Proceedings of the
International Geometry Center
Vol. 10, no. 3-4 (2017) pp. 7–16

Some remarks concerning strongly separately continuous functions on spaces ℓ_p with $p \in [1, +\infty]$

Olena Karlova, Tomáš Visnyai

Abstract. We give a sufficient condition on strongly separately continuous function f to be continuous on space ℓ_p for $p \in [1, +\infty]$. We prove the existence of an ssc function $f : \ell_\infty \to \mathbb{R}$ which is not Baire measurable. We show that any open set in ℓ_p is the set of discontinuities of a strongly separately continuous real-valued function for $p \in [1, +\infty)$.

1. Introduction

The notion of real-valued strongly separately continuous (ssc) function defined on $\mathbb{R}^n$ was introduced and studied by Dzagnidze in his paper [1]. Later, the authors extended in [7] the notion of the strong separate continuity to functions defined on the Hilbert space ℓ_2 equipped with the norm topology and proved, in particular, that there exists a real-valued ssc function on ℓ_2 which is everywhere discontinuous. Visnyai [8] constructed an ssc function $f : \ell_2 \to \mathbb{R}$ which belongs to the third Baire class and is not quasi-continuous at every point. Moreover, he gave a sufficient condition for a strongly separately continuous function to be continuous on ℓ_2.

In [3] Karlova extended the concept of an ssc function on any $\mathcal{S}$-open subset of a product of topological spaces and investigated ssc functions with open set of discontinuities defined on a special subsets of a product of a sequence of normed spaces. Karlova and Mykhaylyuk obtained a characterization of the set of all points of discontinuity of strongly separately continuous functions defined on subspaces of products of finite-dimensional normed spaces [4].

2010 Mathematics Subject Classification: Primary 26A21, 54C08; Secondary 26B05, 54C30

Keywords: strongly separately continuous function, Baire classification
DOI: http://dx.doi.org/10.15673/tmgc.v10i3-4.769

Karlova and Visnyai proved in [5] that any open set in ℓ_p is the set of discontinuities of a strongly separately continuous real-valued function for $p \in [1, +\infty)$ (see [5, Theorem 4.1]). Unfortunately, the proof of this result contains a gap, which we remove in Theorem 5.3 of this paper.

The Baire classification of ssc functions was investigated in [3] and [5]. It was proved that for every $2 \le \alpha < \omega_1$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ which belongs the α'th Baire class and does not belong to the β'th Baire class on ℓ_p for $\beta < \alpha$, $p \in [1, +\infty)$.

In this paper we continue to study ssc functions defined on spaces ℓ_p with $p \in [1, +\infty]$. In Section 3 we give a sufficient condition on ssc function f defined on ℓ_p to be continuous. Further, we prove in Section 4 that there exists an ssc function $f : \ell_\infty \to \mathbb{R}$ which is not Baire measurable. Section 5 contains a result on a construction of ssc functions with open set of discontinuities.

2. Definitions and notations

We denote by ℓ_p, $p \in [1, +\infty)$, the normed space consisting of all sequences $x = (x_k)_{k=1}^{\infty}$ of reals such that $\sum_{k=1}^{\infty} |x_k|^p < +\infty$ endowed with the standard norm $\|\cdot\|_p$ defined by the rule

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

for all $x = (x_k)_{k=1}^{\infty} \in \ell_p$.

Let ℓ_∞ be the space of all bounded sequences of reals with the norm

$$\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$$

for all $x = (x_k)_{k=1}^{\infty} \in \ell_\infty$.

