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Generalized Fokker-Planck Equation for the Nanoparticle Magnetic Moment Driven by Poisson White Noise

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We derive the generalized Fokker-Planck equation for the probability density function of the nanoparticle magnetic moment driven by Poisson white noise. Our approach is based on the reduced stochastic Landau-Lifshitz equation in which this noise is included into the effective magnetic field. We take into account that the magnetic moment under the noise action can change its direction instantaneously and show that the generalized equation has an integro-differential form.

Keywords: Stochastic Landau-Lifshitz equation, Poisson white noise, Probability density function, Generalized Fokker-Planck equation.

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1. INTRODUCTION

The study of the physical properties of nanoparticle materials, especially their magnetic properties, is of great importance. This is substantiated by the fact that the magnetic nanoparticles and their ensembles are widely used in many areas of science, engineering and medicine [1-3]. Moreover, the investigation of the magnetic properties of nanomaterials can significantly expand the scope of their application, e.g., in magnetic recording, sensors based on the effect of giant magnetoresistance, magnetic refrigerators, bio-medical technologies, etc.

Due to continuous internal and external fluctuations that are an integral part of real systems, the dynamics of the nanoparticle magnetic moment is random. In some cases, it can be described by the stochastic Landau-Lifshitz (LL) equation in which the fluctuations are modeled by a white noise. A great amount of effort has been devoted to the study of the role of thermal fluctuations approximated by Gaussian white noise. In this case, the nanoparticle magnetic moment evolves as a continuous-time Markov process whose conditional probability density satisfies the ordinary Fokker-Planck (FP) equation [4-6]. Within a given approach, some aspects of the effects of magnetic relaxation, switching of magnetization and induced magnetization are considered in Refs. [7-12].

However, although the Gaussian white noise is widely applicable (because of the central limit theorem), in some cases the use of white noises with other statistics is more preferable. For example, such a situation occurs when the magnetic moment under strong fluctuations changes its direction so fast that this process can be considered as instantaneous. To describe this behavior, it is reasonable to use the LL equation in which the fluctuations are modeled by Poisson and/or Lévy noises. The dynamics of the magnetic moment driven by these noises is Markovian and, in contrast to the case of Gaussian white noise, is discontinuous. In this paper, we derive the generalized FP equation associated with the reduced LL equation driven by Poisson white noise defined as a random sequence of delta pulses whose counting process is Poisson.

2. MODEL AND BASIC EQUATIONS

We deal with a simple model of single-domain ferromagnetic nanoparticles that are characterized by the magnetic moment $\mathbf{m} = \mathbf{m}(t)$ of a constant length $m = |\mathbf{m}| = \mathrm{const}$. It is assumed that the rotational dynamics of the vector $\mathbf{m}$ is described by the following stochastic LL equation:

$$\frac{d\mathbf{m}}{dt} = -\gamma\mathbf{m}\times(\mathbf{H}_{\mathrm{eff}}+\mathbf{h}) - \frac{\alpha\gamma}{m}\mathbf{m}\times(\mathbf{m}\times\mathbf{H}_{\mathrm{eff}}), \quad (2.1)$$

where $\gamma(>0)$ is the gyromagnetic ratio, $\alpha(>0)$ is the damping parameter, the cross denotes the vector product, $\mathbf{H}_{\mathrm{eff}} = -\partial W/\partial\mathbf{m}$ is the effective magnetic field acting on the magnetic moment, W is the magnetic energy of a particle, and $\mathbf{h} = \mathbf{h}(t)$ is a random magnetic field. Introducing the dimensionless time $\tau = \gamma H_0 t$ and energy $E = W/mH_0$ (H_0 is a characteristic magnetic field), the LL equation (2.1) in spherical coordinates can be written as a system of two equations

$$\begin{aligned} d\theta &= f_1 d\tau + d\eta_1, \\ d\varphi &= f_2 d\tau + \frac{1}{\sin\theta} d\eta_2. \end{aligned} \quad (2.2)$$

