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A $(CHR)_3$-flat trans-Sasakian manifold

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Abstract. In [4] M. Prvanovic considered several curvaturelike tensors defined for Hermitian manifolds. Developing her ideas in [3], we defined in an almost contact Riemannian manifold another new curvaturelike tensor field, which is called a contact holomorphic Riemannian curvature tensor or briefly $(CHR)_3$-curvature tensor. Then, we mainly researched $(CHR)_3$-curvature tensor in a Sasakian manifold. Also we proved, that a conformally $(CHR)_3$-flat Sasakian manifold does not exist.

In the present paper, we consider this tensor field in a trans-Sasakian manifold. We calculate the $(CHR)_3$-curvature tensor in a trans-Sasakian manifold. Also, the $(CHR)_3$-Ricci tensor ρ_3 and the $(CHR)_3$-scalar curvature τ_3 in a trans-Sasakian manifold have been obtained.

Moreover, we define the notion of the $(CHR)_3$-flatness in an almost contact Riemannian manifold. Then, we consider this notion in a trans-Sasakian manifold and determine the curvature tensor, the Ricci tensor and the scalar curvature. We proved that a $(CHR)_3$-flat trans-Sasakian manifold is a generalized η-Einstein manifold.

Finally, we obtain the expression of the curvature tensor with respect to the Riemannian metric g of a trans-Sasakian manifold, if the latter is $(CHR)_3$-flat.

Анотація. У [4] М. Прванович розглянуто декілька кривиноподібних тензорів, визначених для ермітових різновидів. Розвиваючи її ідеї у [3], ми ввводимо в майже контактному рімановому многовиді ще одне нове кривиноподібне тензорне поле, яке називається тензором контактної голоморфної ріманової кривини або коротко тензором $(CHR)_3$-кривини. Далі ми, головним чином, досліджуємо тензор $(CHR)_3$-кривини сасакійового многовиду. Крім того, ми доводимо, що конформно $(CHR)_3$-плоского сасакійового многовиду не існує.

У цій роботі ми розглядаємо це тензорне поле на транссасакійовому многовиді. Нами обчислено тензор $(CHR)_3$-кривини транссасакійового многовиду. Також були отримані $(CHR)_3$-Річчі тензор ρ_3 та $(CHR)_3$-скалярна кривина τ_3 транссасакійового многовиду.

Keywords: $(CHR)_3$-curvature tensor, trans-Sasakian manifold, $(CHR)_3$-flat almost contact Riemannian manifold, (generalized) η-Einstein manifold.

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Крім того, ми вводимо поняття $(CHR)_3$-плоских майже контактних ріманових многовидів. Нами розглянуто $(CHR)_3$-плоскі транс-сасакійові многовиди, для яких ми знаходимо тензор кривини, тензор Річчі та скалярну кривину. Нами доведено, що $(CHR)_3$-плоский транссасакійовий многовид є узагальненим η-ейнштейновим многовидом.

Нарешті, нами отримно вирази тензору кривини ріманової метрики g транссасакійового многовиду, якщо останній є $(CHR)_3$-плоским.

1. Almost contact Riemannian manifolds

A real $(2n+1)$-dimensional differentiable Riemannian manifold (M^{2n+1}, g) is said to be an *almost contact Riemannian manifold* if it has a $(1,1)$-tensor φ and a 1-form η which satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\varphi X) = 0, \qquad \eta(\xi) = 1, \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

for any $Y, X \in TM^{2n+1}$, where ξ is defined by

$$g(\xi, X) = \eta(X)$$

and TM^{2n+1} is the tangent bundle of M^{2n+1}.

From $(1.1)_3$, the vector field ξ is unit and we call this vector field the *structure vector field* of the almost contact Riemannian manifold. Next, in an almost contact Riemannian manifold M^{2n+1} we define a 2-form F by

$$F(X, Y) = g(\varphi X, Y) \tag{1.3}$$

for all $X, Y \in TM^{2n+1}$. Then the 2-form F is skew-symmetric and we call this tensor field the *fundamental* 2*-form* of this almost contact Riemannian manifold. Hereafter, we write the same φ instead of F.

An almost contact manifold M^{2n+1} is called *trans-Sasakian* if the fundamental form φ satisfies

$$\begin{aligned}(\nabla_X \varphi)(Y, Z) = \alpha\{g(X, Y)\eta(Z) - g(X, Z)\eta(Y)\}+\\ + \beta\{\varphi(X, Y)\eta(Z) - \varphi(X, Z)\eta(Y)\},\end{aligned} \tag{1.4}$$

for certain smooth functions α and β on M^{2n+1} and for all tangetn vectors $X, Y, Z \in TM^{2n+1}$, where ∇ means the covariant differentiation with respect to g. In that case we will say that a trans-Sasakian structure is of *type* (α, β) or of an (α, β)*-type*, [5].

Remark 1.1. A $(-1, 0)$-type (resp. $(0, 1)$-type) trans-Sasakian manifold is a Sasakian (resp. a Kenmotsu) manifold.