If $p \in [1, +\infty]$, $x^0 \in \ell_p$ and $\delta > 0$, then we write

$$B_p(x^0, \delta) = \{x \in \ell_p : \|x - x^0\|_p < \delta\}.$$

Definition 2.1. Let $p \in [1, +\infty]$, $x^0 = (x_k^0)_{k=1}^{\infty} \in \ell_p$ and $(Y, |\cdot - \cdot|)$ be a metric space. A function $f : \ell_p \to Y$ is said to be

- *separately continuous at a point x^0 with respect to the k-th variable* if the function $\varphi_k : \mathbb{R} \to Y$,

$$\varphi_k(t) = f(x_1^0, \dots, x_{k-1}^0, t, x_{k+1}^0, \dots)$$

  for all $t \in \mathbb{R}$, is continuous at x_k^0.
- *strongly separately continuous at a point x^0 with respect to the k-th variable* if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x = (x_k)_{k=1}^{\infty} \in B_p(x^0, \delta)$$

$$|f(x_1,\dots,x_k,\dots) - f(x_1,\dots,x_{k-1},x_k^0,x_{k+1},\dots)| < \varepsilon. \tag{2.1}$$

If f is strongly separately continuous at x^0 with respect to each variable, then f is said to be *strongly separately continuous at* x^0. Moreover, f is *(strongly) separately continuous on* ℓ_p if it is (strongly) separately continuous at each point of ℓ^p.

It is easy to see that

continuity ⇒ strong separate continuity ⇒ separate continuity.

None of the converse implications is true as the following examples show.

Example 2.2. Let

$$f(x_1,x_2,\dots) = \begin{cases} \dfrac{x_1\cdot x_2}{x_1^2+x_2^2}, & x_1^2+x_2^2\neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

The function $f\colon \ell_p\to\mathbb{R}$ is separately continuous on ℓ_p for every $p\in[1,+\infty]$, but is not strongly separately continuous at $(0,0,\dots)$ for any $p\in[1,+\infty]$ (see remarks after Theorem 3.1).

Example 2.3. Let $A=\{x=(x_k)_{k=1}^\infty\in\ell_p : |\{k : x_k\in\mathbb{Q}\}|<\aleph_0\}$. We put

$$f(x_1,x_2,\dots) = \begin{cases} 1, & x\in A,\\ 0, & \text{otherwise.}\end{cases}$$

The function $f\colon \ell_p\to\mathbb{R}$ is strongly separately continuous on ℓ_p, but is everywhere discontinuous for every $p\in[1,+\infty]$.

Proof. Fix $p\in[1,+\infty]$. It is easy to see that both A and $\ell_p\setminus A$ are everywhere dense in ℓ_p. This imply that f is everywhere discontinuous on ℓ_p. Moreover, if x and y differs in at most one coordinate, then $x\in A$ if and only if $y\in A$. Therefore, $|f(x)-f(y)|=0$ and (2.1) holds. □

3. Continuity of ssc functions

We will prove in this section the sufficient condition on strongly separately continuous functions to be continuous on spaces ℓ_p.

For $p\in[1,+\infty]$, $x,y\in\ell_p$ and $n\in\mathbb{N}$ we put

$$\gamma_p^n(x,y) = \begin{cases} \sum\limits_{k>n}|x_k-y_k|^p, & p<+\infty,\\ \sup\limits_{k>n}|x_k-y_k|, & p=+\infty.\end{cases}$$

Theorem 3.1. *Let* $p \in [1, +\infty]$, $x^0 = \left(x_k^0\right)_{k=1}^{\infty} \in \ell_p$, $(Y, |\cdot - \cdot|)$ *be a metric space, and* $f : \ell_p \to Y$ *be a strongly separately continuous function at* x^0. *If for every* $\varepsilon > 0$ *there exist* $\delta > 0$ *and* $K \in \mathbb{N}$ *such that*

$$\gamma_p^K(x^0, y) < \delta \Rightarrow |f(y_1, \dots) - f\left(y_1, \dots, y_K, x_{K+1}^0, \dots\right)| < \varepsilon \tag{3.1}$$

for all $y = (y_k)_{k=1}^{\infty} \in \ell_p$, *then* f *is continuous at* x^0.

