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Warped product semi-slant submanifolds in locally conformal Kaehler manifolds II

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Abstract. In 1994 N. Papaghiuc introduced the notion of semi-slant submanifold in a Hermitian manifold which is a generalization of CR- and slant-submanifolds, [4], [10]. In particular, he considered this submanifold in Kaehlerian manifolds, [13]. Then, in 2007, V. A. Khan and M. A. Khan considered this submanifold in a nearly Kaehler manifold and obtained interesting results, [9].

Recently, we considered semi-slant submanifolds in a locally conformal Kaehler manifold and we gave a necessary and sufficient conditions of the two distributions (holomorphic and slant) be integrable. Moreover, we considered these submanifolds in a locally conformal Kaehler space form.

In the last paper, we defined 2-kind warped product semi-slant submanifolds in almost hermitian manifolds and studied the first kind submanifold in a locally conformal Kaehler manifold. Using Gauss equation, we derived some properties of this submanifold in an locally conformal Kaehler space form, [3], [11].

In this paper, we consider same submanifold with the parallel second fundamental form in a locally conformal Kaehler space form. Using Codazzi equation, we partially determine the tensor field P which defined in (1.2), see Theorem 4.1. Finally, we show that, in the first type warped product semi-slant submanifold in a locally conformal space form, if it is normally flat, then the shape operators A satisfy some special equations, see Theorem 5.2.

Анотація. В 1994 році Н Папагіук ввів поняття напівпохилого (semi-slant) підмноговиду що є зануреним у ермітовий многовид. Такі підмноговиди є узагальненням CR-підмноговидів та похилих (slant) підмноговидів. На цих многовидах дотичне розшарування є прямою сумою голоморфного та похилого розподілів, [4], [10]. Зокрема, він розглядав таку структуру як підмноговид келерового многовиду, [13]. Згодом, у 2007 році В. А. Хан, та М. А. Хан досліджували такий підмноговид, занурений у наближено келеровий многовид, та отримали цікаві результати, [9].

Нещодавно, автором було досліджено напівпохилі підмноговиди занурені у локально конформно-келерові многовиди і отримано необхідні та

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достатні умови інтегровності обох розподілів (голоморфного та похилого). Крім того, вивчались напівпохилі підмноговиди локально конформно-келерової просторової форми.

В останній статті автором введено два типи напівпохилих підмноговидів, що є викривленими добутками, занурених у майже ермітові многовиди та досліджено підмноговиди першого типу в локально конформно-келерових многовидах. Використовуючи рівняння Гауса, ми отримали деякі властивості такого підмноговиду локально конформно-келерової просторової форми, [3], [11].

В представлені роботі роглядаються напівпохилі підмноговиди локально конформно-келерових просторових форм, які мають паралельну другу фундаментальну форму. За допомогою рівняння Кодацці знайдено вигляд тензору P, що визначений у (1.2) (див. Теорему 4.1). Отримано умови на оператор Вейнгартена A за яких напівпохилий підмноговид локально конформно-келерової просторової форми є викривленим добутком з плоскою нормальною зв'язністю, (див. Теорему 5.2).

1. Introduction

A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is called a *locally conformal Kaehler* (an l.c.K. manifold) if each point $x \in \tilde{M}$ has an open neighborhood U with a differentiable function $\rho : U \to \mathbb{R}$ such that $\tilde{g}^* = e^{-2\rho}\tilde{g}|_U$ is a Kaehlerian metric on U, that is, $\nabla^* J = 0$, where J is an almost complex structure, $\tilde{g}$ is a Hermitian metric, ∇^* is the covariant differentiation with respect to $\tilde{g}^*$, and $\mathbb{R}$ is a real number space, [14], [10].

Proposition 1.1. *A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is an l.c.K.-manifold if and only if there exists a global closed 1-form α, called **Lee form**, satisfying*

$$(\tilde{\nabla}_V J)U = -\tilde{g}(\alpha^\sharp, U)JV + \tilde{g}(V, U)\beta^\sharp + \tilde{g}(JV, U)\alpha^\sharp - \tilde{g}(\beta^\sharp, U)V$$

for any $V, U \in T\tilde{M}$, where $\tilde{\nabla}$ denotes the covariant differentiation with respect to $\tilde{g}$, $\alpha^\sharp$ is the dual vector field of α, the 1-form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β, and $T\tilde{M}$ is the tangent bundle of $\tilde{M}$.

An l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is called an *l.c.K.-space form* if it has a constant holomorphic sectional curvature. Then, [9], the Riemannian curvature tensor $\tilde{R}$ with respect to $\tilde{g}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by the following formula:

$$\begin{aligned}4\tilde{R}(X, Y, Z, W) = c\{&\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W) + \\ &+ \tilde{g}(JX, W)\tilde{g}(JY, Z) - \tilde{g}(JX, Z)\tilde{g}(JY, W) - \\ &- 2\tilde{g}(JX, Y)\tilde{g}(JZ, W)\} +\end{aligned}$$

$$
\begin{aligned}
&+3\big\{P(X,W)\tilde{g}(Y,Z)-P(X,Z)\tilde{g}(Y,W)+\\
&\qquad\qquad+\tilde{g}(X,W)P(Y,Z)-\tilde{g}(X,Z)P(Y,W)\big\}+\\
&-P(JX,W)\tilde{g}(JY,Z)+P(JX,Z)\tilde{g}(JY,W)+\\
&-\tilde{g}(JX,W)P(JY,Z)+\tilde{g}(JX,Z)P(JY,W)+\\
&+2\{P(JX,Y)\tilde{g}(JZ,W)+\tilde{g}(JX,Y)P(JZ,W)\}
\end{aligned}
\tag{1.1}
$$

for any $X,Y,Z,W\in T\tilde{M}$, where P is defined by

$$P(X,Y)=-(\tilde{\nabla}_X\alpha)Y-\alpha(X)\alpha(Y)+\tfrac{1}{2}\|\alpha\|^2\tilde{g}(X,Y), \tag{1.2}$$

for any $X,Y\in T\tilde{M}$, where $\|\alpha\|$ is the length of the Lee form α with respect to g.

Let $(M,g)=M_1\otimes_f M_2$ be a warped product Riemannian manifold of (M_1,g_1) and (M_2,g_2) with a warping function f, [12]. Then g is given by

$$g(U,V)=e^{f^2}g_1(\pi_{1*}U,\pi_{1*}V)+g_2(\pi_{2*}U,\pi_{2*}V) \tag{1.3}$$

for any $U,V\in TM$, where π_1 (resp. π_2) denotes the projection operator of M to M_1 (resp. M_2) and π_{1*} (resp π_{2*}) is the differential of π_1 (resp. π_2).

Let ∇, ∇_1 and ∇_2 be the covariant differentiation with respect to g, g_1 and g_2, respectively. Then we have from (1.3)

$$
\begin{aligned}
&\nabla_XY=\nabla_{1X}Y-f^2e^{f^2}g_1(X,Y)(\triangle_2\log f),\\
&\nabla_XZ=\nabla_ZX=f^2(Z\log f)X,\\
&\nabla_ZW=\nabla_{2Z}W
\end{aligned}
\tag{1.4}
$$

for any $X,Y\in TM_1$ and $Z,W\in TM_2$, where we put

$$(\triangle_2\log f)(Z)=(d_2\log f)(Z).$$

By virtue of (1.3) and (1.4), the curvature tensor form $R(X,Y,Z,W)$ is given by

$$
\begin{aligned}
R(X_1,X_2,X_3,X_4)&=e^{f^2}\Big[R_1(X_1,X_2,X_3,X_4)-\\
&\quad-f^4e^{f^2}\|\nabla_2\log f\|^2\big\{g_1(X_1,X_4)g_1(X_2,X_3)-\\
&\qquad\qquad-g_1(X_1,X_3)g_1(X_2,X_4)\big\}\Big],\\
R(X_1,Z_1,Z_2,X_2)&=-f^2e^{f^2}\big\{(2+f^2)(Z_2\log f)(Z_1\log f)+\\
&\qquad\qquad+\nabla_{2Z_1}\nabla_{2Z_2}\log f\big\}g_1(X_1,X_2),\\
R(Z_1,Z_2,Z_3,Z_4)&=R_2(Z_1,Z_2,Z_3,Z_4),\\
Other&=0,
\end{aligned}
\tag{1.5}
$$