In a trans-Sasakian manifold of (α,β)-type, we know, [5], that

$$\begin{aligned}\nabla_X\xi &= -\alpha\varphi X+\beta\{X-\eta(X)\xi\},\\ (\nabla_X\eta)(Y) &= -\alpha g(\varphi X,Y)+\beta g(\varphi X,\varphi Y),\\ R(X,Y,Z,\xi) &= (X\alpha)g(\varphi Y,Z)-(Y\alpha)g(\varphi X,Z)\\ &\quad -(X\beta)g(\varphi Y,\varphi Z)+(Y\beta)g(\varphi X,\varphi Z)\\ &\quad +(\alpha^2-\beta^2)A(X,Y,Z)-2\alpha\beta A(X,Y,\varphi Z),\\ \rho(X,\xi) &= \{2n(\alpha^2-\beta^2)-(\xi\beta)\}\eta(X)-\alpha(\varphi X)-(2n-1)(X\beta)\end{aligned} \tag{1.5}$$

for any $X,Y\in TM^{2n+1}$, where ρ is the Ricci tensor with respect to g and $A(X,Y,Z)$ is defined as

$$A(X,Y,Z)=g(Z,Y)\eta(X)-g(Z,X)\eta(Y) \tag{1.6}$$

for any $X,Y,Z\in TM^{2n+1}$.

The following equations (1.7), (1.8), (1.9), (1.10) and (1.11) are very useful for calculations of the $(CHR)_3$-curvature tensor in a trans-Sasakian manifold.

By virtue of (1.4) and the Bianci identity, we have

$$\begin{aligned}&-R(X,Y,Z,\varphi W)+R(X,Y,W,\varphi Z)=\\ &\quad=(X\alpha)A(Z,W,Y)-(Y\alpha)A(Z,W,X)+\\ &\qquad+(X\beta)A(Z,W,\varphi Y)-(Y\beta)A(Z,W,\varphi X)+\\ &\quad+(\alpha^2-\beta^2)\big\{g(Y,W)g(\varphi X,Z)-g(Y,Z)g(\varphi X,W)-\\ &\qquad-g(X,W)g(\varphi Y,Z)+g(X,Z)g(\varphi Y,W)\big\}+\\ &\quad+2\alpha\beta\big\{g(\varphi X,W)g(\varphi Y,Z)-g(\varphi Y,W)g(\varphi X,Z)+\\ &\qquad+g(X,Z)g(Y,W)-g(X,W)g(Y,Z)\big\}\end{aligned} \tag{1.7}$$

for any $X,Y,Z,W\in TM^{2n+1}$. By virtue of (1.6), we can easily obtain

$$\begin{aligned}&R(\varphi X,Y,Z,\varphi W)+R(\varphi X,Y,\varphi Z,W)=\\ &\quad=-(\varphi X\alpha)A(Z,W,Y)+(Y\alpha)B(Z,W,X)+\\ &\qquad+(\varphi X\beta)A(Z,W,Y)-(Y\beta)A(Z,W,X)+\\ &\quad+(\alpha^2-\beta^2)\{g(\varphi X,Z)g(\varphi Y,W)-g(\varphi X,W)g(\varphi Y,Z)\\ &\qquad\quad+g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\ &\qquad\quad+g(Y,W)\eta(X)\eta(Z)-g(Y,Z)\eta(X)\eta(W)\}\\ &\quad+2\alpha\beta\{g(X,W)g(\varphi Y,Z)-g(X,Z)g(\varphi Y,W)\\ &\qquad\quad+g(Y,Z)g(\varphi X,W)-g(Y,W)g(\varphi X,Z)\\ &\qquad\quad+g(\varphi Y,W)\eta(X)\eta(Z)-g(\varphi Y,Z)\eta(X)\eta(W),\end{aligned} \tag{1.8}$$

and

$$\begin{aligned} R(X,Y,\varphi Z,\varphi W) = R(X,Y,Z,W)+ \\ + (X\alpha)A(Z,W,\varphi Y) - (Y\alpha)A(Z,W,\varphi X) \\ - (X\beta)A(Z,W,Y) + (Y\beta)A(Z,W,X) \\ + (\alpha^2-\beta^2)\big\{g(Y,W)g(X,Z) - g(X,W)g(Y,Z) \\ - g(\varphi X,Z)g(\varphi Y,W) + g(\varphi X,W)g(\varphi Y,Z)\big\} \\ + 2\alpha\beta\big\{g(\varphi Y,W)g(X,Z) - g(\varphi X,W)g(Y,Z) \\ - g(\varphi Y,Z)g(X,W) + g(\varphi X,Z)g(Y,W)\big\} \end{aligned} \tag{1.9}$$

for any $X,Y,Z,W \in TM^{2n+1}$.

Moreover, we have from the above equation

$$\begin{aligned} R(\varphi X,\varphi Y,\varphi Z,\varphi W) = R(X,Y,Z,W) \\ + (Z\alpha)A(X,Y,\varphi W) - (W\alpha)A(X,Y,\varphi Z) \\ - (Z\beta)A(X,Y,W) + (W\beta)A(X,Y,Z) \\ - (\varphi X\alpha)A(Z,W,Y) + (\varphi Y\alpha)A(Z,W,X) \\ - (\varphi X\beta)A(Z,W,\varphi Y) + (\varphi Y\beta)A(Z,W,\varphi X) \\ + (\alpha^2-\beta^2)\{A(X,Y,W)\eta(Z) - A(X,Y,Z)\eta(W)\} \\ + 2\alpha\beta\Big[2\big\{g(X,W)g(\varphi Y,Z) - g(X,Z)g(\varphi Y,W) \\ + g(Y,W)g(\varphi X,Z) - g(Y,Z)g(\varphi X,W)\big\} \\ - A(X,Y,\varphi W)\eta(Z) + A(X,Y,\varphi Z)\eta(W)\Big] \end{aligned} \tag{1.10}$$

for any $X,Y,Z,W \in TM^{2n+1}$.