Proof. Fix $\varepsilon > 0$. According to the assumption there exists $\delta_0 > 0$ and $K \in \mathbb{N}$ such that the inequality

$$\gamma_p^K(x^0, y) < \delta_0$$

implies the inequality

$$|f(y_1, y_2, \dots) - f\left(y_1, \dots, y_K, x_{K+1}^0, x_{K+2}^0, \dots\right)| < \frac{\varepsilon}{2}$$

for all $y \in \ell_p$. Since f is strongly separately continuous at the point x^0, for every $k \in \{1, 2, \dots, K\}$ there exists $\delta_k > 0$ such that

$$|f(x_1, \dots, x_k, \dots) - f\left(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots\right)| < \frac{\varepsilon}{2K}$$

for all $x \in B_p(x^0, \delta_k)$. We put

$$\delta = \begin{cases} \min\left\{\sqrt[p]{\delta_0}, \delta_1, \dots, \delta_K\right\}, & p < \infty, \\ \min\left\{\delta_0, \delta_1, \dots, \delta_K\right\}, & p = \infty. \end{cases}$$

Let us take $x = (x_k)_{k=1}^{\infty} \in B_p\left(x^0, \delta\right)$ and observe that

$$\left(x_1^0, \dots, x_k^0, x_{k+1}, \dots\right) \in B_p\left(x^0, \delta\right)$$

for every $k \in \{1, \dots, K\}$. It follows that

$$\begin{aligned} &\left|f(x_1, x_2, \dots) - f\left(x_1^0, x_2^0, \dots\right)\right| \le \left|f(x_1, x_2, \dots) - f\left(x_1^0, x_2, \dots\right)\right| + \\ &\quad + \left|f\left(x_1^0, x_2, x_3, \dots\right) - f\left(x_1^0, x_2^0, x_3, \dots\right)\right| + \dots + \\ &\quad + \left|f\left(x_1^0, \dots, x_{K-1}^0, x_K, x_{K+1}, \dots\right) - f\left(x_1^0, \dots, x_{K-1}^0, x_K^0, x_{K+1}, \dots\right)\right| + \\ &\quad + \left|f\left(x_1^0, \dots, x_{K-1}^0, x_K^0, x_{K+1}, \dots\right) - f\left(x_1^0, \dots, x_K^0, x_{K+1}^0, x_{K+2}^0, \dots\right)\right| < \\ &\quad < K \cdot \frac{\varepsilon}{2K} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, f is continuous at x^0. □

Now we are ready to show that the function f from Example 2.2 is not strongly separately continuous at $x^0 = (0, 0, \dots)$. Assume the contrary and observe that for $K = 2$ we have $|f(y_1, y_2, \dots) - f(y_1, y_2, 0, \dots)| = 0$ for all $y \in \ell_p$. It follows that condition (3.1) holds for any $\varepsilon > 0$ and for any $\delta > 0$. Therefore, f has to be continuous at x^0 by Theorem 3.1, a contradiction.

As a straightforward corollary from Theorem 3.1 we obtain the next result.

Theorem 3.2. *Let* $p \in [1, +\infty]$, $(Y, |\cdot - \cdot|)$ *be a metric space and* $f : \ell_p \to Y$ *be a strongly separately continuous function. If*

$$\forall x \in \ell_p \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \exists K \in \mathbb{N}$$
$$|f(y_1, y_2, \dots) - f(y_1, \dots, y_K, x_{K+1}, x_{K+2}, \dots)| < \varepsilon$$

for all $y \in \ell_p$ *with* $\gamma_p^K(x, y) < \delta$, *then* f *is continuous on* ℓ_p.

4. Baire classification of ssc-functions

Let us recall the definition of Baire classes of functions. We denote the collection of all continuous maps $f : X \to Y$ between topological spaces X and Y by $\mathrm{B}_0(X, Y)$. Assume that the classes $\mathrm{B}_\xi(X, Y)$ are already defined for all $0 \le \xi < \alpha$, where $\alpha < \omega_1$. Then $f : X \to Y$ is said to be *of the* α*-th Baire class*, $f \in \mathrm{B}_\alpha(X, Y)$, if f is a pointwise limit of a sequence of maps $f_n \in \mathrm{B}_{\xi_n}(X, Y)$, where $\xi_n < \alpha$.

