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Automorphisms of Kronrod-Reeb graphs of Morse functions on 2-sphere

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Abstract. Let M be a compact two-dimensional manifold, $f \in C^{\infty}(M,\mathbb{R})$ be a Morse function, and Γ_f be its Kronrod-Reeb graph. Denote by

$$\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}\}$$

the orbit of f with respect to the natural right action of the group of diffeomorphisms $\mathcal{D}$ on $C^{\infty}(M,\mathbb{R})$, and by $\mathcal{S}(f) = \{h \in \mathcal{D} \mid f \circ h = f\}$ the corresponding stabilizer of this function. It is easy to show that each $h \in \mathcal{S}(f)$ induces a homeomorphism of Γ_f. Let also $\mathcal{D}_{\mathrm{id}}(M)$ be the identity path component of $\mathcal{D}(M)$, $\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M)$ be group of diffeomorphisms of M preserving f and isotopic to identity map, and G_f be the group of homeomorphisms of the graph Γ_f induced by diffeomorphisms belonging to $\mathcal{S}'(f)$. This group is one of the key ingredients for calculating the homotopy type of the orbit $\mathcal{O}(f)$.

Recently, the authors described the structure of groups G_f for Morse functions on all orientable surfaces distinct from 2-torus and 2-sphere. The present paper is devoted to the case $M = S^2$. In this situation Γ_f is always a tree, and therefore all elements of the group G_f have a common fixed subtree $\mathrm{Fix}(G_f)$, which may even consist of a unique vertex. Our main result calculates the groups G_f for all Morse functions $f\colon S^2 \to \mathbb{R}$ whose fixed subtree $\mathrm{Fix}(G_f)$ consists of more than one point.

Анотація. Нехай M – компактна двовимірна поверхня, $f\colon M \to \mathbb{R}$ – функція Морса і Γ_f – її граф Кронрода-Ріба. Визначимо природну праву дію групи $\mathcal{C}^{\infty}$ дифеоморфізмів $\mathcal{D}(M)$ поверхні M на $\mathcal{C}^{\infty}(M,\mathbb{R})$ за правилом: результат дії дифеоморфізма $h \in \mathcal{D}(M)$ на $f \in \mathcal{C}^{\infty}(M,\mathbb{R})$ – це композиція $f \circ h\colon M \to \mathbb{R}$. Для $f \in \mathcal{C}^{\infty}(M,\mathbb{R})$ позначимо через $\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\}$ та $\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$ – відповідно орбіту та стабілізатор цієї функції. Легко показати, що кожен $h \in \mathcal{S}(f)$ індукує деякий гомеоморфізм $\rho(h)$ графа Γ_f, а відповідність $h \mapsto \rho(h)$ є гомоморфізмом з $\mathcal{S}(f)$ в групу гомеоморфізмів Γ_f. Нехай $\mathcal{D}_{\mathrm{id}}(M)$ – тотожна компонента зв'язності групи $\mathcal{D}(M)$, $\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M)$ – підгрупа в $\mathcal{D}_{\mathrm{id}}(M)$ що складається з дифеоморфізмів які зберігають f

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і орієнтацію M та ізотопні до тотожного відображення, і G_f – група автоморфізмів графа Кронрода-Ріба функції f індукованих дифеоморфізмами з $\mathcal{S}'(f)$. Остання група є скінченною і відіграє ключову роль для обчислення гомотопічного типу орбіти $\mathcal{O}(f)$.

В попередній статті автори описали алгебраїчну структуру груп G_f для функцій Морса на всіх орієнтовних поверхнях відмінних від тора T^2 та 2-сфери S^2. Дана робота присвячена випадку $M = S^2$. Відмітимо, що для кожної функції Морса $f\colon S^2 \to \mathbb{R}$ її граф Γ_f завжди є деревом, а тому всі елементи групи G_f мають спільне нерухому піддерево $\mathrm{Fix}(G_f)$, яке може складатись з однієї вершини. Основний результат даної статті обчислює групи G_f для всіх функцій Морса $f\colon S^2 \to \mathbb{R}$, у яких нерухоме піддерево $\mathrm{Fix}(G_f)$ містить більше ніж одну точку.

Отримані результати мають місце також для більш широкого ніж морсівські класу функцій $f\colon S^2 \to \mathbb{R}$, що є гладко еквівалентними до однорідних многочленів без кратних множників в околі кожної своєї критичної точки.

