = f$ and $h|_{B'} = g$. Theorem 2.3 is completed. □

5. Striped atlas

Let Z be a two-dimensional topological manifold.

Definition 5.1. A *striped atlas* on Z is a map $q : Z_0 \to Z$ having the following properties:

(1) $Z_0 = \bigsqcup\limits_{\lambda\in\Lambda} S_\lambda$ is at most countable family of mutually disjoint strips;

(2) q is a *quotient* map, which means that it is continuous, surjective, and has the property that a subset $U \subset Z$ is open if and only if $q^{-1}(U) \cap S_\lambda$ is open in S_λ for each $\lambda \in \Lambda$;

(3) there exist two disjoint families $\mathcal{X} = \{X_\gamma\}_{\gamma\in\Gamma}$ and $\mathcal{Y} = \{Y_\gamma\}_{\gamma\in\Gamma}$ of mutually disjoint boundary intervals of Z_0 enumerated by the same set of indexes Γ such that
   - (a) q is injective on $Z_0 \setminus (\mathcal{X} \cup \mathcal{Y})$;
   - (b) $q(X_\gamma) = q(Y_\gamma)$ for each $\gamma \in \Gamma$;
   - (c) the restrictions $q|_{X_\gamma} : X_\gamma \to q(X_\gamma)$ and $q|_{Y_\gamma} : Y_\gamma \to q(Y_\gamma)$ are embeddings with closed images;

Definition 5.2. A surface Z admitting a striped atlas will be called a *striped* surface.

Notice that a striped surface Z is a non-compact two-dimensional manifold which can be non-connected and non-orientable, and each of its boundary component is an open interval.

Moreover, Z admits a one-dimensional foliation obtained from canonical foliations on the corresponding model strips S_λ. We will call it the *canonical* foliation associated to the striped atlas q and denote by Δ. Evidently, each leaf of Δ is a homeomorphic image of $\mathbb{R}$ and is also a closed subset of Z.

Definition 5.3. We will say that a foliated surface (Z, Δ) admits a *striped structure* if there exists a striped atlas $q : Z_0 \to Z$ which maps each leaf of the canonical foliation of each strip in Z_0 onto some leaf of Δ.

Remark 5.4. Due to (c) for each $\gamma \in \Gamma$ one get the following "gluing" homeomorphism $\phi_\gamma : Y_\gamma \to X_\gamma$ defined by

$$\phi_\gamma = \bigl(q|_{X_\gamma}\bigr)^{-1} \circ q|_{Y_\gamma}. \tag{5.1}$$

Therefore one can think that a striped surface is obtained from a family of model strips by gluing them along certain boundary intervals by homeomorphisms ϕ_γ. It is allowed that two strips are glued along more than one pair of boundary components. Moreover, one may glue together boundary components of the same strip S.

Definition 5.5. Two striped atlases $q : Z_0 \to Z$ and $q' : Z'_0 \to Z'$ on striped surfaces Z and Z' will be called *equivalent* if there exist two foliated homeomorphisms $h : Z_0 \to Z'_0$ and $k : Z \to Z'$ making commutative the following diagram:

$$\begin{array}{ccc} Z_0 & \xrightarrow{h} & Z'_0 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ Z & \xrightarrow[k]{} & Z' \end{array} \tag{5.2}$$

Turning back to the definition of a striped surface notice that for each $\gamma \in \Gamma$ the intervals X_γ, Y_γ are horizontal, and so

$$X_\gamma = (a, b) \times \{x_\gamma\}, \qquad\qquad Y_\gamma = (c, d) \times \{y_\gamma\}$$

for some $x_\gamma, y_\gamma \in \{u, v\}$ and $a, b, c, d \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ and $c < d$. Hence $\phi_\gamma : Y_\gamma \to X_\gamma$ can be written as follows:

$$\phi_\gamma(s, y_\gamma) = (\psi_\gamma(s), x_\gamma), \quad s \in (c, d), \tag{5.3}$$

where $\psi_\gamma : (c, d) \to (a, b)$ is a certain homeomorphism.

Remark 5.6. Notice that if $a < b$ and $c < d$, then there exist exactly two *affine* homeomorphisms $\psi^+, \psi^- : (c, d) \to (a, b)$ given by

$$\psi^+(t) = \frac{b-a}{d-c}(t-c) + a, \qquad \psi^-(t) = \frac{a-b}{d-c}(t-c) + b, \tag{5.4}$$

for $t \in (a, b)$. Evidently, ψ^+ preserves the orientation and ψ^- reverses it.

Definition 5.7. A striped atlas $q : Z_0 \to Z$ will be called *affine* if the following two conditions hold:

(a) Z_0 consists of model strips only;
(b) each gluing map $\phi_\gamma : Y_\gamma \to X_\gamma$, $\gamma \in \Gamma$, is *affine*, that is the homeomorphism ψ_γ in (5.3) is given by either of the formulas from (5.4).

Theorem 5.8. *Each striped atlas $q : Z_0 = \bigsqcup_{\lambda\in\Lambda} S_\lambda \to Z$ on a striped surface Z is equivalent to an affine one. Moreover, if Z_0 consists of model strips only, then there exists a foliated homeomorphism $h : Z_0 \to Z_0$ such that the composition $q' = q \circ h : Z_0 \to Z$ is an affine atlas.*

Proof. First we show that q is equivalent to an atlas consisting of model strips only. By Proposition 2.2 for every strip S_λ there exists a model strip $S'_\lambda \subset S_\lambda$ and a foliated homeomorphism $h_\lambda : S'_\lambda \to S_\lambda$. Put $Z'_0 = \bigsqcup\limits_{\lambda\in\Lambda} S'_\lambda$ and define a homeomorphism $h : Z'_0 \to Z_0$ by $h|_{S'_\lambda} = h_\lambda$, $\lambda \in \Lambda$. Then $q' = q \circ h : Z'_0 \to Z$ is an atlas on Z consisting of model strips and the pair (h, id_Z) is an equivalence between q' and q.

Assume now that each strip S_λ in Z_0 is model. For each $\gamma \in \Gamma$ let $\sigma_\gamma : Y_\gamma \to X_\gamma$ be a unique affine homeomorphism preserving or reversing orientation mutually with ϕ_γ.