By virtue of $R(X,Y,Z,W) = R(Z,W,X,Y)$ for any $X,Y,Z,W \in TM^{2n+1}$, we have from (1.10)

$$\begin{aligned} \{(X\alpha) + (\varphi X\beta)\}A(Z,W,\varphi Y) - \{(Y\alpha) + (\varphi Y\beta)\}A(Z,W,\varphi X) \\ - \{(Z\alpha) + (\varphi Z\beta)\}A(X,Y,\varphi W) + \{(W\alpha) + (\varphi W\beta)\}A(X,Y,\varphi Z) \\ - \{(X\beta) - (\varphi X\alpha)\}A(Z,W,Y) + \{(Y\beta) - (\varphi Y\alpha)\}A(Z,W,X) \\ + \{(Z\beta) - (\varphi Z\alpha)\}A(X,Y,W) - \{(W\beta) - (\varphi W\alpha)\}A(X,Y,Z) \\ = 4\alpha\beta\Big[2\big\{g(X,W)g(\varphi Y,Z) - g(X,Z)g(\varphi Y,W) \\ + g(Y,Z)g(\varphi X,W) - g(Y,W)g(\varphi X,Z)\big\} \\ + A(Z,W,\varphi Y)\eta(X) - A(Z,W,\varphi X)\eta(Y)\Big] \end{aligned} \tag{1.11}$$

for any $X,Y,Z,W \in TM^{2n+1}$. Thus we have

Proposition 1.2. *In an (α,β)-type trans-Sasakian manifold M^{2n+1}, the functions α and β satisfy* (1.11).

2. $(CHR)_3$-curvature tensor in a trans-Sasakian manifold

In this section, we consider the $(CHR)_3$-curvature tensor in a trans-Sasakian manifold.

The $(CHR)_3$-curvature tensor in an almost contact Riemannian manifold is defined by

$$\begin{aligned}
16(CHR)_3(X,Y,Z,W) = & \\
= 3\{&R(X,Y,Z,W)+R(\varphi X,\varphi Y,Z,W) \\
&+R(X,Y,\varphi Z,\varphi W)+R(\varphi X,\varphi Y,\varphi Z,\varphi W)\} \\
-&R(X,Z,\varphi W,\varphi Y)-R(\varphi X,\varphi Z,W,Y) \\
&-R(X,W,\varphi Y,\varphi Z)-R(\varphi X,\varphi W,Y,Z) \\
&+R(\varphi X,Z,\varphi W,Y)+R(X,\varphi Z,W,\varphi Y) \\
&+R(\varphi X,W,Y,\varphi Z)+R(X,\varphi W,\varphi Y,Z) \\
+&\eta(X)P(Z,W,Y)-\eta(Y)P(Z,W,X) \\
&+\eta(Z)P(X,Y,W)-\eta(W)P(X,Y,Z) \\
&+\eta(X)\eta(W)Q(Y,Z)-\eta(X)\eta(Z)Q(Y,W) \\
&+\eta(Y)\eta(Z)Q(W,X)-\eta(Y)\eta(W)Q(Z,X),
\end{aligned} \tag{2.1}$$

where we put

$$\begin{aligned}
P(X,Y,Z)=3\{R(X,Y,Z,\xi)+R(\varphi X,\varphi Y,Z,\xi)\}+R(\varphi X,\varphi Z,Y,\xi) \\
+R(\varphi Z,\varphi Y,X,\xi)-R(X,\varphi Z,\varphi Y,\xi)-R(\varphi Z,Y,\varphi X,\xi)
\end{aligned} \tag{2.2}$$

and

$$Q(X,Y)=3R(\xi,X,Y,\xi)-R(\xi,\varphi X,\varphi Y,\xi). \tag{2.3}$$

for any $X,Y,Z\in TM^{2n+1}$, [3]. We call this tensor field a *contact holomorphic Riemannian curvature tensor* or briefly $(CHR)_3$*-curvature tensor* in an almost contact Riemannian manifold. Hereafter, we assume that all vector fields are elements of TM^{2n+1}.

Now, to calculate $(CHR)_3$-curvature tensor in a trans-Sasakian manifold M^{2n+1}, we separate this tensor field as the following 5-parts:

$$\text{(I)}\qquad R(X,Y,Z,W)+R(\varphi X,\varphi Y,Z,W)+ \\ +R(X,Y,\varphi Z,\varphi W)+R(\varphi X,\varphi Y,\varphi Z,\varphi W),$$

$$\text{(II)}\qquad R(X,Z,\varphi W,\varphi Y)+R(\varphi X,\varphi Z,W,Y)+ \\ +R(X,W,\varphi Y,\varphi Z)+R(\varphi X,\varphi W,Y,Z),$$

$$\text{(III)} \qquad R(\varphi X, Z, \varphi W, Y) + R(X, \varphi Z, W, \varphi Y) + \\ + R(\varphi X, W, Y, \varphi Z) + R(X, \varphi W, \varphi Y, Z),$$

$$\text{(IV)} \qquad \eta(X)P(Z, W, Y) - \eta(Y)P(Z, W, X) + \\ + \eta(Z)P(X, Y, W) - \eta(W)P(X, Y, Z),$$

$$\text{(V)} \qquad \eta(X)\eta(W)Q(Y, Z) - \eta(X)\eta(Z)Q(Y, W) + \\ + \eta(Y)\eta(Z)Q(W, X) - \eta(Y)\eta(W)Q(Z, X).$$