Let $0 \le \alpha < \omega_1$, X be a metrizable space, Y is a topological space and let Z be a locally convex space. According to Rudin' result [6] each map $f : X \times Y \to Z$, which is continuous with respect to the first variable and is of the α-th Baire class with respect to the second one, belongs to the $(\alpha + 1)$-th Baire class on $X \times Y$. It is easy to prove the corollary of the Rudin Theorem (see [3, Proposition 3.1]): if $n \in \mathbb{N}$, $X_1, \dots, X_n$ are metrizable spaces and Z is a locally convex space, then every separately continuous map $f : \prod\limits_{i=1}^{n} X_i \to Z$ belongs to the $(n-1)$-th Baire class.

On the other hand, it was proved in [3, Corollary 2.8] that any strongly separately continuous map $f : \prod\limits_{i=1}^{n} X_i \to Z$ is continuous. Therefore, it is interesting to study Baire classification of ssc functions defined on subsets of products of infinitely many factors, in particular, on spaces ℓ_p.

Definition 4.1. A subset $A \subseteq X$ of a Cartesian product $X = \prod_{k=1}^\infty X_k$ of sets $X_1, X_2, \dots$ is called $\mathcal{S}$*-open* [3], if

$$\{x = (x_k)_{k=1}^\infty \in X : |\{k : x_k \neq a_k\}| \le 1\} \subseteq A \tag{4.1}$$

for all $a = (a_k)_{k=1}^\infty \in A$.

Notice that any space ℓ_p as a subset of the set $\mathbb{R}^\omega$ of all sequences is an example of $\mathcal{S}$-open set.

Proposition 4.2. *For every* $p \in [1, +\infty]$ *there exists an* $\mathcal{S}$*-open set* $A \subseteq \ell_p$ *which is not Borel measurable.*

Proof. Firstly, we consider the case $p < +\infty$. Define a relation $\sim$ on ℓ_p in the following way:

$$x \sim y \Leftrightarrow \text{ the set } \{k \in \mathbb{N} : x_k \neq y_k\} \text{ is finite}$$

for all $x = (x_k)_{k=1}^{\infty}, y = (y_k)_{k=1}^{\infty} \in \ell_p$. Clearly, $\sim$ defines the equivalence relation on ℓ_p. Consider a partition $(\sigma_i : i \in I)$ of ℓ_p on the equivalence classes σ_i.

It is not hard to verify that $|I| = \mathfrak{c}$. Then there are $2^{\mathfrak{c}}$ many sets of the form $\bigcup_{i\in J} \sigma_i$, where $J \subseteq I$.

On the other hand, since ℓ_p is separable, it is a second countable space. Hence, the cardinality of the collection of all open subsets of ℓ_p is $\mathfrak{c}$. Therefore, the cardinality of the collection of all Borel measurable sets in ℓ_p is also equal to $\mathfrak{c}$. Consequently, there exists a set $J \subset I$ such that the union

$$A = \bigcup_{i \in J} \sigma_i$$

is not Borel measurable.

Let $a = (a_k)_{k=1}^{\infty} \in A$ and $x = (x_k)_{k=1}^{\infty} \in \ell_p$ be a sequence which differs from a in at most one coordinate. Since $a \in \sigma_i$ for some $i \in J$, there exists a point $y = (y_k)_{k=1}^{\infty} \ell_p$ such that $\sigma_i = [y]$ and $|\{k \in \mathbb{N} : a_k \neq y_k\}| < \aleph_0$. Clearly, $|\{k \in \mathbb{N} : x_k \neq y_k\}| < \aleph_0$. Therefore, $x \in \sigma_i \subseteq A$. Hence, the set A is $\mathcal{S}$-open.