1. Introduction

Let M be a compact two-dimensional manifold and $\mathcal{D}(M)$ the group of diffeomorphisms of M. Then there exists a natural right action

$$\phi\colon \mathcal{C}^{\infty}(M,\mathbb{R}) \times \mathcal{D}(M) \to \mathcal{C}^{\infty}(M,\mathbb{R})$$

of this group on the space of smooth functions on M defined by the formula $\phi(f,h) = f\circ h$. For $f \in \mathcal{C}^{\infty}(M,\mathbb{R})$ denote by

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f\circ h = f\}$$

its *stabilizer* with respect to the specified action.

Definition 1.1. Let $\mathcal{F}(M,\mathbb{R})$ be the subset of $\mathcal{C}^{\infty}(M,\mathbb{R})$ consisting of maps $f\colon M \to \mathbb{R}$ such

(1) f takes constant values on the connected components of the boundary ∂M and has no critical points on ∂M;

(2) for each critical point z of f there are local coordinates (x,y) in which $z = (0,0)$ and $f(x,y) = f(z) + g_z(x,y)$, where $g_z\colon \mathbb{R}^2 \to \mathbb{R}$ is a homogeneous polynomial without multiple factors.

Notice that every critical point of $f \in \mathcal{F}(M,\mathbb{R})$ is isolated.

A function $f \in \mathcal{F}(M,\mathbb{R})$ is called *Morse*, if $\deg g_z = 2$ for each critical point z of f. In that case, due to Morse Lemma, one can assume that $g_z(x,y) = \pm x^2 \pm y^2$.

We will denote by $\mathcal{M}(M,\mathbb{R})$ the space of all Morse maps $M \to \mathbb{R}$.

Homotopy types of stabilizers and orbits of Morse functions and functions from $\mathcal{F}(M,\mathbb{R})$ were studied in [8], [9], [10], [1], [2], [3], [4], [5], [6].

Let $f \in \mathcal{C}^\infty(M,\mathbb{R})$, $\mathit{\Gamma}_f$ be a partition of the surface M into the connected components of level sets of this function, and $p : M \to \mathit{\Gamma}_f$ be the canonical factor-mapping, associating to each $x \in M$ the connected component of the level set $f^{-1}(f(x))$ containing that point.

Endow $\mathit{\Gamma}_f$ with the factor topology with respect to the mapping p: so a subset $A \subset \mathit{\Gamma}_f$ will be regarded as open if and only if its inverse image $p^{-1}(A)$ is open in M. Then f induces the function $\hat{f} \colon \mathit{\Gamma}_f \to \mathbb{R}$, such that $f = \hat{f} \circ p$.

It is well known, that if $f \in \mathcal{F}(M,\mathbb{R})$, then $\mathit{\Gamma}_f$ has a structure of a one-dimensional CW-complex called the *Kronrod-Reeb graph*, or simply the *graph* of f. The vertices of this graph correspond to critical connected components of level sets of f and connected components of the boundary of the surface. By the *edge* of $\mathit{\Gamma}_f$ we will mean an *open* edge, that is, a one-dimensional cell.

Denote by $\mathcal{H}(\mathit{\Gamma}_f)$ the group of homeomorphisms of $\mathit{\Gamma}_f$. Notice that each element of the stabilizer $h \in \mathcal{S}(f)$ leaves invariant each level set of f, and therefore induces a homeomorphism $\rho(h)$ of the graph of f, so that the following diagram is commutative:

$$\begin{array}{ccccc} M & \xrightarrow{\;p\;} & \mathit{\Gamma}_f & \xrightarrow{\;\hat{f}\;} & \mathbb{R} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \rho(h)} & & \| \\ M & \xrightarrow{\;p\;} & \mathit{\Gamma}_f & \xrightarrow{\;\hat{f}\;} & \mathbb{R} \end{array} \qquad (1.1)$$

Moreover, the correspondence $h \mapsto g(h)$ is a homomorphism of groups $\rho : \mathcal{S}(f) \to \mathcal{H}(\mathit{\Gamma}_f)$.

Let also $\mathcal{D}_{\mathrm{id}}(M)$ be the path component of the identity map id_M in $\mathcal{D}(M)$. Put

$$\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M) \qquad\qquad G_f = \rho(\mathcal{S}'(f)).$$

Thus, G_f is the group of automorphisms of the Kronrod-Reeb graph of f induced by diffeomorphisms of the surface preserving the function and isotopic identity.