Then we know that

$$16(CHR)_3(X, Y, Z, W) = 3\text{(I)} - \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)}.$$

Using (1.9) and (1.10) we get that

$$\begin{aligned} \text{(I)} = {} & 4R(X, Y, Z, W) + \\ & + \{(X\alpha) - (\varphi X\beta)\}A(Z, W, \varphi Y) - \{(Y\alpha) - (\varphi Y\beta)\}A(Z, W, \varphi X) \\ & + 2(Z\alpha)A(X, Y, \varphi W) - 2(W\alpha)A(X, Y, \varphi Z) - \\ & - \{(X\beta) + (\varphi X\alpha)\}A(Z, W, Y) + \{(Y\beta) + (\varphi Y\alpha)\}A(Z, W, X) \\ & - 2(Z\beta)A(X, Y, W) + 2(W\beta)A(X, Y, Z) \\ & + (\alpha^2 - \beta^2)\Big[2\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + \\ & \qquad + g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W)\} \\ & \qquad\qquad + A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y)\Big] \\ & + 2\alpha\beta\Big[2\{g(Y, Z)g(\varphi X, W) + g(X, Z)g(\varphi Y, W) \\ & \qquad + g(X, W)g(\varphi Y, Z) - g(Y, W)g(\varphi X, Z)\} \\ & \qquad\qquad + A(X, Y, \varphi Z)\eta(W) - A(X, Y, \varphi W)\eta(Z)\Big]. \end{aligned} \tag{2.4}$$

Using (1.9), we obtain

$$
\begin{aligned}
-(\mathrm{II}) = 2R(X,Y,Z,W) &+ (X\alpha)\{A(Z,Y,\varphi W) - A(W,Y,\varphi Z)\} \\
&- (Y\alpha)\{A(Z,X,\varphi W) - A(W,X,\varphi Z)\} \\
&+ (Z\alpha)\{A(X,W,\varphi Y) - A(Y,W,\varphi X)\} \\
&- (W\alpha)\{A(X,Z,\varphi Y) - A(Y,Z,\varphi X)\} \\
&- (X\beta)\{A(Z,Y,W) - A(W,Y,Z)\} \\
&+ (Y\beta)\{A(Z,X,W) - A(W,X,Z)\} \\
&- (Z\beta)\{A(X,W,Y) - A(Y,W,X)\} \\
&+ (W\beta)\{A(X,Z,Y) - A(Y,Z,X)\} \\
&+ 2(\alpha^2 - \beta^2)\big\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\
&\quad - g(\varphi X,W)g(\varphi Y,Z) + g(\varphi X,Z)g(\varphi Y,W) \\
&\quad - 2g(\varphi X,Y)g(\varphi Z,W)\big\}.
\end{aligned}
\tag{2.5}
$$

(III) is separated as $(A)+(B)$, where we put

$$
\begin{aligned}
(A) &= R(\varphi X,Z,\varphi W,Y) + R(\varphi X,W,Y,\varphi Z) \\
&= -\{R(\varphi X,\varphi W,Y,Z) + R(\varphi X\varphi Z,W,Y)\} \\
&\quad - \{R(\varphi X,Y,Z,\varphi W) + R(\varphi X,Y,\varphi Z,W)\},
\end{aligned}
$$

$$
(B) = R(X,\varphi Z,W,\varphi Y) + R(X,\varphi W,\varphi Y,Z).
$$

By virtue of (1.9), we obtain

$$
\begin{aligned}
&-\{R(\varphi X,\varphi W,Y,Z) + R(\varphi X\varphi Z,W,Y)\} = R(Z,Y,Z,W)+ \\
&\quad + (Y\alpha)\{A(X,Z,\varphi W) - A(X,W,\varphi Z)\}+ \\
&\quad + (Z\alpha)A(X,W,\varphi Y) - (W\alpha)A(X,Z,\varphi Y)+ \\
&\quad + (Y\beta)\{A(X,W,Z) - A(X,Z,W)\}- \\
&\quad - (Z\beta)A(X,W,Y) + (W\beta)A(X,Z,Y) \\
&\quad + (\alpha^2-\beta^2)\Big\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\
&\qquad + g(\varphi X,Z)g(\varphi Y,W) - g(\varphi X,W)g(\varphi Y,Z) \\
&\qquad - 2g(\varphi X,Y)g(\varphi Z,W)\Big\} \\
&\quad + 2\alpha\beta\Big\{g(X,Z)g(\varphi Y,W) - g(Y,W)g(\varphi X,Z) \\
&\qquad + g(Y,Z)g(\varphi X,W) - g(X,W)g(\varphi Y,Z) \\
&\qquad - 2g(X,Y)g(\varphi Z,W)\Big\}.
\end{aligned}
\tag{2.6}
$$