Now let $p = +\infty$. For every $r \in \mathbb{R}$ we write

$$B_r = \{x \in \ell_1 : \|x\|_1 \leq r\}$$

and show that B_r is closed in ℓ_∞. Suppose that $\|x\|_1 = \sum_{k=1}^{\infty} |x_k| > r$. There exists a number $m \in \mathbb{N}$ such that

$$\sum_{k=1}^{m} |x_k| > r.$$

Since the map $s : \mathbb{R}^m \to \mathbb{R}$, $s(y_1, \dots, y_m) = \sum_{k=1}^{m} |y_k|$ is continuous at $(x_1, \dots, x_m)$, there exists $\delta > 0$ such that

$$|x_k - y_k| < \delta \text{ for every } k \in \{1, \dots, m\} \implies \sum_{k=1}^{m} |y_k| > r.$$

Then

$$B_\infty(x, \delta) \subseteq \ell_\infty \setminus B_r.$$

Therefore, $\ell_\infty \setminus B_r$ is open in ℓ_∞ and hence B_r is closed.

Now let G be an open subset of ℓ_1. Then there exists a sequence $(\delta_n)_{n=1}^{\infty}$ of reals and $(x_n)_{n=1}^{\infty}$ of points from ℓ_1 such that $G = \bigcup_{n=1}^{\infty} B_1(x_n, \delta_n)$. It follows that G is an F_σ-subset of ℓ_∞. Consequently, every Borel measurable subset of ℓ_1 is Borel measurable in ℓ_∞.

Conversely, if $U = B_\infty(0,1) \cap \ell_1 = \{x \in \ell_1 : \sup_{k\in\mathbb{N}} |x_k| < 1\}$, then for $x \in U$ we have that

$$B_1(x, 1 - \|x\|_\infty) = \{y \in \ell_1 : \sum_{k=1}^{\infty} |y_k - x_k| < 1 - \|x\|_\infty\} \subseteq U.$$

This implies that every open set in ℓ_∞ is open in ℓ_1. Hence, the collections of all Borel measurable sets in ℓ_1 and in ℓ_∞ coincide.

According to the previous arguments, there is an $\mathcal{S}$-open subset A of ℓ_1 which is not Borel measurable. Then A is not Borel measurable in ℓ_∞. □

Theorem 4.3. *For every $p \in [1, +\infty]$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ such that $f \notin \bigcup_{\alpha<\omega_1} \mathrm{B}_\alpha(\ell_p, \mathbb{R})$.*

Proof. Fix $p \in [1, +\infty]$. By Proposition 4.2 we can find an $\mathcal{S}$-open subset $A \subseteq \ell_p$ which is not Borel measurable. For all $x \in \ell_p$ we put

$$f(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Notice that $f \notin \bigcup_{\alpha<\omega_1} \mathrm{B}_\alpha(\ell_p, \mathbb{R})$, since the set $A = f^{-1}(1)$ is not Borel measurable.

Since $f(x) = f(y)$ whenever y differs from x in at most finitely many coordinates, f is strongly separately continuous on ℓ_p. □

5. Discontinuities of ssc functions

By $C(f)$ ($D(f)$) we denote the set of all points of continuity (discontinuity) of a map f.

We start with two simple facts.

Lemma 5.1. *Let X be a topological space, $\varphi : X \to \mathbb{R}$ be a continuous function, $g : X \to \mathbb{R}$ be a bounded function and $f : X \to \mathbb{R}$ be a function such that $f(x) = \varphi(x) \cdot g(x)$ for all $x \in X$. Then $\varphi^{-1}(0) \subseteq C(f)$.*

Proof. Fix $x_0 \in \varphi^{-1}(0)$ and $\varepsilon > 0$. Let $C > 0$ be a real number such that $|g(x)| \le C$ for all $x \in X$. Since φ is continuous at x_0, we can find a neighborhood U of x_0 such that $|\varphi(x)| < \frac{\varepsilon}{C}$ for all $x \in U$. Then

$$|f(x) - f(x_0)| = |\varphi(x) \cdot g(x)| < \frac{\varepsilon}{C} \cdot C = \varepsilon$$

for all $x \in U$. □

Lemma 5.2. *For any $p \in [1, +\infty)$ the set*

$$D = \Big\{x = (x_k)_{k=1}^{\infty} \in \ell_p : \sum_{k=1}^{\infty} \sqrt{|x_k|} = +\infty\Big\}$$

is dense in ℓ_p.