Remark 1.2. Since $\hat{f}$ is monotone on edges of $\mathit{\Gamma}_f$, it is easy to show that G_f is a finite group. Moreover, if $g(E) = E$, for some $g \in G$ and an edge E of the graph $\mathit{\Gamma}_f$, then $g(x) = x$ for all $x \in E$.

Since G_f is finite and ρ is continuous, it follows that ρ reduces to an epimorphism $\rho_0 \colon \pi_0\mathcal{S}'(f) \to G_f$ of the group $\pi_0\mathcal{S}'(f)$ path components of $\mathcal{S}'(f)$ being an analogue of the mapping class group for f-preserving diffeomorphisms.

Algebraic structure of the group $\pi_0\mathcal{S}'(f)$ of connected components of $\mathcal{S}'(f)$ for all $f \in \mathcal{F}(M,\mathbb{R})$ on orientable surfaces M distinct from 2-torus and 2-sphere is described in [11], and the structure of its factor group G_f is investigated in [7]. These groups play an important role in computing the homotopy type of the path component $\mathcal{O}_f(f)$ of the orbit of f, see also [8], [9], [1], [2], [3].

The purpose of this note is to describe the groups G_f for a certain class of smooth functions on 2-sphere.

The main result Theorem 1.4 reduces computation of G_f to computations of similar groups for restrictions of f to some disks in the 2-sphere. As noted above the latter calculations were described in [7].

First we recall a variant of the well known fact about automorphisms of finite trees from graphs theory.

Lemma 1.3. *Let Γ be a finite contractible one-dimensional CW-complex («a topological tree»), G be a finite group of its cellular homeomorphisms, and* $\mathrm{Fix}(G)$ *be the set of common fixed points of all elements of the group G. Then* $\mathrm{Fix}(G)$ *is either a contractible subcomplex or consists of a single point belonging to some edge E an open 1-cell), and in the latter case there exists $g \in G$ such that $g(E) = E$ and g changes the orientation of E.* □

Suppose $f\colon \mathcal{S}^2 \to \mathbb{R}$ belongs to $\mathcal{F}(S^2,\mathbb{R})$. Then it is easy to show that Γ_f is a tree, *i.e.*, a finite contractible one-dimensional CW-complex, and by Remark 1.2 G_f is a finite group of cellular homeomorphisms of Γ_f. Therefore, for G_f, the conditions of Lemma 1.3 are satisfied. Note that according to Remark 1.2 the second case of Lemma 1.3 is impossible, and hence G_f has a fixed subtree.

In this paper we consider the case when the fixed subtree of the group G_f contains more than one vertex, *i.e.* has at least one edge.

Let us also mention that $\mathcal{D}_{\mathrm{id}}(S^2)$ coincides with the group $\mathcal{D}^+(S^2)$ of diffeomorphisms of the sphere preserving orientation, [12]. Therefore $\mathcal{S}'(f)$ consists of diffeomorphisms of the sphere preserving the function f and the orientation of $\mathcal{S}^2$.

Theorem 1.4. *Let $f \in \mathcal{F}(S^2,\mathbb{R})$. Suppose that all elements of the group G_f have a common fixed edge E. Let $x \in E$ be an arbitrary point and A and B be the closures of the connected components of $S^2 \setminus p^{-1}(x)$, see Figure 1.1. Then*

1. *A and B are 2-disks being invariant with respect to $\mathcal{S}'(f)$;*
2. *the restrictions $f|_A \in \mathcal{F}(A,\mathbb{R})$ and $f|_B \in \mathcal{F}(B,\mathbb{R})$;*
3. *the map $\phi\colon G_f \to G_{f|_A} \times G_{f|_B}$ defined by $\phi(\gamma) = (\gamma|_{\Gamma_A}, \gamma|_{\Gamma_B})$ is an isomorphism of groups.*

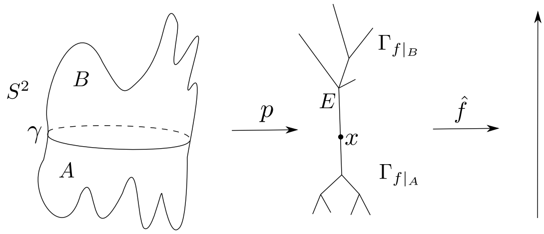

FIGURE 1.1.