Thus we have from (2.5) and (2.6)

$$
\begin{aligned}
(A) = R(X, Y, Z, W)+ \\
&+ (\varphi X\alpha)A(Z, W, Y) - 2(Y\alpha)g(\varphi Z, W)\eta(X) \\
&\quad + (Z\alpha)A(X, W, \varphi Y) - (W\alpha)A(X, Z, \varphi Y) \\
&+ (\varphi X\beta)A(Z, W, \varphi Y) + 2(Y\beta)A(Z, W, X) \\
&\quad - (Z\beta)A(X, W, Y) + (W\beta)A(X, Z, Y) \\
&+ (\alpha^2 - \beta^2)\Big[2\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} \\
&\qquad - A(Z, W, Y)\eta(X)\Big] \\
&+ 2\alpha\beta\Big[2\{g(X, Z)g(\varphi Y, W) - g(X, W)g(\varphi Y, Z) \\
&\qquad - g(X, Y)g(\varphi Z, W)\} - A(Z, W, \varphi Y)\eta(X)\Big].
\end{aligned}
$$

Since, (B) is the equation which change $X \Leftrightarrow Y$ and $Z \Leftrightarrow W$ in (A), we have

$$
\begin{aligned}
(B) = R(X, Y, Z, W) &+ (\varphi Y\alpha)A(W, Z, X) - 2(X\alpha)g(\varphi W, Z)\eta(Y) \\
&+ (W\alpha)A(Y, Z, \varphi X) - (Z\alpha)A(Y, W, \varphi X) \\
&\quad + (\varphi Y\beta)A(W, Z, \varphi X) + 2(X\beta)A(W, Z, Y) \\
&\qquad - (W\beta)A(Y, Z, X) + (Z\beta)A(Y, W, X) \\
&+ (\alpha^2 - \beta^2)\Big[2\big\{g(Y, W)g(X, Z) - g(Y, Z)g(X, W)\big\} - A(W, Z, X)\eta(Y)\Big] \\
&+ 2\alpha\beta\Big[2\big\{g(Y, W)g(\varphi X, Z) - g(Y, Z)g(\varphi X, W) \\
&\qquad - g(X, Y)g(\varphi W, Z)\big\} - A(W, Z, \varphi X)\eta(Y)\Big].
\end{aligned}
$$

By virtue of the above two equations, we obtain

$$
\begin{aligned}
(\mathrm{III}) = 2R(X, Y, Z, W) &+ (\varphi X\alpha)A(Z, W, Y) - (\varphi Y\alpha)A(W, Z, X) \\
&+ 2(X\alpha)g(\varphi Z, W)\eta(Y) - 2(Y\alpha)g(\varphi Z, W)\eta(X) \\
&\quad - (Z\alpha)\{A(X, Y, \varphi W) - 2g(\varphi X, Y)\eta(W)\} \\
&\qquad + (W\alpha)\{A(X, Y, \varphi Z) - 2g(\varphi X, Y)\eta(Z)\} \\
&+ (\varphi X\beta)A(Z, W, \varphi Y) - (\varphi Y\beta)A(Z, W, \varphi X) \\
&\quad - 2(X\beta)A(Z, W, Y) + 2(Y\beta)A(Z, W, X) \\
&\qquad - (Z\beta)A(X, Y, W) + (W\beta)A(X, Y, Z)
\end{aligned}
$$

$$
\begin{aligned}
&+(\alpha^2-\beta^2)\Big[4\big\{g(X,Z)g(Y,W)-g(X,W)g(Y,Z)-g(\varphi X,Y)g(\varphi Z,W)\big\}\\
&\qquad\qquad -A(Z,W,Y)\eta(X)+A(Z,W,X)\eta(Y)\Big]\\
&+2\alpha\beta\Big[2\big\{g(X,Z)g(\varphi Y,W)-g(X,W)g(\varphi Y,Z)\\
&\qquad +g(Y,W)g(\varphi X,Z)-g(Y,Z)g(\varphi X,W)\big\}\\
&\qquad\qquad -A(Z,W,\varphi X)\eta(Y)+A(Z,W,\varphi X)\eta(Y)\Big].
\end{aligned}
$$

Next, to calculate (IV) in a trans-Sasakian manifold, we have to get $P(X,Y,Z)$ which defined by (2.2) in a trans-Sasakian manifold. By virtue of (1.5) we obtain that

$$
\begin{aligned}
P(X,Y,Z)=&\,4\{(X\alpha)g(\varphi Y,Z)-(Y\alpha)g(\varphi X,Z)\}\\
&-2\{(X\beta)g(\varphi Y,\varphi Z)-(Y\beta)g(\varphi X,\varphi Z)\}\\
&-4\{(\varphi X\alpha)g(\varphi Y,\varphi Z)-(\varphi Y\alpha)g(\varphi X,\varphi Z)\}\\
&-2\{(\varphi X\beta)g(\varphi Y,Z)-(\varphi Y\beta)g(\varphi X,Z)\}\\
&+4(\varphi Z\beta)g(\varphi X,Y)+2(\alpha^2-\beta^2)A(X,Y,Z)-8\alpha\beta A(X,Y,\varphi Z).
\end{aligned}
$$

