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THE GEOMETRIC MODEL OF MAXIMAL SEMIPARALLEL SPACE-LIKE 3-DIMENSIONAL SUBMANIFOLDS IN PSEUDO-EUCLIDEAN SPACE

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Parallel submanifolds in pseudo-Euclidean spaces are characterized locally by the system $\bar{\nabla} h = 0$. Submanifolds satisfying the integrability condition $\bar{R} \circ h = 0$ of this system are called semiparallel; geometrically they are 2nd order envelopes of the parallel submani-folds. It is shown that in E_s^n with $s > 0$ do exist not totally geodesic maximal semiparallel space-like 3-submanifolds. The existence of such 3-dimensional Riemannian submanifolds and geometry of the corresponding parallel submanifolds are investigated.

1. A submanifold M^m in $N_s^n(c)$ is called *semiparallel* if $\bar{R}(X;Y)h = 0$ (this is the integrability condition of the system $\bar{\nabla} h = 0$ which characterizes a parallel submanifold). Here $\bar{R}$ is the curvature operator of the van der Waerden-Bortolotti connection $\bar{\nabla}$ ($\bar{\nabla} = \nabla \oplus \nabla^{\perp}$) and h is the second fundamental form. The space-like submanifold M^m in E_s^n is said to be *maximal submanifold*, if the mean curvature vector H is identically zero. We are trying to describe a geometric model of maximal semiparallel space-like 3-dimensional submanifolds in E_s^n, bearing in mind that every semiparallel surface is a 2nd order envelope of the parallel ones (according to the result [1]).

2. Semiparallel submanifolds in Euclidean space have been studied by several authors (e.g.[2]-[5]). In pseudo-Euclidean space it is done for the time-like surfaces in [6] and for the space-like surfaces in [7].

Let $\{x;\ e_I\}$, $(I = 1, 2, \dots, n)$ be the moving frame in E_s^n, i.e. a free element of the frame bundle in E_s^n. At a point $x \in M^m$ the *tangent vector space* $T_x M^m$ is a vector subspace of $T_x[E_s^n]$ and has an orthogonal compliment $T_x^{\perp} M^m$ in the letter, which is a (*n*-*m*)-dimensional vector space, called the *normal vector space* of the submanifolds M^m at *x*. The linear span of all *h*(*X;X*) in a given point $x \in M^m$ is called the *principal normal subspace* $N_x M^m$ of the submanifold M^m at *x*, and its dimension is denoted by n_1.

The moving frame is said to be adapted to a space-like submanifold M^m in E_s^n, if to take, $e_i \in T_x M^m$, $e_\alpha \in T_x^{\perp} M^m$, where $i, j = 1, \dots, m$; $\alpha, \beta = m + 1, \dots, n$. The farme vectors belonging to the normal space $T_x^{\perp} M^m = N_x M^m \oplus N_x^{\perp} M^m$ can be taken so that $e_a \in N_x M^m$; $e_\xi \in N_x^{\perp} M^m$, where $a = m + 1, \dots, m + n_1$; $e_\xi = m + n_1 + 1, \dots, n$. Denoting scalar composition of the frame vectors e_I and e_J, as usually, $\langle e_I, e_J \rangle = g_{IJ}$, one has $g_{i\alpha} = 0$ and it can be taken $g_{ij} = \delta_{ij}$; moreover let denote $\langle e_\alpha, e_\alpha \rangle = \varepsilon_\alpha$ and $\langle e_\alpha, e_\beta \rangle = g_{\alpha\beta}$, $\alpha \neq \beta$. In the well-known formulae

$$dx = e_I \omega^I, \qquad de_I = e_J \omega_I^J, \qquad d\omega^I = \omega^J \wedge \omega_J^I, \qquad d\omega_J^I = \omega_K^I \wedge \omega_J^K$$

(where the point x is identified with its radius-vector) there hold $\omega^\alpha = 0$ and

$$\omega_i^j = -\omega_j^i,\ \ g_{\alpha\beta}\omega_i^\beta + \omega_\alpha^i = 0,\ \ dg_{\alpha\beta} = g_{\gamma\beta}\omega_\alpha^\gamma + g_{\alpha\gamma}\omega_\beta^\gamma. \quad (1)$$