Proof. Fix $p \in [1, +\infty)$, $x \in \ell_p$ and $\delta > 0$. We find $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |x_k|^p < \left(\frac{\delta}{2}\right)^p \qquad \text{and} \qquad \sum_{k=N+1}^{\infty} \frac{1}{k^{2p}} < \left(\frac{\delta}{2}\right)^p.$$

Let

$$y = \Big(x_1, \dots, x_N, \tfrac{1}{(N+1)^2}, \tfrac{1}{(N+2)^2}, \dots\Big).$$

Clearly, $y \in D$. Moreover,

$$\|x - y\|_p \le \left(\sum_{k=N+1}^{\infty} \left(\frac{1}{k^2}\right)^p\right)^{\frac{1}{p}} + \left(\sum_{k=N+1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, D is dense in ℓ_p. □

Theorem 5.3. *For any $p \in [1, +\infty)$ and for any open nonempty set $G \subseteq \ell_p$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ such that $D(f) = G$.*

Proof. Fix $p \in [1, +\infty)$. Let $\varnothing \neq G \subseteq \ell_p$ be an open set and $F = \ell_p \setminus G$.

For every $x = (x_k)_{k=1}^{\infty} \in \ell_p$ we put

$$\varphi(x) = \begin{cases} \min\{d_\infty(x, F), 1\}, & F \neq \varnothing, \\ 1, & F = \varnothing, \end{cases}$$

$$g(x) = \begin{cases} \exp(-\sum_{k=1}^{\infty} \sqrt{|x_k|}), & x \in \ell_{1/2}, \\ 1, & \text{otherwise}, \end{cases}$$

and let

$$f(x) = \varphi(x) \cdot g(x).$$

Then $F \subseteq C(f)$ by Lemma 5.1.

Now we show that $G \subseteq D(f)$. Assume that $x^0 \in G$. Then $f(x^0) > 0$. We put $\varepsilon = \frac{1}{2} f(x^0)$ and take an arbitrary $\delta > 0$.

Since the set $D = \{x \in \ell_p : \sum_{k=1}^{\infty} \sqrt{|x_k|} = +\infty\}$ is dense in ℓ_p by Lemma 5.2, there exists $x = (x_n)_{n\in\mathbb{N}} \in \ell_p$ such that

$$\|x - x^0\|_p < \frac{\delta}{2} \text{ and } x \in D.$$

Take a number N such that

$$\sum_{n=1}^{N} \sqrt{|x_n|} > \ln\Big(\frac{2}{f(x^0)}\Big) \qquad \text{and} \qquad \sum_{n=N+1}^{\infty} |x_n|^p < \Big(\frac{\delta}{2}\Big)^p.$$

We put

$$y = (x_1, \dots, x_N, 0, 0, \dots).$$

Then $y \in \ell_{\frac{1}{2}}$ and

$$\|y - x^0\|_p \le \|y - x\|_p + \|x - x^0\|_p =$$
$$= \Big(\sum_{n=N+1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}} + \|x - x^0\|_p < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

But

$$f(x^0) - f(y) = f(x^0) - \varphi(y) \cdot \exp\Big(-\sum_{n=1}^{N} \sqrt{|x_n|}\Big) >$$
$$> f(x^0) - \exp\Big(-\sum_{n=1}^{N} \sqrt{|x_n|}\Big) > f(x^0) - \frac{f(x^0)}{2} = \varepsilon,$$

which implies that f is discontinuous at x^0. Therefore, $D(f) = G$.