Using the above equation, we get

$$
\begin{aligned}
(\mathrm{IV})=&\,2\big\{2(X\alpha)-(\varphi X\beta)\big\}A(Z,W,\varphi Y)-2\big\{2(Y\alpha)-(\varphi Y\beta)\big\}A(Z,W,\varphi X)\\
&+2\big\{2(Z\alpha)-(\varphi Z\beta)\big\}A(X,Y,\varphi W)-2\big\{2(W\alpha)-(\varphi W\beta)\big\}A(X,Y,\varphi Z)\\
&-2\big\{2(\varphi X\alpha)+(X\beta)\big\}A(Z,W,Y)+2\big\{2(\varphi Y\alpha)+(Y\beta)\big\}A(Z,W,X)\\
&-2\big\{2(\varphi Z\alpha)+(Z\beta)\big\}A(X,Y,W)+2\big\{2(\varphi W\alpha)+(W\beta)\big\}A(X,Y,Z)\\
&+4\Big\{(\varphi Y\beta)g(\varphi Z,W)\eta(X)-(\varphi X\beta)g(\varphi Z,W)\eta(Y)\\
&\qquad +(\varphi W\beta)g(\varphi X,Y)\eta(Z)-(\varphi Z\beta)g(\varphi X,Y)\eta(W)\Big\}\\
&+4(\alpha^2-\beta^2)\big\{A(Z,W,Y)\eta(X)-A(Z,W,X)\eta(Y)\big\}.
\end{aligned}
$$

Finally, we calculate (V) in a trans-Sasakian manifold.

By virtue of $(1.5)_3$, we have

$$
\begin{aligned}
R(\xi,X,Y,\xi)=&\,\{(\xi\alpha)+2\alpha\beta\}g(\varphi X,Y)+\\
&+\{(\alpha^2-\beta^2)-(\xi\beta)\}g(\varphi X,\varphi Y).
\end{aligned}
\tag{2.7}
$$

In (2.7), the left hand side is symmetric with respect to X and Y. So we have

Proposition 2.1. *In a trans-Sasakian manifold, the condition*

$$(\xi\alpha) + 2\alpha\beta = 0$$

holds.

Thus, (2.7) is written as

$$R(\xi, X, Y, \xi) = \{(\alpha^2 - \beta^2) - (\xi\beta)\}\, g(\varphi X, \varphi Y). \tag{2.8}$$

By virtue of (2.8), we can easily obtain that

$$R(\xi, X, Y, \xi) = R(\xi, \varphi X, \varphi Y, \xi).$$

Thus we have from the above equation

$$Q(X, Y) = 2\{(\alpha^2 - \beta^2) - (\xi\beta)\}\, g(\varphi X, \varphi Y). \tag{2.9}$$

Thus we have from (2.9)

$$(\mathrm{V}) = -2\{(\alpha^2 - \beta^2) - (\xi\beta)\}\, \{A(Z, W, Y)\eta(X) - A(Z, W, X)\eta Y)\}.$$

By virtue of (1.11), (I), (II), (III), (IV) and (V), the $(CHR)_3$-curvature tensor in a trans-Sasakian manifold is written as follows:

$$\begin{aligned}
16(CHR)_3(X, Y, Z, W) &= 16R(X, Y, Z, W) \\
&+ (X\alpha)\{7A(Z, W, \varphi Y) + 4g(\varphi Z, W)\eta(Y)\} \\
&- (Y\alpha)\{7A(Z, W, \varphi X) + 4g(\varphi Z, W)\eta(X)\} \\
&+ (Z\alpha)\{7A(X, Y, \varphi W) + 4g(\varphi X, Y)\eta(W)\} \\
&- (W\alpha)\{7A(X, Y, \varphi Z) + 4g(\varphi X, Y)\eta(Z)\} \\
&- 5\big\{(\varphi X\alpha)A(Z, W, Y) - (\varphi Y\alpha)A(Z, W, X) \\
&\quad + (\varphi Z\alpha)A(X, Y, W) - (\varphi W\alpha)A(X, Y, Z)\big\} \\
&- 9\big\{(X\beta)A(Z, W, Y) - (Y\beta)A(Z, W, X) \\
&\quad + (Z\beta)A(X, Y, W) - (W\beta)A(X, Y, Z)\big\} \\
&- (\varphi X\beta)\{3A(Z, W, \varphi Y) + 4g(\varphi Z, W)\eta(Y)\} \\
&+ (\varphi Y\beta)\{3A(Z, W, \varphi X) + 4g(\varphi Z, W)\eta(X)\} \\
&- (\varphi Z\beta)\{3A(X, Y, \varphi W) + 4g(\varphi X, Y)\eta(W)\} \\
&+ (\varphi W\beta)\{3A(X, Y, \varphi Z) + 4g(\varphi X, Y)\eta(Z)\} \\
&+ (\alpha^2 - \beta^2)\Big[12\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} \\
&\quad + 4\{g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) \\
&\quad - 2g(\varphi X, Y)g(\varphi Z, W)\}\Big] \\
&+ 2(\xi\beta)\{A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y)\}.
\end{aligned} \tag{2.10}$$

From the above equation, we can easily obtain the $(CHR)_3$-Ricci tensor ρ_3 and the $(CHR)_3$-scalar curvature τ_3 as

$$\begin{aligned} 8\rho_3(X,Y) &= 8\rho(X,Y) + (5n+3)\big[\{(\varphi X)\alpha\}\eta(Y) + \{(\varphi Y)\alpha\}\eta(X)\big] \\ &+ (9n-1)\{(X\beta)\eta(Y) + (Y\beta)\eta(X)\} \\ &- 4(\alpha^2-\beta^2)\Big\{(3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y)\Big\} \\ &+ 2(\xi\beta)\{4g(X,Y) - (n+3)\eta(X)\eta(Y)\}. \end{aligned} \tag{2.11}$$

and

$$\tau_3 = \tau - (3n+1)n(\alpha^2-\beta^2) + 4n(\xi\beta), \tag{2.12}$$

where τ denotes the scalar curvature with respect to g.