The equations $\omega^\alpha = 0$ lead to $\omega_i^\alpha = h_{ij}^\alpha\omega^j$, $h_{ij}^\alpha = h_{ji}^\alpha$. Let h_{ijk}^α denote the covariant derivative of h_{ij}^α defined by

$$\bar\nabla h_{ij}^\alpha \left(\equiv dh_{ij}^\alpha - h_{kj}^\alpha\omega_i^k - h_{ik}^\alpha\omega_j^k + h_{ij}^\beta\omega_\beta^\alpha\right) = h_{ijk}^\alpha\omega^k,\ \ h_{ijk}^\alpha = h_{ikj}^\alpha. \quad (2)$$

The relationship

$$\bar\nabla h_{ijk}^\alpha \wedge \omega^k = \bar\Omega \circ h_{ij}^\alpha, \quad (3)$$

where

$$\bar\Omega \circ h_{ij}^\alpha = -h_{kj}^\alpha\Omega_i^k - h_{ik}^\alpha\Omega_j^k + h_{ij}^\beta\Omega_\beta^\alpha, \quad (4)$$

can be obtained from the previous by exterior differentiation. In formulae (4)

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = -g_{\alpha\beta}\omega_i^\alpha \wedge \omega_j^\beta, \quad (5)$$

$$\Omega_\beta^\alpha = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha = -\textstyle\sum_i g_{\alpha\gamma}\omega_i^\gamma \wedge \omega_i^\beta \quad (6)$$

are the curvature 2-forms of the Levi-Civita connection ∇ and the normal connection $\nabla^\perp$, respectively. Together they represent the curvature 2-forms of the van der Waerden-Bortolotti connection $\bar\nabla$. Remark here, that $\Omega_i^j = -\Omega_j^i$ and that exterior differentiation leads from (1) to the following relationships $g_{\gamma\beta}\Omega_\alpha^\beta + g_{\alpha\gamma}\Omega_\beta^\gamma = 0$.

Due to (2) and (3) the parallelity and semiparallelity conditions are, respectively (see [8]),

$$dh_{ij}^\alpha - h_{kj}^\alpha\omega_i^k - h_{ik}^\alpha\omega_j^k + h_{ij}^\beta\omega_\beta^\alpha = 0, \quad (7)$$

$$h_{kj}^\alpha\Omega_i^k + h_{ik}^\alpha\Omega_j^k - h_{ij}^\beta\Omega_\beta^\alpha = 0. \quad (8)$$

If we denote $H_{ij,kl} = g_{\alpha\beta}h_{ij}^\alpha h_{kl}^\beta$ and $h_{ij} = h_{ij}^\alpha e_\alpha$, then (8) is equivalent to

$$\textstyle\sum_k\left(h_{kj}H_{i[p;q]k} + h_{ik}H_{j[p;q]k} - H_{ij,k[p}h_{q]k}\right) = 0. \quad (9)$$

In fact, according to the theory of minimal submanifolds in E^n, it is known that every minimal semiparallel submanifold is totally geodesic (see [2] and [8]). Hence the class of all such submanifolds are very small. There exist minimal semiparallel time-like surfaces (strings) in E_1^n (see [6]) and maximal semiparallel space-like surfaces in E_s^n (see [7]), which are not totally geodesic.

3. The conditions $H = 0$ and (8) give together that maximal semiparallel space-like M^3 in E_s^n occurs if $\dim N_x M^3 = 5$.