Let S_λ, $\lambda \in \Lambda$, be a model strip from Z_0 and $\partial S_\lambda = \sqcup_{\alpha\in\mathbf{A}} I_\alpha$ be the family of its boundary intervals. We will now define a certain homeomorphism $h_\lambda : \partial S_\lambda \to \partial S_\lambda$ preserving each I_α with its orientation. If $I_\alpha = X_\gamma$ for some $\gamma \in \Gamma$, then we set

$$h_\lambda = \phi_\gamma \circ \sigma_\gamma^{-1} : X_\gamma \to X_\gamma,$$

otherwise put h_λ to be the identity map id_{I_α}.

Then h_λ satisfies assumptions of Theorem 2.3 and therefore extends to a foliated homeomorphism $h_\lambda : S_\lambda \to S_\lambda$. Hence we get a homeomorphism $h : Z_0 \to Z_0$ defined by $h|_{S_\lambda} = h_\lambda$.

Then one easily checks that the map $q' = q \circ h : Z_0 \to Z$ is a striped atlas for Z. Moreover, q' glues the same strips along the same boundary intervals and in the same directions as q, but its gluing maps

$$\phi'_\gamma = \left(q'|_{X_\gamma}\right)^{-1} \circ q'|_{Y_\gamma} : Y_\gamma \to X_\gamma$$

differs from the ones of q. It follows from the following commutative diagram:

$$\begin{tikzcd}
Y_\gamma \arrow[r, "h|_{Y_\gamma}=\mathrm{id}_{Y_\gamma}"'] \arrow[d, "\phi'_\gamma"'] \arrow[rr, bend left, "q'|_{Y_\gamma}"] & Y_\gamma \arrow[r, "q|_{Y_\gamma}"'] \arrow[d, "\phi_\gamma"] \arrow[dl, "\sigma_\gamma"'] & q(Y_\gamma) \arrow[d, equal] \\
X_\gamma \arrow[r, "h|_{X_\gamma}=\phi_\gamma\circ\sigma_\gamma^{-1}"] \arrow[rr, bend right, "q'|_{X_\gamma}"'] & X_\gamma \arrow[r, "q|_{X_\gamma}"] & q(X_\gamma)
\end{tikzcd}$$

that $\phi'_\gamma = \sigma_\gamma$ is affine. Hence q' is an affine striped atlas for Z. $\square$

6. Graph of a striped atlas

Let $q : \sqcup_{\lambda\in\Lambda} S_\lambda \to Z$ be a striped atlas on Z. We will now associate to q a certain graph G which encodes a "combinatorial" information about gluing strips via q. It was firstly considered in [12] for a special class of "rooted tree like" striped surfaces. That graph may have multiple edges and loops and also half-open edges.

(1) The *vertices* of G are strips of $\bigsqcup\limits_{\lambda\in\Lambda} S_\lambda$.
(2) It will be convenient to call each boundary interval X of some strip S_λ a *half-edge incident to the vertex* S_λ. The set of all half edges of $\partial_\pm S_\lambda$ will be denoted by $d_\pm(S_\lambda)$. We also put $d(S_\lambda) = d_-(S_\lambda) \cup d_+(S_\lambda)$.
(3) The *edges* of G are of the following two types.
    (a) If two strips S_1 and S_2 are glued along their boundary intervals X_γ and Y_γ, then we assume that the *vertices* S_1 and S_2 of G are connected by an edge e_γ. Thus formally, an edge e_γ is an unordered pair of half edges (X_γ, Y_γ) and will be called a *closed* edge of G.
    (b) If X is a boundary interval of some strip S_λ which is not glued to any other interval, so it represents a boundary interval of Z, then we assume that X is a half-open edge with one vertex S_λ.

We also add to G the information about directions of gluing boundary intervals, and the disposition of boundary intervals along each strip.

For a homeomorphism $f : (a, b) \to (c, d)$ define a number $\mathbf{or}(f) = +1$ if f preserves orientation and $\mathbf{or}(f) = -1$ otherwise. It is evident, that if $g : (c, d) \to (e, f)$ is another homeomorphism, then $\mathbf{or}(g\circ f) = \mathbf{or}(g)\cdot\mathbf{or}(f)$.

(4) To each closed edge (X_γ, Y_γ) corresponding to the gluing of boundary components $\phi_\gamma : Y_\gamma \to X_\gamma$ we associate the number $\sigma(X_\gamma, Y_\gamma) := \mathbf{or}(\phi_\gamma)$ and call it the *orientation of gluing*.
(5) Recall that for each strip S_λ the set of its boundary intervals is at most countable partially ordered set being a disjoint union of two linearly ordered subsets corresponding to $\partial_- S_\lambda$ and $\partial_+ S_\lambda$ respectively. Therefore we have a *linear order* on each of the sets $d_-(S_\lambda)$ and $d_+(S_\lambda)$ of all half-edges incident to the *vertex* S_λ of G.

Thus, "very formally", a graph of a striped atlas is the following object

$$G = (\Lambda, H, \xi, \sigma)$$

where

- Λ is a set, called the set of *vertices* of G.
- $H = \bigsqcup\limits_{\lambda\in\Lambda} \Bigl(d_{-1}(\lambda) \sqcup d_{+1}(\lambda)\Bigr)$ is a family of mutually disjoint at most countable linearly ordered sets, $d_{-1}(\lambda)$ and $d_{+1}(\lambda)$, called *half-edges incident to* λ. We also denote $d(\lambda) = d_-(\lambda) \sqcup d_+(\lambda)$.
- $\xi : H \to H$ is an involution, i.e. a bijection such that $\xi^2 = \mathrm{id}_H$. In this case if $X \neq \xi(X)$ for some $X \in H$, then the *unordered* pair $\{X, \xi(X)\}$ is called a *closed edge* of G. Otherwise X is fixed point of ξ and is called a *half-open* edge of G.

- $\sigma : E \to \{\pm 1\}$ is a map from the set
$$E = \big\{\{X, \xi(X)\} \mid X \in H,\ X \neq \xi(X)\big\}$$
of all closed edges of G to $\{\pm 1\}$, called *orientation of gluing*.
Equivalently, σ can be regarded as a map $\sigma : H \setminus \mathrm{Fix}(\xi) \to \{\pm 1\}$ such that $\sigma \circ \xi = \sigma$.