for any $X_1, X_2, X_3, X_4 \in TM_1$ and $Z_1, Z_2, Z_3, Z_4 \in TM_2$, where R_1 and R_2 be the Riemannian curvature forms with respect to g_1 and g_2, respectively. Next, using (1.5), the Ricci tensor $\rho(U,V)$ is separated as

$$\begin{aligned}
\rho(X_1,X_2) &= \rho_1(X_1,X_2) - f^2 e^{f^2}\{(2+n_1 f^2)\|\nabla_2 \log f\|^2 + \\
&\qquad + \delta_2 d_2 \log f\} g_1(X_1,X_2), \\
\rho(X_1,Z_1) &= 0, \\
\rho(Z_1,Z_2) &= \rho_2(Z_1,Z_2) - n_1 f^2\big\{(2+f^2)(\nabla_{2Z_1}\log f)(\nabla_{2Z_2}\log f) + \\
&\qquad + \nabla_{2Z_1}\nabla_{2Z_2}\log f\big\},
\end{aligned}$$

where ρ_1 (resp. ρ_2) denotes the Ricci tensor with respect to g_1 (resp.g_2). Finally, if we respectively put τ, τ_1 and τ_3 the scalar curvature with respect to g, g_1 and g_2. It easily follows that

$$\tau = e^{f^2}\tau_1 + \tau_2 - (n_1-1)n_1 f^4 \|\nabla_2 \log f\|^2.$$

2. Semi-slant-submanifolds in an almost Hermitian manifold.

In generall, for a Riemannian manifold $(\tilde{M}, \tilde{g})$ and its Riemannian submanifold (M, g) we know the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \qquad \tilde{\nabla}_X N = -A_N X + \nabla^{\perp}{}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where ∇ is the covariant differentiation with respect to g, σ is the second fundamental form, and A_N is the shape operator or the fundamental tensor of Weingarten with respect to N and $\nabla^{\perp}$ is normal connection, [5], [6]. Also the following identity holds true:

$$\tilde{g}(A_N Y, X) = \tilde{g}(\sigma(Y,X), N).$$

The Codazzi equation and the Ricci equation are respectively given by

$$\tilde{R}(U,V,W,N_1) = \tilde{g}((\bar{\nabla}_U\sigma)(V,W) - (\bar{\nabla}_V\sigma)(U,W), Z), \tag{2.1}$$

$$\tilde{R}(U,V,N_1,N_2) = R^{\perp}(U,V,N_1,N_2) - \tilde{g}([A_{N_1}, A_{N_2}]U, V) \tag{2.2}$$

for all $U, V, W, Z \in TM$ and $N_1, N_2 \in T^{\perp}M$, where $R^{\perp}$ is the normal curvature tensor, and

$$(\bar{\nabla}_U\sigma)(V,W) = \nabla^{\perp}{}_U\sigma(V,W) - \sigma(\nabla_U V, W) - \sigma(V, \nabla_U W).$$

The second fundamental form σ is called *parallel* if it satisfies $\bar{\nabla}\sigma = 0$ identically. Also a submanifold M is said to be totally geodesic whenever $\sigma = 0$ on M.

Let $(M, g) = (M_1, g_1) \otimes_f (M_2, g_2)$ be a warped product submanifold of $(\tilde{M}, \tilde{g})$. Then the induced metric tensor g of $\tilde{g}$ is given by

$$g(U, V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V)$$

for any $U, V \in TM$. A warped product submanifold M is called M_1 (resp. M_2) *geodesic* if the second fundamental form satisfies $\sigma(X, Y) = 0$ (resp. $\sigma(Z, W) = 0$) for all $X, Y \in TM_1$ and $Z, W \in TM_2$. Moreover M is said to be *mixed totally geodesic* if the second fundamental form σ satisfies $\sigma(X, Z) = 0$ for all $X \in TM_1$ and $Z \in TM_2$.

By virtue of (1.5) and the Gauss equation, we have

$$\begin{aligned}
\tilde{R}(X_1, X_2, X_3, X_4) &= e^{f^2}\big\{R_1(X_1, X_2, X_3, X_4) - \\
&\quad - f^4 e^{f^2} \|\log f\|^2 (g_1(X_1, X_4) g_1(X_2, X_3) - g_1(X_1, X_3) g_1(X_2, X_4)\big\} + \\
&\quad + \tilde{g}\big(\sigma(X_1, X_4), \sigma(X_2, X_3)\big) - \tilde{g}\big(\sigma(X_1, X_3), \sigma(X_2, X_4)\big), \\
\tilde{R}(X_1, X_2, X_3, Z_1) &= \tilde{g}(\sigma(X_1, Z_1), \sigma(X_2, X_3)) - \tilde{g}\big(\sigma(X_1, X_3), \sigma(X_2, Z_1)\big), \\
\tilde{R}(X_1, X_2, Z_1, Z_2) &= \tilde{g}(\sigma(X_1, Z_2), \sigma(X_2, Z_1)) - \tilde{g}\big(\sigma(X_1, Z_1), \sigma(X_2, Z_2)\big), \\
\tilde{R}(X_1, Z_1, Z_2, X_2) &= -f^2 e^{f^2}\big\{(2 + f^2)(Z_1 \log f)(Z_2 \log f) + \\
&\quad + \nabla_{2Z_2} \nabla_{2Z_1} \log f\big\} g_1(X_1, X_2) + \tilde{g}\big(\sigma(X_1, X_2), \sigma(Z_1, Z_2)\big) - \\
&\quad - \tilde{g}\big(\sigma(X_1, Z_2), \sigma(Z_1, X_2)\big), \\
\tilde{R}(X_1, Z_1, Z_2, Z_3) &= \tilde{g}\big(\sigma(X_1, Z_3), \sigma(Z_1, Z_2)\big) - \tilde{g}\big(\sigma(X_1, Z_2), \sigma(Z_1, Z_3)\big), \\
\tilde{R}(Z_1, Z_2, Z_3, Z_4) &= R_2(Z_1, Z_2, Z_3, Z_4) + \tilde{g}\big(\sigma(Z_1, Z_4), \sigma(Z_2, Z_3)\big) - \\
&\quad - \tilde{g}\big(\sigma(Z_1, Z_3), \sigma(Z_2, Z_4)\big),
\end{aligned}$$

for all $X_1, X_2, X_3, X_4 \in TM_1$ and $Z_1, Z_2, Z_3, Z_4 \in TM_2$.

For a vector field $U \in TM$, the angle between JU and TM is called the *Wirtinger angle* of U.

A differentiable distribution $\mathcal{D}^\theta : x \to \mathcal{D}^\theta{}_x$ on M is said to be a *slant* one if for each $U_x \in \mathcal{D}^\theta{}_x$, the Wirtinger angle of U_x is constant ($= \theta$) for any $x \in M$. In this case, the Wirtinger angle is said to be the *slant angle*. In particular, if TM is slant, then the submanifold is called *slant* as well. A slant submanifold is holomorphic (resp. totally real) if its slant angle $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$). A slant submanifold is said to be *proper* if it is not holomorphic nor totally real.

A submanifold M in $\tilde{M}$ is called *semi-slant* if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ on M satisfying the following conditions

(i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = D_x$ for each $x \in M$ and
(ii) the complementary orthogonal distribution $\mathcal{D}^\theta : x \to \mathcal{D}^\theta{}_x \subset T_xM$ is slant with slant angle θ, where T_xM means the tangent vector space of M at x, [7].

Remark 2.1. A semi-slant submanifold is a CR-submanifold if the slant angle is equal to $\frac{\pi}{2}$, e.g. [1], [2], [8].

In a submanifold M of an almost Hermitian manifold $\tilde{M}(J, \tilde{g})$, for all $U \in TM$ and $\xi \in T^\perp M$, we write

$$JU = TU + FU, \qquad J\xi = t\xi + f\xi,$$

where TU (resp. FU) is the tangential (resp. normal) component of JU and $t\xi$ (resp. $f\xi$) is the tangential (resp. normal) component of $J\xi$.