Existance. Due to dimension of the principal normal subspace one has at the point $x \in M^3$ a linear dependence between vectors h_{ij}, and there exist six coefficients ϑ^{ij} so that $h_{ij}\vartheta^{ij} = 0$, $\sum\left(\vartheta^{ij}\right)^2 \neq 0$. Here h_{ij} are components of a vector valued symmetric tensor field, hence ϑ^{ij} are components of a symmetric tensorfield to a multiplier. Now the vectors e_1, e_2, e_3 can be taken at the point x that this dependency transforms into $h_{11}\vartheta^{11} + h_{22}\vartheta^{22} + h_{33}\vartheta^{33} = 0$, which together with maximal supposition leads (after renumbering, if needed) to

$$h_{33} = -h_{11} - h_{22}. \qquad (10)$$

The five vectors $h_{11}, h_{22}, h_{12}, h_{13}$ and h_{23} must be here linearly independent and can be taken so that: $h_{11} = e_4,\ h_{22} = e_5,\ h_{12} = e_6,\ h_{13} = e_7,\ h_{23} = e_8$.

Thus the semiparallelity condition (9) gives that all $H_{ij,kl}$ are zero, i.e. the metric of the principal normal subspace $N_x M^3$ vanishes completely and the frame vectors can be taken so that

$$\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 0, \qquad \varepsilon_9 = \varepsilon_{10} = \varepsilon_{11} = \varepsilon_{12} = \varepsilon_{13} = 0,$$

$$g_{4,9} = g_{5,10} = g_{6,11} = g_{7,12} = g_{8,13} = 0.$$

It can be obtained by the appropriate choice of remaining frame vectors so that all ohter $g_{\alpha\beta} = 0$ are zero. Now the Pfaff system is

$$\begin{array}{lllll}
\omega_1^4 = \omega^1, & \omega_1^5 = 0, & \omega_1^6 = \omega^2, & \omega_1^7 = \omega^3, & \omega_1^8 = 0, \\
\omega_2^4 = 0, & \omega_2^5 = \omega^2, & \omega_2^6 = \omega^1, & \omega_2^7 = 0, & \omega_2^8 = \omega^3, \\
\omega_3^4 = -\omega^3, & \omega_3^5 = -\omega^3, & \omega_3^6 = 0, & \omega_3^7 = \omega^1, & \omega_3^8 = \omega^2,
\end{array}$$

$\omega_i^{\bar a} = \omega_i^{\xi} = 0$, where $a = \{4, \ldots, 8\}$, $\bar a = \{9, \ldots, 13\}$, $\xi = \{14, \ldots, n\}$. The exterior

differentiation of $\omega_i^{\bar a} = 0$, using the Cartans lemma and relations (1) leads to $\omega_a^{\bar b} = 0$. The others relations in the Pfaff system by exterior differentiation gives