By virtue of $(1.5)_4$ and (2.11), we easily have

$$8\rho_3(X,\xi) = 5(n-1)\Big\{(\varphi X)\alpha) - 7(n-1)\big\{(X\beta) - (\xi\beta)\eta(X)\big\}\Big\}. \tag{2.13}$$

3. $(CHR)_3$-flat trans-Sasakian Manifolds

An almost contact Riemannian manifold is called $(CHR)_3$-*flat* if the $(CHR)_3$-curvature tensor equals to zero on M^{2n+1}.

Let us consider a $(CHR)_3$-flat trans-Sasakian manifold. Then the left hand side of (2.10) is zero.

Moreover, if the $(CHR)_3$-curvature tensor is flat, then the $(CHR)_3$-Ricci tensor and the $(CHR)_3$-scalar are flat. So, by virtue of (2.11) and (2.12), we respectively have

$$\begin{aligned} 8\rho(X,Y) &+ (5n+3)\Big[\{(\varphi X)\alpha\}\eta(Y) + \{(\varphi Y)\alpha\}\eta(X)\Big] + \\ &+ (9n-1)\{(X\beta)\eta(Y) + (Y\beta)\eta(X)\} - \\ &- 4(\alpha^2-\beta^2)\{(3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y)\} + \\ &+ 2(\xi\beta)\{4g(X,Y) - (n+3)\eta(X)\eta(Y)\} = 0 \end{aligned} \tag{3.1}$$

and

$$\tau - (3n+1)n(\alpha^2-\beta^2) + 4n(\xi\beta) = 0. \tag{3.2}$$

We know from (2.13)

$$5\{(\varphi X)\alpha\} - 7\{(X\beta) - (\xi\beta)\eta(X)\} = 0. \tag{3.3}$$

From the above equation, we get

$$\begin{aligned} &\{(\varphi X)\alpha\}\eta(Y) + \{(\varphi Y)\alpha\}\eta(X) = \\ &\quad + \frac{7}{5}\{(X\beta)\eta(Y) + (Y\beta)\eta(X)\} - \frac{14}{5}(\xi\beta)\eta(X)\eta(Y). \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.1), we get

$$\begin{aligned}\rho(X,Y) = {} & \frac{(3n-1)(\alpha^2-\beta^2)-2(\xi\beta)}{2}g(X,Y) \\ & + \frac{1}{2}\Big\{\frac{2(10n+9)}{5}(\xi\beta)+(n+1)(\alpha^2-\beta^2)\Big\}\eta(X)\eta(Y) \qquad (3.5) \\ & - \frac{2(5n+1)}{5}\Big\{d\beta(X)\eta(X)+d\beta(Y)\eta(X)\Big\}.\end{aligned}$$

Thus we have

Theorem 3.1. *A $(CHR)_3$-flat trans-Sasakian manifold is a generalized η-Einstein manifold.*

Remark 3.2. The notion of a generalized η-Einstein manifold is defined by A. A. Shaikh and Y. Matsuyama, [5]. Moreover, M. C. Chaki called this manifold a *generalized quasi-Einstein manifold*, [1], [2].

From the above theorem, we can easily obtain

Corollary 3.3. *A $(CHR)_3$-flat trans Sasakian manifold is η-Einstein if and only if the function β is constant. Then the Ricci tensor ρ and the scalar curvature τ with respect to g are written as*

$$\rho(X,Y) = (\alpha^2-\beta^2)\big\{\tfrac{3n-1}{2}g(X,Y)+\tfrac{n+1}{2}\eta(X)\eta(Y)\big\} \qquad (3.6)$$

and

$$\tau = -(3n+1)n(\alpha^2-\beta^2).$$

By virtue of Remark 1.1 and the above corollary, we get

Corollary 3.4. *In a $(CHR)_3$-flat Sasakian, resp. Kenmotsu, manifold, the Ricci tensor ρ and the scalar curvature τ with respect to g satisfy*

$$\rho(X,Y) = \tfrac{3n-1}{2}g(X,Y)+\tfrac{n+1}{2}\eta(X)\eta(Y),$$

resp.

$$\rho(X,Y) = -\big\{\tfrac{3n-1}{2}g(X,Y)+\tfrac{n+1}{2}\eta(X)\eta(Y)\big\}$$

and

$$\tau = -(3n+1)n \qquad (\textit{resp.} \quad \tau=(3n+1)n). \qquad (3.7)$$

Now, from (3.3), we obtain

$$\begin{aligned} & -5\{(\varphi X)\alpha\}A(Z,W,Y)+5\{(\varphi Y)\alpha\}A(Z,W,X)- \\ & \quad -5\{(\varphi Z)\alpha\}A(X,Y,W)+5\{(\varphi W)\alpha\}A(X,Y,Z)- \\ & -9\Big\{(X\beta)A(Z,W,Y)-(Y\beta)A(Z,W,X)+\end{aligned}$$

$$
\begin{aligned}
&+ (Z\beta)A(X,Y,W) - (W\beta)A(X,Y,Z)\Big\} = \\
&= -16\Big\{(X\beta)A(Z,W,Y) - (Y\beta)A(Z,W,X) + \\
&+ (Z\beta)A(X,Y,W) - (W\beta)A(X,Y,Z)\Big\} + \\
&+ 14(\xi\beta)\{A(Z,W,Y)\eta(X) - A(Z,W,X)\eta(Y)\}.
\end{aligned}
$$