Corollary 1.5. *For any $a, b \in \mathcal{F}(D^2, \mathbb{R})$ there exists $f \in \mathcal{F}(S^2, \mathbb{R})$ such that G_f has an fixed subtree consisting of more than one vertex and*

$$G_f \cong G_a \times G_b.$$

Sketch of proof. One can assume that a takes maximal value on ∂D^2, while b takes a minimal value on ∂D^2. Now regard $S^2 = \{x^2 + y^2 + z^2 = 1\}$ as the unit sphere in $\mathbb{R}^3$. Let also $A = S^2 \cap \{z \leq 0\}$ and $B = S^2 \cap \{z \geq 0\}$ be the lower and upper hemispheres respectively. Then one can assume that the function a is defined on A, the function b is defined on B, they coincide on the big circle $A \cap B$, and define a function $f \in \mathcal{F}(S^2, \mathbb{R})$. It then easily follows that G_f has a fixed edge containing the point corresponding to $A \cap B$, whence by Theorem 1.4 we get an isomorphism $G_f \cong G_a \times G_b$. We leave the details for the reader. □

For the structure of the groups G_f for $f \in \mathcal{F}(D^2, \mathbb{R})$ we refer the reader to the paper [7].

2. PROOF OF THEOREM 1.4

(1) By assumption x belongs to the open edge E. Therefore $p^{-1}(x)$ is a regular connected component of some level set of the function f, that is, a simple closed curve. Then, by Jordan Theorem, $p^{-1}(x)$ divides the sphere into two connected components whose closures are homeomorphic to two-dimensional disks. Consequently, A and B are two-dimensional disks.

Let as show that A and B are invariant with respect to $\mathcal{S}'(f)$, *i.e.*, $h(A) = A$ and $h(B) = B$ for each $h \in \mathcal{S}'(f)$. Denote

$$\mathit{\Gamma}_A = p(A) \qquad\qquad \mathit{\Gamma}_B = p(B).$$

Then

$$\mathit{\Gamma}_A \cup \mathit{\Gamma}_B = \mathit{\Gamma} \qquad\qquad \mathit{\Gamma}_A \cap \mathit{\Gamma}_B = \{x\}.$$

By definition, $\rho(h)(x) = x$, whence $\rho(h)$ either preserves both Γ_A and Γ_B or interchange them. We claim that

$$\rho(h)(\Gamma_A) = \Gamma_A \qquad\qquad \rho(h)(\Gamma_B) = \Gamma_B.$$

Indeed suppose $\rho(h)(\Gamma_A) = \Gamma_B$. Since $\rho(h)$ is fixed on E, it follows that

$$\rho(h)(\Gamma_A \cap E) = \Gamma_A \cap E,$$

whence

$$\rho(h)(\Gamma_A \cap E) = \rho(h)(\Gamma_A) \cap \rho(E) = \Gamma_B \cap E \neq \Gamma_A \cap E,$$

which contradicts to our assumption. Thus Γ_A and Γ_B are invariant with respect to the group G_f.

Now we can show that A and B are also invariant with respect to h. By virtue of the commutativity of the diagram (1.1) $\rho(h)(p(y)) = p(h(y))$ for all $y \in \Gamma$. In particular:

$$p(h(A)) = \rho(h)(p(A)) = \rho(h)(\Gamma_A) = \Gamma_A.$$

Therefore, $h(A) = p^{-1}(\Gamma_A) = A$. The proof for B is similar. Thus, A and B are invariant with respect to $\mathcal{S}'(f)$.

(2) Notice that the function f takes a constant value on the simple closed curve $p^{-1}(x)$ being a common boundary of disks A and B, and does not contain critical points of f. Therefore, the restrictions $f|_A, f|_B$ satisfy the conditions 1) and 2) the Definition 1.1, and so they belong to $\mathcal{F}(A,\mathbb{R})$ and $\mathcal{F}(B,\mathbb{R})$ respectively.

(3) We should prove that the map $\phi\colon G_f \to G_{f|_A} \times G_{f|_B}$ defined by formula $\phi(\gamma) = (\gamma|_{\Gamma_A}, \gamma|_{\Gamma_B})$ is an isomorphism.