$$\omega_4^4 \wedge \omega^1 + (\omega_1^2 + \omega_6^4) \wedge \omega^2 + (2\omega_1^3 + \omega_7^4) \wedge \omega^3 = 0,$$
$$(\omega_1^2 + \omega_6^4) \wedge \omega^1 + \omega_5^4 \wedge \omega^2 + (\omega_2^3 + \omega_8^4) \wedge \omega^3 = 0,$$
$$(2\omega_1^3 + \omega_7^4) \wedge \omega^1 + (\omega_2^3 + \omega_8^4) \wedge \omega^2 - (\omega_4^4 + \omega_5^4) \wedge \omega^3 = 0,$$
$$\omega_4^5 \wedge \omega^1 + \left(\omega_6^5 - \omega_1^2\right) \wedge \omega^2 + (\omega_1^3 + \omega_7^5) \wedge \omega^3 = 0,$$
$$\left(\omega_6^5 - \omega_1^2\right) \wedge \omega^1 + \omega_5^5 \wedge \omega^2 + \left(2\omega_2^3 + \omega_8^5\right) \wedge \omega^3 = 0,$$
$$(\omega_1^3 + \omega_7^5) \wedge \omega^1 + \left(2\omega_2^3 + \omega_8^5\right) \wedge \omega^2 - \left(\omega_4^5 + \omega_5^5\right) \wedge \omega^3 = 0,$$
$$(\omega_4^6 - 2\omega_1^2) \wedge \omega^1 + \omega_6^6 \wedge \omega^2 + (\omega_2^3 + \omega_7^6) \wedge \omega^3 = 0,$$
$$\omega_6^6 \wedge \omega^1 + (2\omega_1^2 + \omega_5^6) \wedge \omega^2 + (\omega_1^3 + \omega_8^6) \wedge \omega^3 = 0,$$
$$(\omega_2^3 + \omega_7^6) \wedge \omega^1 + (\omega_1^3 + \omega_8^6) \wedge \omega^2 - (\omega_4^6 + \omega_5^6) \wedge \omega^3 = 0,$$
$$(\omega_4^7 - 2\omega_1^3) \wedge \omega^1 + (\omega_6^7 - \omega_2^3) \wedge \omega^2 + \omega_7^7 \wedge \omega^3 = 0,$$
$$(\omega_6^7 - \omega_2^3) \wedge \omega^1 + \omega_5^7 \wedge \omega^2 + (\omega_1^2 + \omega_8^7) \wedge \omega^3 = 0,$$
$$\omega_7^7 \wedge \omega^1 + (\omega_1^2 + \omega_8^7) \wedge \omega^2 - (\omega_4^7 - 2\omega_1^3 + \omega_5^7) \wedge \omega^3 = 0,$$
$$\omega_4^8 \wedge \omega^1 + (\omega_6^8 - \omega_1^3) \wedge \omega^2 + (\omega_7^8 - \omega_1^2) \wedge \omega^3 = 0,$$
$$(\omega_6^8 - \omega_1^3) \wedge \omega^1 + (\omega_5^8 - 2\omega_2^3) \wedge \omega^2 + \omega_8^8 \wedge \omega^3 = 0,$$
$$(\omega_7^8 - \omega_1^2) \wedge \omega^1 + \omega_8^8 \wedge \omega^2 - (\omega_4^8 + \omega_5^8 - 2\omega_2^3) \wedge \omega^3 = 0,$$
$$\omega_4^{\xi} \wedge \omega^1 + \omega_6^{\xi} \wedge \omega^2 + \omega_7^{\xi} \wedge \omega^3 = 0,$$
$$\omega_6^{\xi} \wedge \omega^1 + \omega_5^{\xi} \wedge \omega^2 + \omega_8^{\xi} \wedge \omega^3 = 0,$$
$$\omega_7^{\xi} \wedge \omega^1 + \omega_8^{\xi} \wedge \omega^2 - \left(\omega_4^{\xi} + \omega_5^{\xi}\right) \wedge \omega^3 = 0.$$

Applying here the Cartan's lemma one has that the common number of independent coefficients on the right sides is $N = 35 + 7(n-8)$. On the other hand, first, the basis of the secondary forms consists of

$$\begin{array}{ccccc}
\omega_4^4, & \omega_5^4, & \omega_1^2+\omega_6^4, & 2\omega_1^3+\omega_7^4, & \omega_2^3+\omega_8^4, \\
\omega_4^5, & \omega_5^5, & \omega_6^5-\omega_1^2, & \omega_1^3+\omega_7^5, & 2\omega_2^3+\omega_8^5, \\
\omega_4^6-2\omega_1^2, & 2\omega_1^2+\omega_5^6, & \omega_6^6, & \omega_2^3+\omega_7^6, & \omega_1^3+\omega_8^6, \\
\omega_4^7-2\omega_1^3, & \omega_5^7, & \omega_6^7-\omega_2^3, & \omega_7^7, & \omega_1^2+\omega_8^7, \\
\omega_4^8, & \omega_5^8-2\omega_2^3, & \omega_6^8-\omega_1^3, & \omega_7^8-\omega_1^2, & \omega_8^8, \\
\omega_4^\xi, & \omega_5^\xi, & \omega_6^\xi, & \omega_7^\xi, & \omega_8^\xi;
\end{array}$$

second, the ranks of the polar systems are: $s_1 = 15 + 3(n-8)$ and $s_1 + s_2$, where $s_2 = 10 + 2(n-8)$, thus the Cartan's number $Q = s_1 + 2s_2 = 35 + +7(n-8) = 7n - 21$. Hence the Cartan's criterion is satisfied and the maximal semiparallel submanifold M^3 in $E_{0,5}^n$ exists with arbitrariness of $7n - 21$ real holomorphic functions of two variables.