Definition 6.1. Let $G = (\Lambda, H, \xi, \sigma)$ and $G' = (\Lambda', H', \xi', \sigma')$ be graphs of striped atlases of some striped surfaces. Then by an *isomorphism* of these graphs we will mean four maps
$$\nu : \Lambda \to \Lambda', \qquad \varepsilon : H \to H', \qquad l, \tau : \Lambda \to \{\pm 1\},$$
having the following properties.

(a) ν and ε are bijections satisfying the identity
$$\varepsilon(d_s(\lambda)) = d'_{\tau(\lambda)\cdot s}(\nu(\lambda))$$
for all $\lambda \in \Lambda$ and $s \in \{\pm 1\}$, where $d'_{\pm 1}(\lambda') \subset H'$ is the set of half edges of G' incident to $\lambda' \in \Lambda$. Moreover, both bijections
$$\varepsilon|_{d_{-1}(\lambda)} : d_{-1}(\lambda) \to d'_{-\tau(\lambda)}(\nu(\lambda)),$$
$$\varepsilon|_{d_{+1}(\lambda)} : d_{+1}(\lambda) \to d'_{\tau(\lambda)}(\nu(\lambda)),$$
are increasing for $l(\lambda) = +1$ and decreasing for $l(\lambda) = -1$.

(b) $\xi' \circ \varepsilon = \varepsilon \circ \xi$, in particular, ε induces a bijection between closed edges of G and G'.

(c) Let $\{X, Y\}$ be a closed edge of G with $X \in d(\lambda)$ and $Y = \xi(X) \in d(\mu)$ for some $\lambda, \mu \in \Lambda$. Then
$$l(\lambda) \cdot \sigma(X, Y) = \sigma'(\varepsilon(X), \varepsilon(Y)) \cdot l(\mu). \tag{6.1}$$

Notice that the set $\mathrm{Aut}(G)$ of all automorphisms of a graph G is a group with respect to the following multiplication: if
$$a' = (\nu', \varepsilon', l', \tau'),\ a = (\nu, \varepsilon, l, \tau)\ \in\ \mathrm{Aut}(G),$$
then their product $a'' = a'a = (\nu'', \varepsilon'', l'', \tau'')$ is defined as follows:
$$\nu'' = \nu' \circ \nu, \qquad \varepsilon'' = \varepsilon' \circ \varepsilon, \tag{6.2}$$
$$l''(\lambda) = l'(\nu(\lambda)) \cdot l(\lambda), \qquad \tau''(\lambda) = \tau'(\nu(\lambda)) \cdot \tau(\lambda), \tag{6.3}$$
for all $\lambda \in \Lambda$.

Let $\mathbf{1} : \Lambda \to \{\pm 1\}$ be the constant function taking value $+1$. Then $(\mathrm{id}_\Lambda, \mathrm{id}_H, \mathbf{1}, \mathbf{1})$ is the unit of $\mathrm{Aut}(G)$ and $(\nu, \varepsilon, l, \tau)^{-1} = (\nu^{-1}, \varepsilon^{-1}, l, \tau)$.

For a set X denote by $\Sigma(X)$ the group of all bijections of X, that is the permutation group on X. For a group A let also A^X be the group of all maps $X \to A$ with respect to the point-wise multiplication. Then the group $\Sigma(X)$ naturally acts from the right on A^X by the rule: the result of the action of a bijection $\nu : X \to X$ from $\Sigma(X)$ on a map $a : X \to A$ belonging to A^X is the composition map

$$a \circ \nu : X \xrightarrow{\nu} X \xrightarrow{a} A.$$

The corresponding semidirect product $A^X \rtimes \Sigma(X)$ is called the *wreath* product of $\Sigma(X)$ and A over X and denoted by $A \wr_X \Sigma(X)$. Thus, by definition, $A \wr_X \Sigma(X)$ is a direct product of sets $A^X \times \Sigma(X)$ with respect to the following multiplication:

$$(a', \nu')(a, \nu) = \bigl((a' \circ \nu) \cdot a,\ \nu' \circ \nu\bigr),$$

where $\cdot$ denotes multiplication in A^X. Notice that there is a natural surjective homomorphism $\eta : A \wr_X \Sigma(X) \to \Sigma(X)$, $\eta(a, \nu) = \nu$, whose kernel is $A^X \times \mathrm{id}_X$. Moreover, we also have an inclusion $\mathbf{1} \times \Sigma(X) \subset A \wr_X \Sigma(X)$, where $\mathbf{1} : X \to A$ is the constant map into the unit of A. In other words the following short exact sequence

$$1 \to A^X \to A \wr_X \Sigma(X) \xrightarrow{\eta} \Sigma(X) \to 1$$

admits a *section* $s : \Sigma(X) \to A \wr_X \Sigma(X)$, $s(\nu) = (\mathbf{1}, \nu)$, i.e. a homomorphism such that $\eta \circ s = \mathrm{id}(\Sigma(X))$.

Rewriting (6.2) and (6.3) in the form:

$$(\nu', \varepsilon', l', \tau')\ (\nu, \varepsilon, l, \tau) = \bigl(\nu' \circ \nu,\ \varepsilon' \circ \varepsilon,\ (l' \circ \nu) \cdot l,\ (\tau' \circ \nu) \cdot \tau\bigr)$$

we see that $\mathrm{Aut}(G)$ is a *subgroup* of

$$\Bigl(\{\pm 1\}^2 \wr_\Lambda \Sigma(\Lambda)\Bigr) \times \Sigma(H).$$

Theorem 6.2. *Each equivalence of striped atlases induces an isomorphism between their graphs. Conversely, each isomorphism between their graphs is induced by some striped atlases equivalence.*

Before proving Theorem 6.2 let us first consider several illustrating examples. To preserve the formalism we need to talk about maps from empty set. As usual, we identify a map $f : A \to B$ between sets with its graph $\{(a, f(a)) \mid a \in A\} \subset A \times B$. Therefore a map $\varnothing \to B$ from empty set is an empty subset of the empty set $\varnothing \times B$.