First we will show that ϕ is correctly defined. Let $\gamma \in G_f = \rho(\mathcal{S}'(f))$, that is, $\gamma = \rho(h)$, where h is a diffeomorphism of the sphere preserving the function f and isotopic to the identity.

We claim that $h|_A \in \mathcal{S}'(f|_A) = \mathcal{S}(f|_A) \cap \mathcal{D}_{\mathrm{id}}(A)$. Indeed, for each point $x \in A$ we have that:

$$f(x) = f|_A(x) = f|_A(h|_A(x)) = f|_A(h(x)) = f(h(x)),$$

which means that $h|_A \in \mathcal{S}(f|_A)$.

Moreover, since h preserves the orientation of the sphere, it follows that $h|_A$ preserves the orientation of the disk A, whence by [12], $h|_A \in \mathcal{D}_{\mathrm{id}}(A)$. Thus $\gamma|_{\Gamma_A} \in G_{f|_A}$. Similarly $\gamma|_{\Gamma_B} \in G_{f|_B}$, and so ϕ is well defined.

Let us now verify that ϕ is an *isomorphism of groups*, that is, a bijective homomorphism. Indeed, if $\delta, \omega \in G_f$, then

$$\begin{aligned}\phi(\delta \circ \omega) &= \big(\delta \circ \omega|_{\Gamma_A},\ \delta \circ \omega|_{\Gamma_B}\big) = \\ &= \big(\delta|_{\Gamma_A},\ \delta|_{\Gamma_B}\big) \circ \big(\omega|_{\Gamma_A},\ \omega|_{\Gamma_B}\big) =\end{aligned}$$

$$= \big(\delta|_{\Gamma_A} \circ \omega|_{\Gamma_A},\ \delta|_{\Gamma_B} \circ \omega|_{\Gamma_B}\big) =$$
$$= \big(\delta \circ \omega|_{\Gamma_A},\ \delta \circ \omega|_{\Gamma_B}\big),$$

sp ϕ is a homomorphism.

Let us show that $\ker\phi = \{\mathrm{id}_\Gamma\}$. Indeed, suppose $\gamma \in \ker\phi$, that is $\gamma|_{\Gamma_A} = \mathrm{id}_{\Gamma_A}$ and $\gamma|_{\Gamma_B} = \mathrm{id}_{\Gamma_B}$. Then γ is fixed on $\Gamma_A \cup \Gamma_B = \Gamma$, and hence it is the identity map.

Surjectivity of $\phi\colon G_f \to G_{f|_A} \times G_{f|_B}$ is implied by the following simple lemma.

Lemma 2.1. *Let $f \in \mathcal{F}(M,\mathbb{R})$. Then for each $\alpha \in G_f$, there exists $a \in \mathcal{S}'(f)$ fixed near the boundary ∂M and such that $\alpha = \rho(a)$.*

Proof. By assumption $\alpha = \rho(h)$ for some $h \in \mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M)$. Since h is isotopic to id_M, it leaves invariant each connected component V of ∂M and preserves its orientation. In particular, the restriction $h|_V : V \to V$ is isotopic to id_V. As f takes constant value on V and has not critical points near V, h can be deformed in $\mathcal{S}(f)$ to a diffeomorphism a fixed near ∂M, *e.g.* [8, Lemma 4.14], whence $\rho(a) = \rho(h) = \alpha$. □

Let $(\alpha,\beta) \in G_{f|_A} \times G_{f|_B}$, then by Lemma 2.1 there exist $a \in \mathcal{S}'(f|_A)$ and $b \in \mathcal{S}'(f|_B)$ fixed near $\partial A = \partial B = p^{-1}(x)$ and such that $\alpha = \rho_A(a)$ and $\beta = \rho_B(b)$. Define h by the following formula:

$$h = \begin{cases} a(x), & x \in A, \\ b(x), & x \in B. \end{cases}$$

Then, h is a diffeomorphism of the sphere, preserving the function and orientation, whence $h \in \mathcal{S}'(f)$.

Moreover if we put $\gamma = \rho(h) \in G_f$, then $\gamma|_{\Gamma_A} = \rho(h|_A) = \alpha$ and $\gamma|_{\Gamma_B} = \rho(h|_B) = \beta$. In other words, $\phi(\gamma) = (\gamma|_{\Gamma_A}, \gamma|_{\Gamma_B}) = (\alpha,\beta)$, *i.e.*, ϕ is surjective and therefore an isomorphism.

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