Geometry. For the corresponding parallel submanifold the equations

$$\omega_1^2+\omega_6^4 = \omega_6^5-\omega_1^2 = \omega_4^6-2\omega_1^2 = 2\omega_1^2+\omega_5^6 = \omega_1^2+\omega_8^7 = \omega_7^8-\omega_1^2 = 0,$$
$$2\omega_1^3+\omega_7^4 = \omega_1^3+\omega_7^5 = \omega_1^3+\omega_8^6 = \omega_4^7-2\omega_1^3 = \omega_6^8-\omega_1^3 = 0,$$
$$\omega_2^3+\omega_8^4 = 2\omega_2^3+\omega_8^5 = \omega_2^3+\omega_7^6 = \omega_6^7-\omega_2^3 = \omega_5^8-2\omega_2^3 = 0,$$
$$\omega_4^4 = \omega_5^4 = \omega_4^5 = \omega_5^5 = \omega_6^6 = \omega_5^7 = \omega_7^7 = \omega_4^8 = \omega_8^8 = 0,$$
$$\omega_4^\xi = \omega_5^\xi = \omega_6^\xi = \omega_7^\xi = \omega_8^\xi = 0$$

are to be added. It can be made $\omega_j^i = 0$ and $\omega^i = du^i$. The derivation formulae are $dx = e_i\omega^i = e_i du^i,\ de_i = h_{ij}\omega^j,\ dh_{ij} = 0,$ with relation (10). So the considered parallel space-like M^3 lies in $E_{0,5}^8$. It spanned by the point x and mutually orthogonal vectors $e_i,\ h_{ij}$. For the principal and the second derivatives of x one has $x_{u^i} = e_i,\ x_{u^iu^i} = h_{ii},\ x_{u^iu^j} = h_{ij},$ whereas all third derivatives are zero. Thus the considered parallel space-like submanifold can be represented by the equation

$$x = \frac{1}{2}[h_{11}((u^1)^2-(u^3)^2) + h_{22}((u^2)^2-(u^3)^2)] +$$
$$+h_{12}u^1u^2 + h_{13}u^1u^3 + h_{23}u^2u^3 + h_{01}u^1 + h_{02}u^2 + h_{03}u^3, \qquad (12)$$

where all coefficients are some constant vectors; the absolute term can be made zero, if to exchange the initial point. It is easy to see that the geodesic lines on this parallel submanifold are parabolas.

4. The results of previous section can be summarized now as follows.

Theorem. *A maximal semiparallel space-like M^3 in pseudo-Euclidean space E_s^n, which is not totally geodesic, is either* (i) *a submanifold in $E_{0,5}^8$ with* *three families of parabola generators and can be represented by the equation (12), where all coefficients here are some constant vectors, or* (*ii*) *a 2nd order envelope of a family, consisting of the submanifolds of the previous class.*

5. The complete classiffication of maximal semiparallel space-like 3-dimensio-nal submanifolds in E^n_s is not available yet. It needs, in particular, to decribe not only geomery of parallel submanifolds, but also geometry of their second order envelopes. Only the first steps are made to study such envelopes in more general cases (see [10]-[15]).

References

[1] Lumiste, Ü. *Semi-symmetric submanifold as the second order envelope of symmetric submanifolds*, *Proc. Estonian Acad. Sci. Phys. Math.*, 1990, **39**, 1-8.

[2] Deprez, J. *Semi-parallel surfaces in Euclidian space*, *J. Geom.*, 1985, **25**, 192-200.

[3] Deprez, J. *Semi-parallel hypersurfaces*, *Rend. Sem. Mat. Univ. Politec. Torino*, 1986, **44**, 303-316.

[4] Lumiste, Ü. *Classiffication of three-dimensional semi-symmetric subma-nifolds in Euclidean spaces*, *Acta Comment. Univ. Tartuensis*, 1990, **899**, 45-56.