Then one can easily check the following relations:

$$\begin{aligned} T^2 + tF &= -I, & f^2 + Ft &= -I \\ FT + fF &= 0, & tF + Tt &= 0. \end{aligned} \tag{2.3}$$

For a semi-slant submanifold M of an almost Hermitian manifold $\tilde{M}$ the tangent bundle TM and the normal bundle $T^\perp M$ of M are decomposed as

$$TM = \mathcal{D} \oplus \mathcal{D}^\theta, \qquad T^\perp M = F\mathcal{D}^\theta \oplus \nu,$$

where ν denotes the orthogonal complementary distribution of $F\mathcal{D}^\theta$ in $T^\perp M$.

Next, for an element $U \in TM$ in a semi-slant submanifold M, we write

$$U = T_1U + T_2U, \tag{2.4}$$

where T_1U (resp. T_2U) denotes the $\mathcal{D}$ (resp. $\mathcal{D}^\theta$) component of U.

It follows from (2.3) and (2.4) that

$$JU = JT_1U + TT_2U + FT_2U, \tag{2.5}$$

where $JT_1U \in \mathcal{D}$, $TT_2U \in \mathcal{D}^\theta$ and $FT_2U \in F\mathcal{D}^\theta \subset T^\perp M$. Thus if we put

$$PU = JT_1U + TT_2U \tag{2.6}$$

for any $U \in TM$, then

$$P^2U = -T_1U - T_2U - tFT_2U \tag{2.7}$$

for any $U \in TM$.

Now we easily get from (2.7) the folowing statement:

Proposition 2.2. *In a semi-slant submanifold of an almost Hermitian manifold $\tilde{M}$, the operator P defined by* (2.6) *is an almost complex structure in the holomorphic distribution $\mathcal{D}$.*

The covariant differentiation $\bar{\nabla}$ of T_1, T_2, T, F, t and f are defined as follows:

$$(\bar{\nabla}_U T_1)V = \nabla_U(T_1V) - T_1\nabla_U V, \qquad (\bar{\nabla}_U T_2)V = \nabla_U(T_2V) - T_2\nabla_U V,$$
$$(\bar{\nabla}_U T)V = \nabla_U(TV) - T\nabla_U V, \qquad (\bar{\nabla}_U F)V = \nabla^{\perp}_U(FV) - F\nabla_U V,$$
$$(\bar{\nabla}_U t)\xi = \nabla_U(t\xi) - t\nabla^{\perp}_U\xi, \qquad (\bar{\nabla}_U f)\xi = \nabla^{\perp}_U(f\xi) - f\nabla^{\perp}_U\xi$$

where $U, V \in TM$ and $\xi \in T^{\perp}M$.

Moreover, if we define the covariant differentiation $\bar{\nabla}$ of P by

$$(\bar{\nabla}_U P)V = \nabla_U(PV) - P\nabla_U V$$

then

$$(\bar{\nabla}_U P)V = (\tilde{\nabla}_U J)T_1V + J(\bar{\nabla}_U T_1)V + (\bar{\nabla}_U T)(T_2V) + \\ + T(\bar{\nabla}_U T_2)V + J\sigma(U, T_1V) - \sigma(U, JT_1V).$$

Write

$$(\tilde{\nabla}_U J)V = \mathcal{P}_U V + \mathcal{Q}_U V,$$

where $\mathcal{P}_U V$ (resp. $\mathcal{Q}_U V$) denotes the tangential (resp. normal) part of $(\tilde{\nabla}_U J)V$.

V. A. Khan and M. A. Khan proved the following statement:

Proposition 2.3. [9]. *The holomorphic distribution $\mathcal{D}$ on a semi-slant submanifold of an almost Hermitian manifold is integrable if and only if*

$$\mathcal{Q}_X Y - \mathcal{Q}_Y X = \sigma(X, TY) - \sigma(Y, TX)$$

for any $X, Y \in \mathcal{D}$. The slant distribution $\mathcal{D}^\theta$ on a semi-slant submanifold of an almost Hermitian manifold is integrable if and only if

$$T_1(\nabla_Z TW - \nabla_W TZ + A_{FZ}W - A_{FW}Z + \mathcal{P}_W Z - \mathcal{P}_Z W) = 0$$

for any $Z, W \in \mathcal{D}^\theta$.

Using these proposition, we proved the following result:

Theorem 2.4. [11]. *(I) The holomorphic distribution $\mathcal{D}$ of a semi-slant submanifold M in an l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is integrable if and only if*

$$\sigma(X, TY) - \sigma(Y, TX) = 2\tilde{g}(TX, Y)\alpha_2^{\sharp}$$

(II) The slant distribution $\mathcal{D}^\theta$ of a semi-slant submanifold M in an l,c,K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is integrable if and only if

$$T_1(\nabla_Z TW - \nabla_W TZ + A_{FZ}W - A_{FW}Z + \\ + \tilde{g}({\alpha_1}^{\sharp}, W)TZ - \tilde{g}({\alpha_1}^{\sharp}, Z)TW + 2\tilde{g}(TW, Z)\alpha_1^{\sharp}) = 0$$

or equivalently

$$T_1\{(\bar\nabla_Z T)W - (\bar\nabla_W T)Z + T[Z,W] + A_{FZ}W - A_{FW}Z + \\ + \tilde g(\alpha_1{}^\sharp, W)TZ - \tilde g(\alpha_1{}^\sharp, Z)TW + 2\tilde g(TW,Z)\alpha_1^\sharp\} = 0,$$

where $Z, W \in \mathcal{D}^\theta$, and $[Z,W]$ is the Lie bracket of Z and W.

3. Warped product semi-slant submanifolds in an l.c.K.-manifold

Let M be a semi-slant submanifold of an almost Hermitian manifold $\tilde M(J,\tilde g)$. Suppose that the distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ are integrable, and let $M_\mathcal{D}$ (resp. $M_{\mathcal{D}^\theta}$) be the maximal integral submanifold of $\mathcal{D}$ (resp. $\mathcal{D}^\theta$). Then M is a product manifold of $M_\mathcal{D}$ and $M_{\mathcal{D}^\theta}$, that is,

$$M = M_\mathcal{D} \otimes M_{\mathcal{D}^\theta}. \tag{3.1}$$

Therefore we can write

$$T\tilde M = TM_\mathcal{D} \oplus TM_{\mathcal{D}^\theta} \oplus FTM_{\mathcal{D}^\theta} \oplus \nu, \tag{3.2}$$

where ν is the complementaly orthogonal subbundle of $FTM_{\mathcal{D}^\theta} = F\mathcal{D}^\theta$ in $T^\perp M$. We will call the submanifold $M_\mathcal{D}$ (resp. $M_{\mathcal{D}^\theta}$) the *holomorphic* (resp. *slant*) *component* of M.

Given a differentiable function f_1 (resp. f_2) on $M_{\mathcal{D}^\theta}$ (resp. $M_\mathcal{D}$)), define the following warped product submanifolds

$$M_1 = M_\mathcal{D} \otimes_{f_1} M_{\mathcal{D}^\theta}, \qquad M_2 = M_{\mathcal{D}^\theta} \otimes_{f_2} M_\mathcal{D}. \tag{3.3}$$

We say that M_1 (resp. M_2) is the *first* (resp. *second*) *type warped product semi-slant submanifold* of an almost Hermitian manifold.

In this paper, we mainly consider the first type warped product semi-slant submanifold in an l.c.K.-manifold.

Let M be the first type warped product semi-slant submanifold in an l.c.K.-manifold $\tilde M$. Then the induced metric tensor g in M of $\tilde M$ is given by

$$g(U,V) = e^{f^2} g_\mathcal{D}(\pi_\mathcal{D} * U, \pi_\mathcal{D} * V) + g_{\mathcal{D}^\theta}(\pi_{\mathcal{D}^\theta} * U, \pi_{\mathcal{D}^\theta} * V) \tag{3.4}$$

for any $U, V \in TM$, where $g_\mathcal{D}$ (resp. $g_{\mathcal{D}^\theta}$) denotes the Riemannian metric on $M_\mathcal{D}$ (resp. $M_{\mathcal{D}^\theta}$), $\pi_\mathcal{D}$ (resp. $\pi_{\mathcal{D}^\theta}$) is the projection operator of M to $M_\mathcal{D}$ (resp. $M_{\mathcal{D}^\theta}$) and f is a certain positive differentiable function on $M_{\mathcal{D}^\theta}$.