From (3.3), we get

$$
\{(\varphi X)\beta\} = -\frac{5}{7}(X\alpha) + \frac{5}{7}(\xi\alpha)\eta(X). \tag{3.8}
$$

From this, we have

$$
\begin{aligned}
&-3\Big[\{(\varphi X)\beta\}A(Z,W,\varphi Y) - \{(\varphi Y)\beta\}A(Z,W,\varphi X) + \\
&+ \{(\varphi Z)\beta\}A(X,Y,\varphi W) - \{(\varphi W)\beta\}A(X,Y,\varphi Z)\Big] + \\
&+ 7\Big\{(X\alpha)A(Z,W,\varphi Y) - (Y\alpha)A(Z,W,\varphi X) + \\
&+ (Z\alpha)A(X,Y,\varphi W) - (W\alpha)A(X,Y,\varphi Z)\Big\} = \\
&= \frac{64}{7}\Big\{(X\alpha)A(Z,W,\varphi Y) - (Y\alpha)A(Z,W,\varphi X) + \\
&+ (Z\alpha)A(X,Y,\varphi W) - (W\alpha)A(X,Y,\varphi Z)\Big\}.
\end{aligned}
$$

Next, since we have

$$
4[(X\alpha) - \{(\varphi X)\beta\}]g(\varphi Z,W)\eta(Y) = \tfrac{48}{7}(X\alpha) - \tfrac{20}{7}(\xi\alpha)g(\varphi Z,W)\eta(X)\eta(Y),
$$

we obtain

$$
\begin{aligned}
&4\big[(X\alpha) - \{(\varphi X)\beta\}\big]g(\varphi Z,W)\eta(Y) - \\
&- 4\big[(Y\alpha) - \{(\varphi Y)\beta\}\big]g(\varphi Z,W)\eta(X) + \\
&+ 4\big[(Z\alpha) - \{(\varphi Z)\beta\}\big]g(\varphi X,Y)\eta(W) - \\
&- 4\big[(W\alpha) - \{(\varphi W)\beta\}\big]g(\varphi X,Y)\eta(Z) = \\
&= \frac{48}{7}\Big\{(X\alpha)g(\varphi Z,W)\eta(Y) - (Y\alpha)g(\varphi Z,W)\eta(X) + \\
&+ (Z\alpha)g(\varphi X,Y)\eta(W) - (W\alpha)g(\varphi X,Y)\eta(Z)\Big\}.
\end{aligned}
$$

Using (3.6), (3.7) and (3.8), the curvature tensor R with respect to g is written as

$$\begin{aligned} R(X,Y,Z,W) = \frac{1}{7}\Big[&(X\alpha)\{4A(W,Z,\varphi Y) + 3g(\varphi W,Z), \eta(Y)\} \\ &- (Y\alpha)\{4A(W,Z,\varphi X) + 3g(\varphi W,Z)\eta(X)\} \\ &+ (Z\alpha)\{4A(Y,X,\varphi W) + 3g(\varphi Y,X)\eta(W)\} \\ &- (W\alpha)\{4A(Y,X,\varphi Z) + 3g(\varphi Y,X)\eta(Z)\}\Big] \\ &+ (X\beta)A(W,Z,Y) - (Y\beta)A(W,Z,X) \\ &- (Z\beta)A(Y,X,W) + (W\beta)A(Y,X,Z) \\ &+ \frac{1}{4}(\alpha^2-\beta^2)\Big[3\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &- g(\varphi X,W)g(\varphi Y,Z) + g(\varphi X,Z)g(\varphi Y,W) \\ &+ 2g(\varphi X,Y)g(\varphi Z,W)\Big] \\ &- \frac{1}{4}\{(\alpha^2-\beta^2) - (\xi\beta)\}\{A(Z,W,Y)\eta(X) - A(Z,W,X)\eta(Y)\}. \end{aligned} \tag{3.9}$$

Thus we get

Theorem 3.5. *If a trans-Sasakian manifold is $(CHR)_3$-flat, the the curvature tensor satisfies* (3.9).

By virtue of Remark 1.1 and the above theorem, we have

Corollary 3.6. *In a $(CHR)_3$-flat Sasakian, resp. Kenmotsu, manifold, the curvature tensor R with respect to g are written by*

$$\begin{aligned} R(X,Y,Z,W) = \frac{1}{4}\Big[&3\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &- g(\varphi X,W)g(\varphi Y,Z) + g(\varphi X,Z)g(\varphi Y,W) + 2g(\varphi X,Y)g(\varphi Z,W)\Big] \\ &- \frac{1}{4}\{A(Z,W,Y)\eta(X) - A(Z,W,X)\eta(Y)\} \end{aligned}$$

resp.

$$\begin{aligned} R(X,Y,Z,W) = -\frac{1}{4}[&3\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &- g(\varphi X,W)g(\varphi Y,Z) + g(\varphi X,Z)g(\varphi Y,W) + 2g(\varphi X,Y)g(\varphi Z,W)\Big] \\ &+ \frac{1}{4}\{A(Z,W,Y)\eta(X) - A(Z,W,X)\eta(Y)\}. \end{aligned}$$

Remark 3.7. The above corollary shows that a $(CHR)_3$-flat Sasakian (resp. Kenmotsu) manifold is a Sasakian (resp. Kenmotsu) space form with zero holomorphic sectional curvature.

Remark 3.8. Of course, we can get (3.2) and (3.5) from Theorem 3.1 directly.

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