Example 6.3. Let $S = \mathbb{R} \times (-1, 1)$ and $q = \mathrm{id}_S : S \to S$ be a striped atlas consisting of one strip, see Figure 6.1(a). Then $\Lambda = \{*\}$ consists of a unique point, $H = \varnothing$, and so $\xi : H \to H$ and $\sigma : H \setminus \mathrm{Fix}(\xi) \to \{\pm 1\}$ are maps of empty set.

Let $(\nu, \varepsilon, l, \tau) \in \mathrm{Aut}(G)$. Then $\nu = \mathrm{id}_\Lambda$ and $\varepsilon = \mathrm{id}_H$ are uniquely determined, while $l, \tau : \{*\} \to \{\pm 1\}$ can be arbitrary maps. It easily follows that $\mathrm{Aut}(G) \cong \{\pm 1\} \times \{\pm 1\}$.

(a) (b)

FIGURE 6.1. Striped atlases consisting of one strip and being the identity homeomorphisms

Example 6.4. Let $S = \mathbb{R} \times (-1, 1) \cup \{(-2, -1) \cup (1, 2)\} \times \{1\}$ and again $q = \mathrm{id}_S : S \to S$ be a striped atlas consisting of one strip, see Figure 6.1(b). Then $\Lambda = \{*\}$ consists of a unique point, $H = \{a, b\} = d_{+1}(*)$, where $a = (-2, -1) \times \{1\}$, $b = (1, 2) \times \{1\}$, and $a < b$ in the sense of the linear order in $d_{+1}(*)$. As these intervals are not glued, we see that $\xi = \mathrm{id}_H : H \to H$ and so $\sigma : H \setminus \mathrm{Fix}(\xi) \to \{\pm 1\}$ is a map from empty set.

Let $x = (\nu, \varepsilon, l, \tau) \in \mathrm{Aut}(G)$. Then $\nu = \mathrm{id}_\Lambda$. Moreover, as $H = d_{+1}(*)$, and so $d_{-1}(*) = \varnothing$, it follows that $\varepsilon(d_{+1}(*)) = d_{+1}(*)$, whence $\tau(*) = +1$.

If $\varepsilon(a) = a$, then $\varepsilon = \mathrm{id}_H$, whence $l(*) = +1$, and so x is the unit of $\mathrm{Aut}(G)$. Suppose $\varepsilon(a) = b$, then $\varepsilon(b) = a$, so ε is an order reversing bijection of $H = d_{+1}(*)$, whence $l(*) = -1$. Thus $\mathrm{Aut}(G)$ consists of two elements, i.e. $\mathrm{Aut}(G) \cong \{\pm 1\}$.

Example 6.5. Let $S = \mathbb{R} \times [0, 1]$, $\phi : \mathbb{R} \times \{0\} \to \mathbb{R} \times \{+1\}$ be a homeomorphism given by $\phi(x, 0) = (x, 1)$, then the quotient $Z = S/\phi$ is an *open cylinder* $\mathbb{R} \times S^1$, and the quotient map $q : S \to Z$ is a striped atlas, see Figure 6.2(a).

In this case $\Lambda = \{*\}$ again consists of a unique point, $H = \{a, b\}$, where $a = \mathbb{R} \times \{0\}$, $b = \mathbb{R} \times \{1\}$, $\xi : H \to H$ is given $\xi(a) = b$, $\xi(b) = a$, and $\sigma : H \to \{\pm 1\}$ is defined by $\sigma(a) = \sigma(b) = \mathbf{or}(\phi) = +1$.

Let $x = (\nu, \varepsilon, l, \tau) \in \mathrm{Aut}(G)$. Then $\nu = \mathrm{id}_\Lambda$. Moreover, since G has a unique edge $\{a, b\}$, ε preserves this edge, whence it follows from (6.1) that $l(a) = l(b)$.

Suppose $\varepsilon(a) = a$, then $\varepsilon = \mathrm{id}_H$, whence $\tau(*) = +1$. Otherwise, $\varepsilon(a) = b$, $\varepsilon(b) = a$, and $\tau(*) = -1$. Notice that in both of those cases, the common value $l(a) = l(b)$ can be taken arbitrary.

This implies that $\mathrm{Aut}(G) \cong \{\pm 1\} \times \{\pm 1\}$.

Example 6.6. Suppose as in the previous example $S = \mathbb{R} \times [0, 1]$, but now $\phi : \mathbb{R} \times \{0\} \to \mathbb{R} \times \{+1\}$ is given by $\phi(x, 0) = (-x, 1)$, and so it reverses orientation. In this case the quotient $Z = S/\phi$ is an *open Möbius band*, see Figure 6.2(b). One easily check that $\mathrm{Aut}(G) \cong \{\pm 1\} \times \{\pm 1\}$ as well.

Proof of Theorem 6.2. Let

$$q : Z_0 = \bigsqcup_{\lambda \in \Lambda} S_\lambda \to Z, \qquad q' : Z'_0 = \bigsqcup_{\lambda' \in \Lambda'} S'_{\lambda'} \to Z'$$

(a) (b)

FIGURE 6.2. Foliated open cylinder and Möbius band

be striped atlases on surfaces Z and Z' respectively, and $G = (\Lambda, H, \xi, \sigma)$ and $G' = (\Lambda', H', \xi', \sigma')$ be the their graphs.

1) Suppose (h, k) is a pair of homeomorphisms defining an equivalence of atlases, so we have a commutative diagram (5.2). Then h induces a bijection between the connected components of Z_0 and Z'_0 which yields a bijection $\nu : \Lambda \to \Lambda'$ between the corresponding sets of indices (being in turn vertices of G and G') such that $h(S_\lambda) = S'_{\nu(\lambda)}$.

In particular, h yields also a bijection between the boundary components of Z_0 and Z'_0 being sets of half edges of G and G'. Thus we get a bijection $\varepsilon : H \to H'$.