Now, let $\tilde{\nabla}$, ∇, $\nabla^{\mathcal{D}}$ and $\nabla^{\mathcal{D}^\theta}$ be the covariant differentiations with respect to $\tilde{g}$, g, $g_{\mathcal{D}}$ and $g_{\mathcal{D}^\theta}$, respectively. Then by (1.4)

$$\begin{aligned} \nabla_X Y &= \nabla^{\mathcal{D}}{}_X Y - f^2 e^{f^2}(\triangle_1 \log f) g_{\mathcal{D}}(X,Y), \\ \nabla_X Z &= \nabla_Z X = f^2(Z \log f)X, \\ \nabla_Z W &= \nabla^{\mathcal{D}^\theta}{}_Z W, \end{aligned} \tag{3.5}$$

for any $X, Y \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\theta$, where we put

$$\triangle_1 \log f = g_{\mathcal{D}^\theta}{}^{ce}(\partial_c \log f)\partial_e.$$

Let M be a semi-slant submanifold with distributions $\mathcal{D}$, $\mathcal{D}^\theta$ be an almost Hermitian manifold $\tilde{M}$, $\dim \mathcal{D} = 2p$, $\dim \mathcal{D}^\perp = q$, and $\dim \nu = 2s$. Then we take the following generalized adopted frame in $\tilde{M}$:

(1) $\{e_1, e_2, \ldots, e_p, e_1^*, e_2^*, \ldots, e_p^*\}$ is an orthonormal frame of $\mathcal{D}$, where $e_i^* = Je_i$ for $i \in \{1, 2, \ldots, p\}$;
(2) $\{e_{2p+1}, e_{2p+2}, \ldots, e_{2p+q}\}$ is an orthonormal frame of $\mathcal{D}^\theta$ such that the vectors Fe_{2p+1}, $Fe_{2p+2}, \ldots, Fe_{2p+q}$ are orthogonal in $F\mathcal{D}^\theta$.
(3) $\{e_{n+q+1}, e_{n+q+2}, \ldots, e_{n+q+s}, e^*_{n+q+1}, e^*_{n+q+2}, \ldots, e^*_{n+q+s}\}$ is an orthonormal frame of ν, where $e^*_{n+q+a} = Je_{n+q+a}$ for $a \in \{1, 2, \ldots, s\}$;
(4) $e^*_{2p+a} = \dfrac{Fe_{2p+a}}{\|Fe_{2p+a}\|}$ for $a \in \{1, 2, \ldots, q\}$.

Hereafter, for a tensor field T of $(0,s)$-type, we write $T_{\mu_1,\mu_2,\ldots,\mu_s}$ instead of $T(e_{\mu_1}, e_{\mu_2}, \ldots, e_{\mu_s})$ with respect to the generalized adapted frame.

In the last paper ([11]), using the Gauss equation, we proved

Proposition 3.1. *In a first type warped product semi-slant submanifold in an l.c.K.-space form, the mean curvature $\|H\|$ satisfies the inequality*

$$\begin{aligned} 4n\|H\|^2 + 8pf^2\Big\{(2p-1)f^2\|\log f\|^2 + 2(2+f^2)\sum_{a=2p+1}^{2p+q}(e_a \log f)^2\Big\} + \\ + (n^2+2n-3q)c + 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\ + 16pf^2\sum_{b,a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}{}_{e_a}\nabla^{\mathcal{D}^\theta}{}_{e_a}\log f + 6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu} + \\ + 6\sum_{a=2p+1}^{2p+q}P_{aa} - 6\sum_{b,a=2p+1}^{2p+q}P(Je_b, e_a)T_{ba} \geq 0. \end{aligned} \tag{3.6}$$

In particular, the equality case of (3.6) *is that our submanifold is totally geodesic. Then we have the following equation for the warping function* f

$$8pf^2\{(2p-1)f^2\|\log f\|^2 + 2(2+f^2)\sum_{a=2p+1}^{2p+q}(e_a \log f)^2\} + \\ + (n^2+2n-3q)c + 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\ + 16pf^2\sum_{b,a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}{}_{e_a}\nabla^{\mathcal{D}^\theta}{}_{e_a}\log f + 6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu} + \\ + 6\sum_{a=2p+1}^{2p+q}P_{aa} - 6\sum_{b,a=2p+1}^{2p+q}P(Je_b, e_a)T_{ba} = 0.$$

From which, we obtain

$$(n^2+2n-3q)c + 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\ + 16pf^2\sum_{b,a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}{}_{e_a}\nabla^{\mathcal{D}^\theta}{}_{e_a}\log f + 6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu} + \\ + 6\sum_{a=2p+1}^{2p+q}P_{aa} - 6\sum_{b,a=2p+1}^{2p+q}P(Je_b, e_a)T_{ba} \leq 0.$$

4. Semi-slant submanifolds with the parallel second fundamental form

Using (1.1), the curvature tensor $\tilde{R}$ of the first type warped product semi-slant submanifold M in an l.c.K.-space form $\tilde{M}(c)$ is separated, with respect to the generalized adapted frame, as

$$\begin{aligned}
4\tilde{R}_{jiha^*} &= 3(P_{ja^*}\delta_{ih} - P_{ia^*}\delta_{jh}),\\
4\tilde{R}_{jih^*a^*} &= -P_{j^*a^*}\delta_{ih} + P_{i^*a^*}\delta_{jh},\\
4\tilde{R}_{ji^*ha^*} &= -3P_{i^*a^*}\delta_{jh} + P_{j^*a^*}\delta_{ih} + 2P_{h^*a^*}\delta_{ji},\\
4\tilde{R}_{ji^*h^*a^*} &= 3P_{ja^*}\delta_{ih} - P_{ia^*}\delta_{jh} - 2P_{ha^*}\delta_{ji},\\
4\tilde{R}_{j^*i^*h^*a^*} &= 3(P_{j^*a^*}\delta_{ih} - P_{i^*a^*}\delta_{jh}),\\
2\tilde{R}_{jiba^*} &= P_{j^*i}F_{ba},
\end{aligned}$$