Now we prove that g is strongly separately continuous. Fix $x^0 \in \ell_p$, $k \in \mathbb{N}$ and $\varepsilon \in (0, 1)$. Let $\delta = \ln^2(1 + \varepsilon)$. Take $x = (x_k)_{k=1}^{\infty} \in B_p(x^0, \delta)$ and $y = (x_1, \dots, x_{k-1}, x_k^0, x_k, \dots) \in B_p(x^0, \delta)$. If $x \notin \ell_{1/2}$, then $y \notin \ell_{1/2}$. In this case $|g(x) - g(y)| = 0 < \varepsilon$. Assume that $x \in \ell_{1/2}$. Then $y \in \ell_{1/2}$ and

$$|g(x) - g(y)| = \Big|\exp\Big(-\sum_{n=1}^{\infty} \sqrt{|x_n|}\Big) - \exp\Big(-\sum_{n=1}^{\infty} \sqrt{|y_n|}\Big)\Big| <$$
$$< \Big|\exp\Big(\sum_{n=1}^{\infty} (\sqrt{|y_n|} - \sqrt{|x_n|})\Big) - 1\Big| =$$
$$= \Big|\exp(\sqrt{|x_k^0|} - \sqrt{|x_k|}) - 1\Big|.$$

It follows that

$$|g(x) - g(y)| = \exp\Big(\big|\sqrt{|x_k|} - \sqrt{|x_k^0|}\,\big|\Big) - 1 < \exp(\sqrt{\delta}) - 1 = \varepsilon$$

in the case $\sqrt{|x_k^0|} - \sqrt{|x_k|} \geq 0$, or

$$|g(x) - g(y)| < 1 - \exp\big(-\sqrt{\delta}\big) < \varepsilon,$$

otherwise. Hence, g is strongly separately continuous at x^0 with respect to the k'th variable.

Finally, f is strongly separately continuous on ℓ_p as a product of two ssc functions (see Theorem 3 from [2]). □

In connection with Example 2.3 and Theorem 5.3 the following question is natural and open.

Question 5.4. *Let $G \subseteq \ell_\infty$ be an open nonempty set. Does there exist a strongly separately continuous function $f : \ell_\infty \to \mathbb{R}$ such that $D(f) = G$?*

References

[1] O. Dzagnidze. Separately continuous function in a new sense are continuous. *Real Anal. Exchange*, 24:695–702, 1998/1999.

[2] O. Karlova. Some properties of strongle separately continuous functions on products. *Bukovinian J. Math.*, 2(2-3):119–125, 2014.

[3] O. Karlova. On Baire classification of strongly separately continuous functions. *Real Anal. Exch.*, 40(2):371–382, 2014/2015.

[4] O. Karlova, V. Mykhaylyuk. A characterization of the discontinuity point set of strongly separately continuous functions on products. *Math. Slovaca*, 66(5):1475–1486, 2016.

[5] O. Karlova, T. Visnyai. On strongly separately continuous functions on sequence spaces. *J. Math. Analysis and Appl.*, 439(1):296–306, 2016.

[6] W. Rudin. Lebesgue first theorem. *Math. Analysis and Appl. Part B (L. Nachbin, ed.), Adv. Math. Studies*, 78:741–747, 1981.

[7] J. Činčura, T. Šalát, T. Visnyai. On separately continuous functions $f : \ell^2 \to \mathbb{R}$. *Acta Acad. Paedagog. Agriensis*, XXXI:11–18, 2004.

[8] T. Visnyai. Strongly separately continuous and separately quasicontinuous functions $f : \ell^2 \to \mathbb{R}$. *Real Anal. Exchange*, 38(2):499–510, 2013.

Received: September 20, 2017, accepted: November 24, 2017.

Olena Karlova
Yurii Fed'kovych Chernivtsi National University, Ukraine
Email: maslenizza.ua@gmail.com

Tomáš Visnyai
Slovak University of Technology in Bratislava, Slovak Republic
Email: tomas.visnyai@stuba.sk