It remains to define the functions $l, \tau : \Lambda \to \{\pm 1\}$. Take $\lambda \in \Lambda$ and consider the restriction $h|_{S_\lambda} : S_\lambda \to S'_{\nu(\lambda)}$. Assume that $\mathrm{Int} S_\lambda = \mathbb{R} \times (a, b)$ and $\mathrm{Int} S_{\nu(\lambda)} = \mathbb{R} \times (c, d)$ for some $a < b, c < d \in \mathbb{R}$. Since $h|_{S_\lambda}$ preserves leaves being horizontal lines, we have that

$$h|_{S_\lambda}(x, y) = (\alpha(x, y), \beta(y))$$

where

- $\alpha : S_\lambda \to \mathbb{R}$ is a continuous function such that for each $y \in (a, b)$ the correspondence $x \mapsto \alpha(x, y)$ is a homeomorphism $\alpha_y : \mathbb{R} \to \mathbb{R}$;
- $\beta : (a, b) \to (c, d)$ is a homeomorphism.

Evidently, all homeomorphisms α_y are increasing or decreasing mutually for all $y \in (a, b)$, i.e. $\mathbf{or}(\alpha_y)$ does not depend on $y \in (a, b)$. Therefore we set

$$l(\lambda) = \mathbf{or}(\alpha_y), \qquad \tau(\lambda) = \mathbf{or}(\beta).$$

We claim that $(\nu, \varepsilon, l, \tau)$ is an isomorphism between graphs G and G' in the sense of Definition 6.1.

Notice that the restriction $h|_{\partial S_\lambda} : \partial S_\lambda \to \partial S'_{\nu(\lambda)}$ is a monotone homeomorphism which easily implies conditions (a) and (b) of Definition 6.1. We leave the verification for the reader and will check condition (c) only.

Let $\{X, Y\}$ be a closed edge of G with $X \in d(\lambda)$ and $Y = \xi(X) \in d(\mu)$ for some $\lambda, \mu \in \Lambda$, and $X' = \varepsilon(X)$ and $Y' = \varepsilon(Y)$. This means that $X \subset \partial S_\lambda$ and $Y \subset \partial S_\mu$ are boundary components with $q(X) = q(Y)$,

$X' = h(X) \subset \partial S'_{\nu(\lambda)}$, and $Y' = h(Y) \subset \partial S'_{\nu(\mu)}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
Y & \xrightarrow[\mathbf{or}(h|_Y)=l(\mu)]{h|_Y} & Y' \\
{\scriptstyle\phi}\Big\downarrow {\scriptstyle \mathbf{or}(\phi)=\sigma(X,Y)} & & {\scriptstyle \mathbf{or}(\phi')=\sigma'(X',Y')}\Big\downarrow{\scriptstyle\phi'} \\
X & \xrightarrow[h|_X]{\mathbf{or}(h|_X)=l(\lambda)} & X'
\end{array}$$

where ϕ and ϕ' are gluing homeomorphisms. Hence

$$\begin{aligned} l(\lambda)\cdot\sigma(X,Y) = \mathbf{or}(h|_Y)\cdot\mathbf{or}(\phi) = \mathbf{or}(\phi\circ h|_Y) = \\ = \mathbf{or}(h|_X\circ\phi') = \mathbf{or}(h|_X)\cdot\mathbf{or}(\phi') = l(\mu)\cdot\sigma'(X',Y'). \end{aligned}$$

2) To prove the converse statement, notice that due to Theorem 5.8, one can assume in addition that both atlases q and q' are affine.

Let $(\nu, \varepsilon, l, \tau)$ be an isomorphism between G and G' in the sense of Definition 6.1.

Let $\lambda \in \Lambda$ and $\lambda' = \nu(\lambda)$. We will now construct a homeomorphism $h_\lambda : S_\lambda \to S'_{\lambda'}$ in the following way.

(i) First suppose $\partial S_\lambda = \varnothing$, that is $d(\lambda) = \varnothing$. Since ε bijectively maps $d(\lambda)$ onto $d'(\lambda')$, it follows that $d'(\lambda') = \varnothing$, and so $\partial S'_{\nu(\lambda)} = \varnothing$ as well. Not loosing generality, one can assume that $S_\lambda = S'_{\lambda'} = \mathbb{R}\times(-1,1)$. Then we define h_λ by the formula:

$$h_\lambda(x,y) = \big(l(\lambda)x, \tau(\lambda)y\big).$$

(ii) Now assume that $\partial S_\lambda \neq \varnothing$. Let $X \in d(\lambda)$ be a half-edge in G incident to the vertex λ, that is X is a boundary component of S_λ. Then $\nu(X)$ is a boundary interval of $S'_{\nu(\lambda)}$. Since we assumed that strips S_λ and $S'_{\nu(\lambda)}$ are model, the intervals X and $\nu(X)$ are bounded. Define h_λ on X to be a unique affine homeomorphism $\psi_X : X \to \nu(X)$ with $\mathbf{or}(\psi_X) = l(\lambda)$.

The family of all $\{\psi_X\}_{X\in d(\lambda)}$ give a homeomorphism $h_\lambda : \partial S_\lambda \to \partial S'_{\nu(\lambda)}$. Due to property (a) of Definition 6.1, h_λ is monotone, and therefore by Theorem 2.3 h extends to a foliated homeomorphism $h_\lambda : S_\lambda \to S'_{\nu(\lambda)}$.

Thus we obtain a foliated homeomorphism $h : \bigsqcup\limits_{\lambda\in\Lambda} S_\lambda \to \bigsqcup\limits_{\lambda'\in\Lambda'} S'_{\lambda'}$ defined by $h|_{S_\lambda} = h_\lambda$ for $\lambda \in \Lambda$.

We claim that h *induces a foliated homeomorphism* $k : Z \to Z'$ *such that the pair* (h,k) *is an equivalence of striped atlases* q *and* q'.

Let $D \subset Z_0$, (resp. $D' \subset Z'_0$), be the set of boundary intervals on which q, (resp. q'), is not injective. Then h yields a homeomorphism of $Z_0 \setminus D$ onto $Z'_0 \setminus D'$, whence the restriction $k : q(Z_0 \setminus D) \to q'(Z'_0 \setminus D')$ must be given by $k = q' \circ h \circ q^{-1}$.