$$
\begin{aligned}
2\tilde{R}_{ji^*ba^*} &= -c\delta_{ji}F_{ba} + P_{ji}F_{ba} + ({T_b}^c P_{ca^*} + {F_b}^c P_{c^*a^*})\delta_{ji},\\
2\tilde{R}_{j^*i^*ba^*} &= -P_{ji^*}F_{ba},\\
4\tilde{R}_{jcba^*} &= 3P_{ja^*}\delta_{cb} - P_{j^*a^*}T_{cb} + P_{j^*b}F_{ca} + 2P_{j^*c}F_{ba},\\
4\tilde{R}_{j^*cba^*} &= 3P_{j^*a^*}\delta_{cb} + P_{ja^*}T_{cb} - P_{jb}F_{ca} - 2P_{jc}F_{ba},\\
4\tilde{R}_{dcba^*} &= c(F_{da}T_{cb} - T_{db}F_{ca} - 2T_{dc}F_{ba}) + 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) -\\
&\quad - ({T_d}^c P_{ca^*} + {F_d}^c P_{c^*a^*})T_{cb} + ({T_d}^e P_{eb} + {F_d}^e P_{e^*b})F_{ca} -\\
&\quad - ({T_c}^e P_{eb} + {F_c}^e P_{e^*b})F_{da} + ({T_c}^e P_{ea^*} + {F_c}^e P_{e^*a^*})T_{db} +\\
&\quad + 2\big\{({T_d}^e P_{ec} + {F_d}^e P_{e^*c})F_{ba} + ({T_b}^e P_{ea^*} + {F_b}^e P_{e^*a^*})T_{dc}\big\},\\
4\tilde{R}_{jihr} &= 3(P_{jr}\delta_{ih} - P_{ir}\delta_{jh}),\\
4\tilde{R}_{jih^*r} &= -P_{j^*r}\delta_{ih} + P_{i^*r}\delta_{jh},\\
4\tilde{R}_{ji^*hr} &= -3P_{i^*r}\delta_{jh} + P_{j^*r}\delta_{ih} + 2P_{h^*r}\delta_{ji},\\
4\tilde{R}_{ji^*h^*r} &= 3P_{jr}\delta_{ih} - P_{ir}\delta_{jh} - 2P_{hr}\delta_{ji},\\
4\tilde{R}_{j^*i^*hr} &= -P_{jr}\delta_{ih} + P_{ir}\delta_{jh},\\
4\tilde{R}_{j^*i^*h^*r} &= 3(P_{j^*r}\delta_{ih} - P_{i^*r}\delta_{jh}),\\
\tilde{R}_{jiar} &= 0,\\
2\tilde{R}_{ji^*ar} &= ({T_a}^e P_{er} + {F_a}^e p_{e^*r})\delta_{ji},\\
\tilde{R}_{j^*i^*ar} &= 0,\\
4\tilde{R}_{jbar} &= 3P_{jr}\delta_{ba} - P_{j^*r}T_{ba},\\
4\tilde{R}_{j^*bar} &= 3P_{j^*r}\delta_{ba} + P_{jr}T_{ba},\\
4\tilde{R}_{cbar} &= 3(P_{cr}\delta_{ba} - P_{br}\delta_{ca}) - ({T_c}^e P_{er} + {F_c}^e {P_c}^e P_{e^*r})T_{ba}\\
&\quad + ({T_b}^e P_{er} + {F_b}^e P_{e^*r})T_{ca} + 2({T_a}^e P_{er} + {F_a}^e P_{e^*r})T_{cb}
\end{aligned}
$$

for any $j, i, h \in \{1, 2, \ldots, p\}$, $d, c, b, a \in \{2p+1, 2p+2, \ldots, 2p+q\}$ and $r \in \{n+q+1, n+q+2, \ldots, m\}$.

Thus, by virtue of previous formulas the Codazzi equation (2.1) is separated as

$$
\begin{aligned}
4\{\tilde{g}(\bar{\nabla}_j \sigma_{ih}, {e_a}^*) - \tilde{g}(\bar{\nabla}_i \sigma_{jh}, {e_a}^*)\} &= 3(P_{ja^*}\delta_{ih} - P_{ia^*}\delta_{jh}),\\
4\{\tilde{g}(\bar{\nabla}_j \sigma_{ih^*}, {e_a}^*) - \tilde{g}(\bar{\nabla}_i \sigma_{jh^*}, {e_a}^*)\} &= -P_{j^*a^*}\delta_{ih} + P_{i^*a^*}\delta_{jh},\\
4\{\tilde{g}(\bar{\nabla}_j \sigma_{i^*h}, {e_a}^*) - \tilde{g}(\bar{\nabla}_{i^*} \sigma_{jh}, {e_a}^*)\} &= -3P_{i^*a^*}\delta_{jh} + P_{j^*a^*}\delta_{ih} + 2P_{h^*a^*}\delta_{ji},\\
4\{\tilde{g}(\bar{\nabla}_j \sigma_{i^*h^*}, {e_a}^*) - \tilde{g}(\bar{\nabla}_{i^*} \sigma_{jh^*}, {e_a}^*)\} &= 3P_{ja^*}\delta_{ih} - P_{ia^*}\delta_{jh} - 2P_{ha^*}\delta_{ji},\\
4\{\tilde{g}(\bar{\nabla}_{j^*} \sigma_{i^*h^*}, {e_a}^*) - \tilde{g}(\bar{\nabla}_{i^*} \sigma_{j^*h^*}, {e_a}^*)\} &= 3(P_{j^*a^*}\delta_{ih} - P_{i^*a^*}\delta_{jh}),
\end{aligned}
$$

$$
\begin{aligned}
2\{\tilde g(\bar\nabla_j\sigma_{ib},{e_a}^*)-\tilde g(\bar\nabla_i\sigma_{jb},{e_a}^*)\}&=P_{j^*i}F_{ba}\\
2\{\tilde g(\bar\nabla_j\sigma_{i^*b},{e_a}^*)-\tilde g(\bar\nabla_{i^*}\sigma_{jb},{e_a}^*)\}&=-c\delta_{ji}F_{ba}+P_{ji}F_{ba}+\\
&\quad+\{{T_b}^cP_{ca^*}+{F_b}^cP_{c^*a^*}\}\delta_{ji},\\
2\{\tilde g(\bar\nabla_{j^*}\sigma_{i^*b},{e_a}^*)-\tilde g(\bar\nabla_{i^*}\sigma_{j^*b},{e_a}^*)\}&=-P_{ji^*}F_{ba},\\
4\{\tilde g(\bar\nabla_j\sigma_{cb},{e_a}^*)-\tilde g(\bar\nabla_c\sigma_{jb},{e_a}^*)\}&=3P_{ja^*}\delta_{cb}-P_{j^*a^*}T_{cb}+\\
&\quad+P_{j^*b}F_{ca}+2P_{j^*c}F_{ba},\\
4\{\tilde g(\bar\nabla_{j^*}\sigma_{cb},{e_a}^*)-\tilde g(\bar\nabla_c\sigma_{j^*b},{e_a}^*)\}&=3P_{j^*a^*}\delta_{cb}+P_{ja^*}T_{cb}-\\
&\quad-P_{jb}F_{ca}-2P_{jc}F_{ba},\\
4\{\tilde g(\bar\nabla_d\sigma_{cb},{e_a}^*)-\tilde g(\bar\nabla_c\sigma_{db},{e_a}^*)\}&=c(F_{da}T_{cb}-T_{db}F_{ca}-2T_{dc}F_{ba})\\
&\quad+3(P_{da^*}\delta_{cb}-P_{ca^*}\delta_{db})-({T_d}^eP_{ea^*}+{F_d}^eP_{e^*a^*})T_{cb}\\
&\quad+({T_d}^eP_{eb}+{F_d}^eP_{be^*})F_{ca}-({T_c}^eP_{eb}+{F_c}^eP_{bc^*})F_{da}\\
&\quad+({T_c}^eP_{ea^*}+{F_c}^eP_{e^*a^*})T_{db}+2\big\{({T_d}^eP_{ec}+{F_d}^cP_{ce^*})F_{ba}\\
&\quad+({T_b}^eP_{ea^*}+{F_b}^eP_{e^*a^*})T_{dc}\big\},\\
4\{\tilde g(\bar\nabla_j\sigma_{ih},e_r)-\tilde g(\bar\nabla_i\sigma_{jh},e_r)\}&=3(P_{jr}\delta_{ih}-P_{ir}\delta_{jh}),\\
4\{\tilde g(\bar\nabla_j\sigma_{ih^*},e_r)-\tilde g(\bar\nabla_i\sigma_{jh^*},e_r)\}&=-P_{j^*r}\delta_{ih}+P_{i^*r}\delta_{jh},\\
4\{\tilde g(\bar\nabla_j\sigma_{i^*h},e_r)-\tilde g(\bar\nabla_{i^*}\sigma_{jh},e_r)\}&=-3P_{i^*r}\delta_{jh}+P_{j^*r}\delta_{ih}+2P_{h^*r}\delta_{ji},\\
4\{\tilde g(\bar\nabla_j\sigma_{i^*h^*},e_r)-\tilde g(\bar\nabla_{i^*}\sigma_{jh^*},e_r)\}&=3P_{jr}\delta_{ih}-P_{ir}\delta_{jh}-2P_{hr}\delta_{ji},\\
4\{\tilde g(\bar\nabla_{j^*}\sigma_{i^*h},e_r)-\tilde g(\bar\nabla_{i^*}\sigma_{j^*h},e_r)\}&=-P_{jr}\delta_{ih}+P_{ir}\delta_{jh},\\
4\{\tilde g(\bar\nabla_{j^*}\sigma_{i^*h^*},e_r)-\tilde g(\bar\nabla_{i^*}\sigma_{j^*h^*},e_r)\}&=3(P_{j^*r}\delta_{ih}-P_{i^*r}\delta_{jh}),\\
\tilde g(\bar\nabla_j\sigma_{ia},e_r)-\tilde g(\bar\nabla_i\sigma_{ja},e_r)&=0,\\
4\{\tilde g(\bar\nabla_j\sigma_{i^*a},e_r)-\tilde g(\bar\nabla_{i^*}\sigma_{ja},e_r)\}&=({T_a}^cP_{cr}+{F_a}^cP_{c^*r})\delta_{ji},\\
\tilde g(\bar\nabla_{j^*}\sigma_{i^*a},e_r)-\tilde g(\bar\nabla_{i^*}\sigma_{j^*a},e_r)&=0,\\
4\{\tilde g(\bar\nabla_j\sigma_{ba},e_r)-\tilde g(\bar\nabla_b\sigma_{ja},e_r)\}&=3P_{jr}\delta_{ba}-P_{j^*r}T_{ba},\\
4\{\tilde g(\bar\nabla_{j^*}\sigma_{ba},e_r)-\tilde g(\bar\nabla_{b^*}\sigma_{ja},e_r)\}&=3P_{j^*r}\delta_{ba}+P_{jr}T_{ba},\\
4\{\tilde g(\bar\nabla_c\sigma_{ba},e_r)-\tilde g(\bar\nabla_b\sigma_{ca},e_r)\}&=3(P_{cr}\delta_{ba}-P_{br}\delta_{ca})-\\
&\quad-({T_c}^eP_{er}+{F_c}^eP_{e^*r})T_{ba}+({T_b}^eP_{er}+{F_b}^eP_{e^*r})T_{ca}+\\
&\quad+2({T_a}^eP_{er}+{F_a}^eP_{e^*r})T_{cb},
\end{aligned}
$$