[5] Lumiste, Ü., Riives, K. *Three-dimensional semi-symmetric submanifolds with axial, planar or spatial point in Euclidean spaces*, *Acta Comment. Univ. Tartuensis*, 1990, **899**, 13-28.

[6] Lumiste, Ü. *Semi-parallel time-like surfaces in Lorentzian spacetime forms*, *Differ. Geom. Appl.*, 1997, **7**, 59-74.

[7] Safiulina, E. *Parallel and semiparallel space-like surfaces in pseudo-Euclidean spaces*, *Proc. Estonian Acad. Sci. Phys. Math.*, 2001, **50**, 16-33.

[8] Lumiste, Ü. Submanifolds with parallel fundamental form, Chapter 7 in *Handbook of Differential Geometry, vol.* (Dillen, F.J.E. and Verstraelen L.C.A., eds.), Elsevier Sc. B. V., Amsterdam, 2000, 779-864.

[9] Lumiste, Ü. *Semiparallel Submanifolds in Space Forms*, Springer-Verlag, New York 2009.

[10] Lumiste, Ü. *Second order envelopes of m-dimensional Veronese submanifolds*, *Acta Comment. Univ. Tartuensis*, 1991, **930**, 35-46.

[11] Lumiste, Ü. *Symmetric orbits of orthogonal Veronese actions and their second order envelopes*, Results in Mathematics, 1995, **27**, 284-301.

[12] Lumiste, Ü. *Second order envelopes of symmetric Segre submanifolds*, *Acta Comment. Univ. Tartuensis*, 1991, **930**, 15-26.

[13] Lumiste, Ü. *Symmetric orbits of the orthogonal Segre action and their se-cond order envelopes*, Rendiconti Semin. Mat. Messina, 1991, Ser II **1**, 142-150.

[14] Lumiste, Ü. *Semi-symmetric envelopes of some symmetric cylindrical submanifolds*, *Proc. Estonian Acad. Sci. Phys. Math.*, 1991, **40**, 245-257.

[15] Riives, K. *Second order envelope of congruent Veronese surfaces in E^6*, *Acta Comment. Univ. Tartuensis*, 1991, **930**, 47-52.

ГЕОМЕТРИЧЕСКАЯ МОДЕЛЬ МАКСИМАЛЬНЫХ ПОЛУПАРАЛЛЕЛЬНЫХ ПРОСТРАНСТВЕННО-ПОДОБНЫХ 3-Х МЕРНЫХ ПОДМНОГООБРАЗИЙ В ПСЕВДО-ЕВКЛИДОВОМ ПРОСТРАНСТВЕ

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В статье ставится задача описать геометрическую модель максимальных полу-параллельных пространственно-подобных 3-х мерных подмногообразий в псевдо-Евклидовом пространстве, используя факт, что полупараллельные под-многообразия являются огибающими 2-го порядка семейства соответствующих параллельных подмногообразий. В результате анализа автор доказывает, что выше названные 3-х мерные подмногообразия существуют с произволом 7n-21 голоморфных функций двух переменных и являются либо подмногообразием с тремя параболическими образующими, либо огибающими 2-го порядка семейства таких подмногообразий.

ГЕОМЕТРИЧНА МОДЕЛЬ МАКСИМАЛЬНИХ НАПІВПАРАЛЕЛЬНИХ ПРОСТОРОВО-ПОДІБНИХ 3-ВИМІРНИХ ПІДМНОГОВИДІВ У ПСЕВДО-ЕВКЛІДОВОМУ ПРОСТОРІ

О.Сафуліна

У статті ставиться задача описати геометричну модель максимальних напів-паралельних просторово-подібних 3-вимірних підмноговидів у псевдо-Евклідовому просторі, використовуючи той факт, що напівпаралельні підмноговиди є обвідними 2-го порядку сім'ї відповідних паралельних підмноговидів. В результаті аналізу автор доводить, що названі вище 3-вимірні підмноговиди існують з довільністю до 7n-21 голоморфних функцій двох змінних і є або підмноговидом з трьома параболічними твірними, або обвідними 2-го порядку сім'ї таких підмноговидів.