Therefore it remains to show that h is "compatible" with q and q' on D and D' in the sense that for each pair of boundary intervals $X \subset \partial S_\lambda$ and $Y \subset \partial S_\mu$ with $q(X) = q(Y)$, we have that $q'(h(X)) = q'(h(Y))$ and the following commutative diagram holds true:

$$\begin{tikzcd} q(X) \arrow[d, equal] & X \arrow[l, "q"'] \arrow[r, "h|_X=\psi_X"] \arrow[d, "\phi"'] & X' \arrow[r, "q'"] \arrow[d, "\phi'"'] & q'(X') \arrow[d, equal] \\ q(Y) & Y \arrow[l, "q"'] \arrow[r, "h|_Y=\psi_Y"] & Y' \arrow[r, "q'"] & q'(Y') \end{tikzcd} \tag{6.4}$$

where ϕ and ϕ' are gluing homeomorphisms. Then for each $z \in D$ we will set $k(q(z)) = q' \circ h(z)$.

In term of graphs we have that $\{X, Y\}$ is a closed edge of G such that $X \in d(\lambda)$, $Y = \xi(X) \in d(\mu)$, $X' = \varepsilon(X)$, and $Y' = \varepsilon(Y)$. Then by (b)

$$Y' = \varepsilon(Y) = \varepsilon \circ \xi(X) = \xi' \circ \varepsilon(X) = \xi'(X)$$

and so $\{X', Y'\}$ is a closed edge of G', that is $q'(h(X)) = q'(h(Y))$.

Then we have the diagram (6.4) but need to check commutativity of its central square consisting of affine homeomorphisms. It follows from (c) that $\mathbf{or}(\phi' \circ \psi_X) = \mathbf{or}(\psi_Y \circ \phi)$. Since $\phi' \circ \psi_X, \psi_Y \circ \phi : X \to Y'$ are *affine* homeomorphisms, it follows that they coincide, and so diagram (6.4) is commutative.

Thus (h, k) is an equivalence of striped atlases inducing given isomorphism $(\nu, \varepsilon, l, \tau)$ between G and G'. □

7. Characterization of a certain class of striped surfaces

Let (Z, Δ) be a foliated surface with countable base, Z/Δ the set of leaves of Δ, and $p : Z \to Z/\Delta$ be the quotient map. We will endow Z/Δ with the quotient topology, so a subset $V \subset Z/\Delta$ is open if and only if $p^{-1}(V)$ is open in Z. Notice that a priori Z/Δ is not even a T_0-space.

For each leaf ω of Δ let $J_\omega = [0, 1)$ if $\omega \subset \partial Z$ and $J_\omega = (-1, 1)$ otherwise. Then a *cross-section* of Δ passing through ω is a continuous map $\gamma : J_\omega \to Z$ such that $\gamma(0) \in \omega$ and for distinct $s, t \in J_\omega$ their images $\gamma(s)$ and $\gamma(t)$ belong to distinct leaves of Δ.

A subset $U \subset Z$ is called *saturated* if it is a union of leaves. For each leaf $\omega \in \Delta$ denote by $\mathrm{c}(\omega)$ the intersection of closures of all saturated neighbourhoods of ω. Evidently $\omega \subset \mathrm{c}(\omega)$.

Definition 7.1. [8] A leaf ω will be called *special* whenever $\omega \neq \mathrm{c}(\omega)$, see Figure 7.1. We will denote by Σ the family of all special leaves of Δ.

In [4] and [3] special leaves were called *branch* points of Z/Δ.

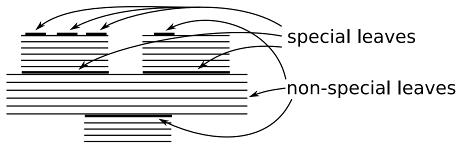

FIGURE 7.1. Special and non-special leaves

Lemma 7.2. *Suppose there exist a striped atlas $q : \bigsqcup\limits_{\lambda\in\Lambda} S_\lambda \to Z$ such that Δ is its canonical foliation. Let also $D = q\Bigl(\bigsqcup\limits_{\lambda\in\Lambda} \partial S_\lambda\Bigr)$ be the union of images of boundary components of strips. Then*

(i) $\Sigma \cup \partial Z \ \subset \ D$;
(ii) Σ, ∂Z, *and* D *are a locally finite families of leaves;*
(iii) Σ, ∂Z, *and* D *are closed subsets of* Z.

Proof. (i) By definition $\partial Z \subset D$. Moreover, one easily check that a leaf $\omega \in \Delta$ is special, i.e. $\omega \subset \Sigma$, if and only if there exists a boundary interval $X \subset \partial_\epsilon S_\lambda$ for some $\lambda \in \Lambda$ and $\epsilon \in \{\pm\}$ such that $q(X) = \omega$ and $X \neq \partial_\epsilon S_\lambda$. Hence $\Sigma \subset D$ as well.

(ii). Evidently, each leaf ω in D has an open neighbourhood containing no other leaves from D. This implies that D is a locally finite family of closed subsets of Z, whence so any subfamily of D. In particular, this holds for Σ and ∂Z.

(iii) follows from (ii), since each leaf of Δ is a closed subset of Z. □

A striped atlas on Z will be called *reduced* whenever $D = \Sigma \cup \partial Z$.

Theorem 7.3. [8, Theorem 3.7]. *Let Z be a striped surface with countable base. Then one of the following statements holds true: either*

(1) *Z is foliated homeomorphic to the open cylinder or Möbius band from Examples 6.5 and 6.6, or*
(2) *Z admits a reduced atlas.*

Idea of proof. We briefly discuss the proof in terms of the graph G of the striped atlas q. It will be convenient to say that an edge $\{X, Y\}$ of G is *unessential* whenever $X = \partial_\epsilon S$ and $Y = \partial_{\epsilon'} S'$ for some *distinct* strips S, S' of the atlas and some $\epsilon, \epsilon' \in \{\pm\}$, see Figure 7.2.

In particular, each unessential edge corresponds to a non-special leaf $\omega \subset D \setminus (\Sigma \cup \partial Z)$. The principal observation of Theorem 7.3 is that gluing S and S' along X and Y gives again a strip S_1, see [8, Lemma 3.2]. Therefore one can replace S and S' in the atlas q with S_1.