for all $j,i,h\in\{1,2,\dots,p\}$, $d,c,b,a\in\{2p+1,2p+2,\dots,2p+q\}$ and $r\in\{n+q+1,n+q+2,\dots,m\}$.

Now we assume that the second fundamental form σ is parallel, that is, $\bar\nabla\sigma=0$. Then we have from the above formulae that the tensor field P_{BA}

is written as

$$
\begin{aligned}
(P_{BA}) &= \begin{pmatrix}
P_{ji} & P_{ji^*} & P_{ja} & P_{ja^*} & P_{jr} & P_{jr^*} \\
P_{j^*i} & P_{j^*i^*} & P_{j^*a} & P_{j^*a^*} & P_{j^*r} & P_{j^*r^*} \\
P_{bi} & P_{bi^*} & P_{ba} & P_{ba^*} & P_{br} & P_{br^*} \\
P_{b^*i} & P_{b^*i^*} & P_{b^*a} & P_{b^*a^*} & P_{b^*r} & P_{b^*r^*} \\
P_{ri} & P_{ri^*} & P_{ra} & P_{ra^*} & P_{sr} & P_{sr^*} \\
P_{s^*i} & P_{s^*i^*} & P_{s^*a} & P_{s^*a^*} & P_{s^*r} & P_{s^*r^*}
\end{pmatrix} \\
&= \begin{pmatrix}
\alpha\delta_{ji} & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha\delta_{ji} & 0 & 0 & 0 & 0 \\
0 & 0 & P_{ba} & 0 & 0 & 0 \\
0 & 0 & 0 & P_{b^*a^*} & P_{b^*r} & P_{b^*r^*} \\
0 & 0 & 0 & P_{sa^*} & P_{sr} & P_{sr^*} \\
0 & 0 & 0 & P_{s^*a^*} & P_{s*r} & P_{sr}
\end{pmatrix}.
\end{aligned} \tag{4.1}
$$

for $p, q \geq 1$ and a certain function α. Thus we have

Theorem 4.1. *In the first type warped product semi-slant submanifold M with parallel second fundamental form σ of an l.c.K.-space form $\tilde{M}(c)$, if p and q are bigger than* 1*, then the tensor field P_{BA} is written by* (4.1)*.*

5. Normally flat warped product semi-slant submanifolds

Finally, we calculate the Ricci equation in a warped product semi-slant submanifold in an l.c.K.-space form.

By virtue of (1.1), we can separate the Riemannian curvature tensor $\tilde{R}(U, V, W, Z)$ for $U, V, W, Z \in \tilde{T}M$ as

$$
\begin{aligned}
2\tilde{R}(X_1, X_2, FZ_1, FZ_2) &= -c\tilde{g}(TX_1, X_2)\tilde{g}(JFZ_1, Z_2) \\
&\quad + P(JX_1, X_2)\tilde{g}(JFZ_1, FZ_2) + P(JFZ_1, FZ_2)\tilde{g}(JX_1, X_2), \\
2\tilde{R}(X_1, X_2, FZ_1, \xi) &= P(JFZ_1, \xi_1)\tilde{g}(JX_1, X_2), \\
2\tilde{R}(X_1.X_2, \xi_1, \xi_2) &= -c\tilde{g}(JX_1, X_2)\tilde{g}(J\xi_1, \xi_2) \\
&\quad + P(JX_1, X_2)\tilde{b}(J\xi_1, \xi_2) + P(J\xi_1, \xi_2)\tilde{g}(JX_1, X_2), \\
4\tilde{R}(X_1, Z_1, FZ_2, FZ_3) &= -P(JX_1, FZ_3)\tilde{g}(FZ_1, FZ_2) \\
&\quad + P(JX_1, FZ_2)\tilde{g}(FZ_1, FZ_3) + 2P(JX_1, Z_1)\tilde{g}(JFZ_2, FZ_3), \\
4\tilde{R}(X_1, Z_1, FZ_2, \xi_1) &= -P(JX_1, \xi_1)\tilde{g}(FZ_1, FZ_2), \\
2\tilde{R}(X_1, Z_1, \xi_1, \xi_2) &= P(JX_1, Z_1)\tilde{g}(J\xi_1, \xi_2), \\
4\tilde{R}(Z_1, Z_2, FZ_3, FZ_4) &= c\{\tilde{g}(FZ_1, FZ_4)\tilde{g}(FZ_1, FZ_3) \\
&\quad - \tilde{g}(FZ_1, FZ_4)\tilde{g}(FZ_1, FZ_3) - 2\tilde{g}(TZ_1, Z_2)\tilde{g}(JFZ_3, FZ_4)\} \\
&\quad - P(JZ_1, FZ_4)\tilde{g}(FZ_2, FZ_3) + P(JZ_1, FZ_3)\tilde{g}(FZ_2, FZ_4)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - P(JZ_2, FZ_3)\tilde{g}(FZ_1, FZ_4) + P(JZ_2, FZ_4)\tilde{g}(FZ_1, FZ_3) \\
&\quad + 2\{P(JZ_1, Z_2)\tilde{g}(JFZ_3, FZ_4) + P(JFZ_3, FZ_4)\tilde{g}(JZ_1, Z_2)\}, \\
4\tilde{R}(Z_1, Z_2, FZ_3, \xi_1) &= -P(JZ_1, \xi)\tilde{g}(FZ_2, FZ_3) \\
&\quad + P(JZ_2, \xi_1)\tilde{g}(FZ_1, FZ_3) + 2P(JFZ_3, \xi_1)\tilde{g}(JZ_1, Z_2), \\
2\tilde{R}(Z_1, Z_2, \xi_1, \xi_2) &= -c\tilde{g}(JZ_1, Z_2)\tilde{g}(J\xi_1, \xi_2) \\
&\quad + P(JZ_1, Z_2)\tilde{g}(J\xi_1, \xi_2) + P(J\xi_1, \xi_2)\tilde{g}(JZ_1, Z_2)
\end{aligned}
$$

for any $X_1, X_2 \in \mathcal{D}$, $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$ and $\xi_1, \xi_2 \in \nu$. Hence we get from (2.2) that the Ricci equation is separated by

$$
\begin{aligned}
&2R^\perp(X_1, X_2, FZ_1, FZ_2) - 2\tilde{g}([A_{FZ_1}, A_{FZ_2}]X_1, X_2) = \\
&\qquad - c\tilde{g}(JX_1, X_2)\tilde{g}(JFZ_1, FZ_2) + P(JX_1, X_2)\tilde{g}(JFZ_1, FZ_2) \\
&\qquad + P(JFZ_1, FZ_2)\tilde{g}(JX_1, X_2), \\
&2R^\perp(X_1, X_2, FZ_1, \xi_1) - 2\tilde{g}([A_{FZ_1}, A_{\xi_1}]X_1, X_2) = P(JFZ_1, \xi)\tilde{g}(JX_1, X_2), \\
&2R^\perp(X_1, X_2, \xi_1, \xi_2) - 2\tilde{g}([A_{\xi_1}, A_{\xi_2}]X_1, X_2) = -c\tilde{g}(JX_1, X_2)\tilde{g}(J\xi_1, \xi_2) + \\
&\qquad + P(JX_1, X_2)\tilde{g}(J\xi_1, \xi_2) + P(J\xi_1, \xi_2)\tilde{g}(JX_1, X_2), \\
&4R^\perp(X_1, Z_1, FZ_2, FZ_3) - 4\tilde{g}([A_{FZ_2}, A_{FZ_3}]X_1, Z_2) = \\
&\qquad = -P(JX_1, FZ_3)\tilde{g}(FZ_1, FZ_2) + P(JX_1, FZ_2)\tilde{g}(FZ_1, FZ_3) + \\
&\qquad + 2P(JX_1, Z_1)\tilde{g}(JFZ_2, FZ_3), \\
&4R^\perp(X_1, Z_1, FZ_2, \xi) - 4\tilde{g}([A_{FZ_2}, A_{\xi_1}]X_1, Z_1) = -P(JX_1, \xi_1)\tilde{g}(FZ_1, FZ_2), \\
&2R^\perp(X_1, Z_1, \xi_1, \xi_2) - 2\tilde{g}([A_{\xi_1}, A_{\xi_2}]X_1, Z_1) = P(JX_1, Z_1)\tilde{g}(J\xi_1, \xi_2), \\
&4R^\perp(Z_1, Z_2, FZ_3, FZ_4) - 4\tilde{g}([A_{FZ_3}, A_{FZ_4}]Z_1, Z_2) = \\
&\qquad = c\{\tilde{g}(FZ_1, FZ_4)\tilde{g}(FZ_2, FZ_3) - \tilde{g}(FZ_1, FZ_3)\tilde{g}(FZ_2, FZ_4) - \\
&\qquad - 2\tilde{g}(TZ_1, Z_2)\tilde{g}(JFZ_3, FZ_4)\} - P(JZ_1, FZ_4)\tilde{g}(FZ_2, FZ_3) + \\
&\qquad + P(JZ_1, FZ_3)\tilde{g}(FZ_2, FZ_4) - P(JZ_2, FZ_3)\tilde{g}(FZ_1, FZ_4) + \\
&\qquad + P(JZ_2, FZ_4)\tilde{g}(FZ_1, FZ_3) + 2\{P(JZ_1, Z_2)\tilde{g}(JFZ_3, FZ_4) + \\
&\qquad + P(JFZ_3, FZ_4)\tilde{g}(TZ_1, Z_2)\}, \\
&4R^\perp(Z_1, Z_2, FZ_3, \xi_1) - 4\tilde{g}([A_{FZ_3}, A_{\xi_1}]Z_1, Z_2) = 2P(JFZ_3, \xi_1)\tilde{g}(TZ_1, Z_2) - \\
&\qquad - P(JZ_1, \xi_1)\tilde{g}(FZ_2, FZ_3) + P(JZ_2, \xi)\tilde{g}(FZ_1, FZ_3), \\
&4R^\perp(Z_1, Z_2, \xi_1, \xi_2) - 2\tilde{g}([A_{\xi_1}, A_{\xi_2}]Z_1, Z_2) = -c\tilde{g}(JZ_1, Z_2)\tilde{g}(J\xi_1, \xi_2) + \\
&\qquad + P(JZ_1, Z_2)\tilde{g}(J\xi_1, \xi_2) + P(J\xi_1, \xi_2)\tilde{g}(JZ_1, Z_2)
\end{aligned}
$$

for any $X_1, X_2 \in \mathcal{D}$, $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$ and $\xi_1, \xi_2 \in \nu$.

Using the above equation, we have, for a generalized adopted frame, the following:

$$
\begin{aligned}
&2R^{\perp}{}_{jib^*a^*} - 2\tilde{g}([A_{b^*}, A_{a^*}]e_j, e_i) = -c\tilde{g}(Je_j, e_i)\tilde{g}(Je_b{}^*, e_a{}^*) + \\
&\qquad + P(Je_j, e_i)\tilde{g}(Je_b{}^*, e_a{}^*) + P(Je_b{}^*, e_a{}^*)\tilde{g}(Je_j, e_i), \\
&2R^{\perp}{}_{jia^*r} - 2\tilde{g}([A_{a^*}, A_r]e_j, e_i) = P(Je_a{}^*, e_r)\tilde{g}(Je_j, e_i), \\
&2R^{\perp}{}_{jisr} - 2\tilde{g}([A_{e_s}, A_{e_r}]e_j, e_i) = -c\tilde{g}(Je_j, e_i)\tilde{g}(Je_s, e_r) + \\
&\qquad + P(Je_j, e_i)\tilde{g}(Je_s, e_r) + P(Je_s, e_r)\tilde{g}(Je_j, e_i), \\
&4R^{\perp}{}_{icb^*a^*} - 4\tilde{g}([A_{b^*}, A_{a^*}]e_i, e_c) = \|Fe_c\|\{-P(Je_i, e_a{}^*)\delta_{cb} + \\
&\qquad + P(Je_i, e_b{}^*)\delta_{ca}\} + 2P(Je_i, e_c)\tilde{g}(Je_b{}^*, e_a{}^*), \\
&4R^{\perp}{}_{iba^*r} - 4\tilde{g}([A_{a^*}, A_r]e_i, e_b) = -\|Fe_b\|P(Je_i, e_r)\delta_{ba}, \\
&2R^{\perp}{}_{iasr} - 2\tilde{g}([A_s, A_r]e_i, e_a) = P(Je_i, e_a)\tilde{g}(Je_s, e_r), \qquad (5.1)\\
&4R^{\perp}{}_{dcb^*a^*} - 4\tilde{g}([A_{b^*}, A_{a^*}]e_d, e_c) = \\
&\quad = c\{\|Fe_c\|\|Fe_d\|(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}) - 2\tilde{g}(Te_d, e_c)\tilde{g}(Je_b{}^*, e_a{}^*)\} - \\
&\quad - \|Fe_c\|\{P(Je_d, e_a{}^*)\delta_{cb} - P(Je_d, e_b{}^*)\delta_{ca}\} - \\
&\quad - \|Fe_d\|\{P(e_c{}^*, e_b{}^*)\delta_{da} - P(Je_c, e_a{}^*)\delta_{cb}\} + \\
&\quad + 2\{\|Fe_c\|P(Je_d, e_c)\tilde{g}(Je_b{}^*, e_a{}^*) + P(Je_b{}^*, e_a{}^*)\tilde{g}(Te_d, e_c)\}, \\
&4R^{\perp}{}_{cba^*r} - 4\tilde{g}([A_{a^*}, A_r]e_c, e_b) = 2P(Je_a{}^*, e_r)\tilde{g}(Te_c, e_b) + \\
&\qquad + \|Fe_c\|\|Fe_b\|\{P(Je_b^*, e_r)\delta_{ca} - P(Je_c{}^*, e_r)\delta_{ba}\}, \\
&2R^{\perp}{}_{basr} - 2\tilde{g}([A_s, A_r]e_b, e_a) = -c\tilde{g}(Te_b, e_a)\tilde{g}(Je_r, e_s) + \\
&\qquad + P(Je_b, e_a)\tilde{g}(Je_s, e_r) + P(Je_s, e_r)\tilde{g}(Te_b, e_a),
\end{aligned}
$$