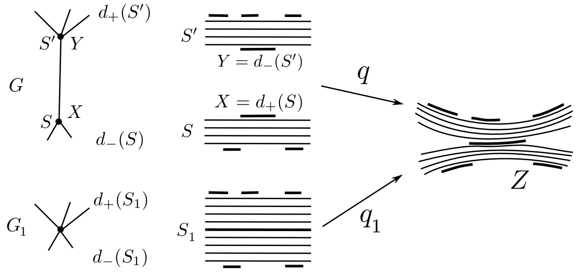

FIGURE 7.2. Reduction of non-special leaves from D

On the graph this means that we replace a closed edge between S and S' with one vertex. That techniques also allows to eliminate even countable paths of such edges. Hence if G does not contain finite cycles of unessential edges, then, using the assumption that Z has a countable base, one can remove all unessential edges and obtain a reduced atlas.

However, if there is a finite cycle of unessential edges, then one can remove all of them but one. This gives two special surfaces: open cylinder and Möbius band from Examples 6.5 and 6.6 in which we glue $X = \partial_- S$ with $Y = \partial_+ S$. But the corresponding closed edge $\{X, Y\}$ is not unessential, since now X and Y belong to *the same* strip. □

Consider the following five conditions on (Z, Δ).

(StrAtlas): Z admits a striped atlas whose canonical foliation is Δ.

(ΣLocFin): The family Σ of all special leaves of Δ is locally finite.

(PrjLocTriv): The quotient map $p : Z \to Z/\Delta$ is a locally trivial fibration and the space of leaves Z/Δ is locally homeomorphic with $[0, 1)$ (though it is not in general a Hausdorff space).

(SatNbh): For each leaf $\omega \in \Delta$ there exist an open Δ-saturated neighbourhood U of ω and a homeomorphism $\eta : \mathbb{R} \times J_\omega \to U$ such that $\eta(\mathbb{R} \times t)$ is a leaf of Δ and $\eta(\mathbb{R} \times 0) = \omega$.

(CrossSect): Each leaf $\omega \in \Delta$ has a cross-section passing through ω.

The following statement summarizes relations between the above properties obtained in [8], [9], [10], and in the present paper.

Theorem 7.4. [9], [10]. *Let (Z, Δ) be a foliated surface satisfying the following two conditions:*

(i) *each leaf of Δ is a non-compact closed subset of Z;*
(ii) *each boundary component of Z is a leaf of Δ.*

Then we have the following implications:

- (PrjLocTriv) $\Rightarrow$ (SatNbh) + (CrossSect)*;*
- (StrAtlas) $\Rightarrow$ (ΣLocFin) $\Rightarrow$ $\big[$(PrjLocTriv) $\Leftrightarrow$ (SatNbh) $\Leftrightarrow$ (CrossSect)$\big]$*;*
- (ΣLocFin) + (SatNbh) $\Rightarrow$ (StrAtlas)*.*

In particular, if either (SatNbh) *or* (CrossSect) *hold, then the conditions* (ΣLocFin) *and* (StrAtlas) *are equivalent.*

Proof. The implication (PrjLocTriv) $\Rightarrow$ (SatNbh) + (CrossSect) and the equivalence of conditions (SatNbh), (PrjLocTriv), and (CrossSect) under assumption (ΣLocFin) is proved in [9, Theorem 2.8].

The implication (StrAtlas) $\Rightarrow$ (ΣLocFin) is contained in statement (ii) of Lemma 7.2.

Finally, the implication (ΣLocFin) + (SatNbh) $\Rightarrow$ (StrAtlas) is established in [10, Theorem 1.8]. $\square$

Remark 7.5. For a striped surface (Z, Δ) with a striped atlas q condition (SatNbh) is equivalent to the requirement that q *does not glue together boundary intervals belonging to the same side of the same strip.* More precisely, if $q(X) = q(Y)$ for some distinct boundary intervals $X \subset \partial_\epsilon S$ and $Y \subset \partial_{\epsilon'} S'$, then either $S \neq S'$ or $\epsilon \neq \epsilon'$.

7.6. **Foliated surface that does not admit a striped atlas.** Consider the sequence $z_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$, of points of y-axis on the plane converging to the origin O and put $K = \{z_n\}_{n\in\mathbb{N}} \cup O$. Let also $Z = \mathbb{R}^2 \setminus K$. Then Z admits a foliation Δ into non-compact leaves being connected components of the intersection of Z with horizontal lines.

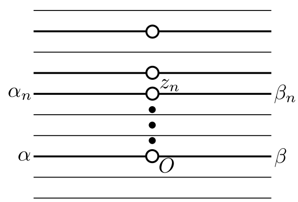

FIGURE 7.3.

Lemma 7.6.1. *The pair* (Z, Δ) *satisfies condition* (CrossSect) *and violates* (ΣLocFin)*. Hence it also violates* (StrAtlas)*, that is* Z *does not admit a striped atlas for which* Δ *is a canonical foliation.*

Proof. (CrossSect). For each leaf $\omega \in \Delta$ there exists a cross-section being just an one vertical interval in Z transversal to ω.

To show that (ΣLocFin) fails, denote

$$\alpha_n = (-\infty, 0) \times z_n, \qquad \beta_n = (0, +\infty) \times z_n,$$

$$\alpha = (-\infty, 0) \times O, \qquad \beta = (0, +\infty) \times O.$$

Then $\Sigma = \{\alpha_n, \beta_n\}_{n\in\mathbb{N}} \cup \{\alpha, \beta\}$ is the family of all special leaves of Δ. Evidently,

$$\mathrm{c}(\alpha_n) = \mathrm{c}(\beta_n) = \{\alpha_n, \beta_n\},\ n \in \mathbb{N}, \qquad \mathrm{c}(\alpha) = \mathrm{c}(\beta) = \{\alpha, \beta\},$$

whence Σ is not locally finite, since α_n converges to α and β_n converges to β. Therefore by Theorem 7.4 Z does not admit a striped atlas with a canonical foliation Δ. □

Notice also that $Z' = Z \setminus \{\alpha, \beta\}$ is disconnected and each of its connected components admits a striped atlas.