where $Fe_a = \|Fe_a\|a^*$, $j, i \in \{1, 2, \ldots, 2p\}$, $d, c, b, a \in \{2p+1, 2p+2, 2p+q\}$, and $s, r \in \{n+q, n+q+1, \ldots, m\}$. By virtue of (2.5), we can formally put

$$Je_j = \sum_{i=1}^{p}(T_j{}^i e_i + T_j{}^{p+i} e_{p+i}) + T_j{}^a e_a + F_j{}^a e_a^* + F_j{}^s e_s,$$

$$Je_{p+j} = \sum_{i=1}^{p}(T_{p+j}^i e_i + T_{p+j}{}^{p+i} e_{p+i}) + T_{p+j}^a e_a + F_{p+j}^a e_a{}^* + F_{p+j}{}^s e_s,$$

$$Je_a = \sum_{i=1}^{p}(T_a{}^i e_i + T_a{}^{p+i} e_{p+i}) + T_a{}^c e_c + F_a{}^c e_c{}^* + F_a{}^s e_s,$$

$$Je_a{}^* = \sum_{i=1}^{p}(t_a{}^i e_i + t_a{}^{p+i} e_{p+i}) + t_a{}^c e_c + f_a{}^c e_c{}^* + f_a{}^s e_s,$$

$$Je_s = \sum_{i=1}^{p}(t_s{}^i e_i + t_s{}^{p+i} e_{p+i}) + t_s{}^a e_a + f_s{}^a e_a{}^* + f_s{}^r e_r,$$

for $a, c \in \{2p+1, 2p+2, \dots, 2p+q\}$, $s, r \in \{n+q+1, n+q+2, \dots, m\}$, and $j \in \{1, 2, .., p\}$. Since, our frame is a generalized adapted one, we know in the above equation

$$\begin{pmatrix} T_j{}^i & T^i_{p+j} & T_a{}^i \\ T_j{}^{p+i} & T_{p+j}{}^{p+i} & T_a{}^{p+i} \\ T_j{}^a & T^a_{p+j} & T_b{}^a \end{pmatrix} = \begin{pmatrix} 0 & -\delta_j{}^i & 0 \\ \delta_j{}^i & 0 & 0 \\ 0 & 0 & T_b{}^a \end{pmatrix}, \tag{5.2}$$

$$\begin{pmatrix} F_j{}^a & F_j{}^s \\ F_{p+j}{}^a & F_{p+j}{}^s \\ F_b{}^a & F_b{}^s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ F_b{}^a & 0 \end{pmatrix}, \tag{5.3}$$

$$\begin{pmatrix} t_a{}^i & t_a{}^{p+i} & t_a{}^c \\ t_s{}^i & t_s{}^{p+i} & t_s{}^a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.4}$$

and

$$\begin{pmatrix} f_a{}^c & f_a{}^s \\ f_s{}^a & f_s{}^r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & f_s{}^r \end{pmatrix}. \tag{5.5}$$

From (5.5), we can easily get that $f_{ba} = 0$ identically.

Theorem 5.1. *With respect to the generalized adapted frame, the tensor field T, F, t and f satisfy* (5.2), (5.3), (5.4) *and* (5.5), *respectively.*

Thus due to (5.3), (5.4) and (5.5) the system of equations (5.1) can be written as follows:

$$2R^{\perp}{}_{jib^*a^*} - 2\tilde{g}([A_{b^*}, A_{a^*}]e_j, e_i) = 0,$$

$$R^{\perp}{}_{jia^*r} - \tilde{g}([A_{a^*}, A_r]e_j, e_i) = 0,$$

$$R^{\perp}{}_{jisr} - \tilde{g}([A_s, A_r]e_j, e_i) = 0,$$

$$4R^{\perp}{}_{icb^*a^*} - 4\tilde{g}([A_{b^*}, A_{a^*}]e_i, e_c) = \|Fe_c\|(P_{i^*a^*}\delta_{cb} - P_{i^*b^*}\delta_{ca}),$$

$$4R^{\perp}{}_{iba^*r} - 4\tilde{g}([A_{a^*}, A_r]e_i, e_b) = \|Fe_b\|P_{i^*r}\delta_{ba},$$

$$2R^{\perp}{}_{iasr} - 2\tilde{g}([A_s, A_r]e_i, e_a) = -P_{i^*a}\tilde{g}(Je_s, e_r),$$

$$\begin{aligned} 4R^{\perp}{}_{dcb^*a^*} - 4\tilde{g}([A_{b^*}, A_{a^*}]e_d, e_c) &= c\|Fe_c\|\|Fe_d\|(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}) + \\ &+ \|Fe_c\|\{(T_d{}^e P_{ea^*} + F_d{}^e P_{e^*a^*})\delta_{cb} - (T_d{}^e P_{eb^*} + F_d{}^e P_{e^*b^*})\delta_{ca}\} - \\ &- \|Fe_d\|\{P_{c^*b^*}\delta_{da} - (T_c{}^e P_{ea^*} + F_c{}^e P_{e^*a^*})\delta_{cb}\}, \end{aligned}$$

$$4R^{\perp}{}_{cba^*r} - 4\tilde{g}([A_{a^*}, A_r]e_c, e_b) = 0,$$

$$\begin{aligned} 2R^{\perp}{}_{basr} - 2\tilde{g}([A_s, A_r]e_b, e_a) &= -cT_{ba}\tilde{g}(Je_r, e_s) + F_b{}^e P_{e^*a}\tilde{g}(Je_s, e_r) + \\ &+ P(Je_s, e_r)T_{ba}, \end{aligned}$$

for $j, i \in \{1, 2, \ldots, p\}$, $d, c, b, a \in \{2p+1, 2p+2, \ldots, 2p+q = n\}$ and $s, r \in \{n+q, n+q+1, .., m\}$. Thus we have

Theorem 5.2. *If the first type warped product semi-slant submanifold in an l.c.K.-space form $\tilde{M}(c)$ is normally flat, that is, $R^{\perp} = 0$, identically, then the shape operators A_λ satisfy*

$$\begin{aligned}
&\tilde{g}([A_{b^*}, A_{a^*}]e_j, e_i) = 0,\\
&\tilde{g}([A_{a^*}, A_r]e_j, e_i) = 0,\\
&\tilde{g}([A_s, A_r]e_j, e_i) = 0,\\
&4\tilde{g}([A_{b^*}, A_{a^*}]e_i, e_c) = -\|Fe_c\|(P_{i^*a^*}\delta_{cb} - P_{i^*b^*}\delta_{ca}),\\
&4\tilde{g}([A_{a^*}, A_r]e_i, e_b) = -\|Fe_b\|P_{i^*r}\delta_{ba},\\
&2\tilde{g}([A_s, A_r]e_i, e_a) = P_{i^*a}\tilde{g}(Je_s, e_r),\\
&4\tilde{g}([A_{b^*}, A_{a^*}]e_d, e_c) = -c\|Fe_c\|\|Fe_d\|(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}) +\\
&\qquad + \|Fe_c\|\{(T_d{}^e P_{ea^*} + F_d{}^e P_{e^*a^*})\delta_{cb} - (T_d{}^e P_{eb^*} + F_d{}^e P_{e^*b^*})\delta_{ca}\} -\\
&\qquad\qquad - \|Fe_d\|\{P_{c^*b^*}\delta_{da} - (T_c{}^e P_{ea^*} + F_c{}^e P_{e^*a^*})\delta_{cb}\},\\
&\tilde{g}([A_{a^*}, A_r]e_c, e_b) = 0,\\
&2\tilde{g}([A_s, A_r]e_b, e_a) = cT_{ba}\tilde{g}(Je_r, e_s) - F_b{}^e P_{e^*a}\tilde{g}(Je_s, e_r) - P(Je_s, e_r)T_{ba},
\end{aligned}$$

for $j, i \in \{1, 2, \ldots, p\}$, $d, c, b, a \in \{2p+1, 2p+2, \ldots, 2p+q = n\}$ *and* $s, r \in \{n+q, n+q+1, .., m\}$.

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