7.7. **Foliation on the plane that does not admit a striped atlas.** We will construct a more complicated example on the plane $\mathbb{R}^2$. Consider the foliation Δ_0 on the strip $S = \mathbb{R} \times [0, 1]$ shown in Figure 7.4(a). As indicated in Figure 7.4(b) it is glued from four strips. For each $k \in \mathbb{N}$

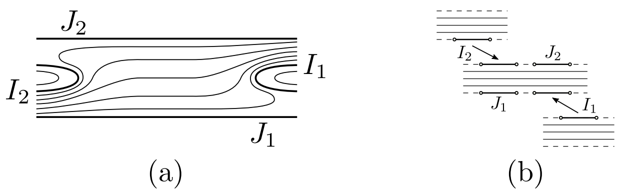

FIGURE 7.4.

define the following strip $S_k = \mathbb{R} \times [\frac{1}{2^k}, \frac{1}{2^{k-1}}]$ and a homeomorphism

$$\phi_n : S \to S_k, \qquad \phi_k(x, y) = (kx, (y+1)/2^k),$$

so it expands strip along x-axis and shrinks it along y-axis.

Denote by Δ_k foliation on S_k being the image of Δ_0 under ϕ_k. Then the union of all Δ_k gives a foliation on $\mathbb{R} \times [0, 1]$ which extends to the foliation on all of $\mathbb{R}^2$ by parallel lines $\mathbb{R} \times y$ for $y \in (-\infty, 0] \cup (1, +\infty)$. We will denote that foliation on $\mathbb{R}^2$ by Δ.

Then $\Sigma = \{\phi_k(I_1), \phi_k(I_2), \phi_k(J_1) = \phi_{k+1}(J_2)\}_{k\in\mathbb{N}}$ is the family of all special leaves of Δ. This set is not locally finite since the leaves $\phi_k(J_1)$ converge to the leaf $\mathbb{R} \times 0$. One easily check that Δ satisfies (CrossSect), whence by Theorem 7.4 $(\mathbb{R}^2, \Delta)$ does not admit a striped atlas.

8. HOMEOTOPY GROUP OF A CANONICAL FOLIATION

Let Z be a connected striped surface with a canonical foliation Δ. Denote by $\mathcal{H}(\Delta)$ the group of all foliated homeomorphisms of (Z, Δ). Thus, by definition, $\mathcal{H}(\Delta)$ consists of all homeomorphism $h : Z \to Z$ such that for each leaf $\omega \in \Delta$ its image $h(\omega)$ is a leaf of Δ as well. Endow $\mathcal{H}(\Delta)$ with the compact open topology and let $\mathcal{H}_0(\Delta)$ be the identity path component of $\mathcal{H}(\Delta)$. It consists of all homeomorphisms $h \in \mathcal{H}(\Delta)$ isotopic to id_Z in $\mathcal{H}(\Delta)$. Then $\mathcal{H}_0(\Delta)$ is a normal subgroup of $\mathcal{H}(\Delta)$ and the quotient $\mathcal{H}(\Delta)/\mathcal{H}_0(\Delta)$ can be identified with the set $\pi_0\mathcal{H}(\Delta)$ of all path components of $\mathcal{H}(\Delta)$, that is $\pi_0\mathcal{H}(\Delta) = \mathcal{H}(\Delta)/\mathcal{H}_0(\Delta)$. This group will be called the *homeotopy* group of the foliation Δ.

Theorem 8.1. c.f. [8, Theorem 4.4]. *Let $q : \bigsqcup_{\lambda\in\Lambda} S_\lambda \to Z$ be a **reduced affine** striped atlas on a **connected** surface Z, G be its graph, and Δ be the corresponding canonical foliation.*

(i) *If Z is foliated homeomorphic either to an open cylinder from Example 6.5 or a Möbius band from Example 6.6 then $\mathcal{H}_{\mathrm{id}}(\Delta)$ is homotopy equivalent to the circle S^1.*
(ii) *Otherwise, $\mathcal{H}_{\mathrm{id}}(\Delta)$ is contractible.*

In all the cases we have an isomorphism $\rho : \pi_0\mathcal{H}(\Delta) \cong \mathrm{Aut}(G)$.

Proof. (i) Suppose Z is either an open cylinder or a Möbius band. Since by Examples 6.5 and 6.6 $\mathrm{Aut}(G) \cong \{\pm1\}\times\{\pm1\}$ we need only to show that $\mathcal{H}_{\mathrm{id}}(\Delta)$ is homotopy equivalent to the circle and $\pi_0\mathcal{H}(\Delta) \cong \{\pm1\} \times \{\pm1\}$ as well. We leave this statement as an exercise for the reader.

(ii). Now let Z be *neither* an open foliated cylinder *nor* a foliated Möbius band. Then the following statement is a reformulation of [8, Theorem 4.4] in terms of striped atlases and their graphs. In particular, it contains (ii).

Lemma 8.1.1. c.f. [8, Theorem 4.4]. *For each $k \in \mathcal{H}(\Delta)$ there exists a unique homeomorphism $h : Z_0 \to Z_0$ such that $q \circ h = k \circ q$, i.e. (h, k) is a self-equivalence of the atlas q. Moreover, $k \in \mathcal{H}_{\mathrm{id}}(\Delta)$ if and only if (h, k) induces the identity automorphism of G.*

Also the group $\mathcal{H}_{\mathrm{id}}(\Delta)$ is contractible. □

It remains to construct an isomorphism $\rho : \pi_0\mathcal{H}(\Delta) \cong \mathrm{Aut}(G)$. Let $k \in \mathcal{H}(\Delta)$ and (h, k) be the self-equivalence of the atlas q. Denote by $\rho(k)$ the automorphism of G induced by (h, k), see Theorem 6.2. Then one easily check that the correspondence $k \mapsto \rho(k)$ is a homomorphism $\rho : \pi_0\mathcal{H}(\Delta) \cong \mathrm{Aut}(G)$.

Moreover, by Theorem 6.2 ρ is surjective, and by Lemma 8.1.1 its kernel is $\mathcal{H}_{\mathrm{id}}(\Delta)$. This gives the required isomorphism

$$\pi_0\mathcal{H}(\Delta) = \mathcal{H}(\Delta)/\mathcal{H}_{\mathrm{id}}(\Delta) = \mathcal{H}(\Delta)/\ker(\rho) \cong \mathrm{Aut}(G).$$

Theorem 8.1 is completed. □

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