Here, $\theta = \theta(t)$ and $\varphi = \varphi(t)$ are the polar and azimuthal angles of $\mathbf{m}$, respectively,

$$\begin{aligned} f_1 &= f_1(\theta,\varphi,\tau) = -\left(\alpha\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)E, \\ f_2 &= f_2(\theta,\varphi,\tau) = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta} - \frac{\alpha}{\sin\theta}\frac{\partial}{\partial\varphi}\right)E, \end{aligned} \quad (2.3)$$

$d\eta_1 = -\sin\varphi d\zeta_x + \cos\varphi d\zeta_y$, $\eta_2 = \sin\theta d\zeta_z - \cos\theta(\cos\varphi d\zeta_x + \sin\varphi d\zeta_y)$, and $d\zeta_\kappa = \int_\tau^{\tau+d\tau} d\tau' h_\kappa(\tau')$ $(\kappa = x,y,z)$ with h_κ being the Cartesian coordinates of the random magnetic field $\mathbf{h}$.

We consider the coordinates h_κ to be independent and identically distributed white noises. Therefore, since these noises are multiplicative, the system of equations (2.2) should be defined more precisely. To this end, we interpret Eqs. (2.2) in the Ito sense [13] (see also Refs. [5,6]). In general, to find the statistical properties of the increments $d\eta_p$ $(p = 1,2)$, it is necessary to solve these equations under condition that the statistical characteristics of h_κ are fixed. However, in order to qualitatively characterize the role of non-Gaussian noises in the dynamics of $\mathbf{m}$, we assume that the statistical properties of $d\eta_p$ are prescribed. According to the definitions [see formulas after Eqs. (2.3)], from the condition $\langle d\zeta_\kappa \rangle = 0$ (the angular brackets denote an average over the realizations of $d\zeta_\kappa$) one gets $\langle d\eta_p \rangle = 0$ and, if $\langle d\zeta_\kappa(\tau) d\zeta_{\kappa'}(\tau') \rangle = 0$ $(\kappa \neq \kappa')$ and $\langle d\zeta_x(\tau) d\zeta_x(\tau') \rangle = \langle d\zeta_y(\tau) d\zeta_y(\tau') \rangle = \langle d\zeta_z(\tau) d\zeta_z(\tau') \rangle$, then $\langle d\eta_p(\tau) d\eta_{p'}(\tau') \rangle = 0$ $(p \neq p')$ and $\langle d\eta_1(\tau) d\eta_1(\tau') \rangle = \langle d\eta_2(\tau) d\eta_2(\tau') \rangle = \langle d\zeta_x(\tau) d\zeta_x(\tau') \rangle$. These properties permit us to consider the increments $d\eta_p$ as independent and identically distributed variables with zero mean and represent them in the form

$$d\eta_p = \int_\tau^{\tau+d\tau} d\tau' \xi_p(\tau'), \tag{2.4}$$

where $\xi_p(\tau)$ are two white noises with identical statistical characteristics. It should be emphasized that such interpretation of noises in Eqs. (2.2) differs from the original one and the stochastic dynamics of $\mathbf{m}$ is also different. Nevertheless, we exploit this toy model to get more insight into the effects of non-Gaussian noises.

In this work we study the case when $\xi(\tau)$ (we omit the index p) is Poisson white noise, i.e., a random sequence of delta pulses defined as

$$\xi(\tau) = \sum_{i=1}^{n(\tau)} z_i \delta(\tau - \tau_i). \tag{2.5}$$

Here, z_i are independent zero-mean random variables with the same probability density $q(z)$, $\delta(\tau)$ is the δ function, $n(\tau)$ is the Poisson counting process characterized by the probability $P\{n(\tau) = n\} = (\nu\tau)^n e^{-\nu\tau}/n!$ (ν is the rate of the process) that $n(\tau) \geq 0$ events occur within the interval (0, τ], and τ_i are the event times. For this noise the probability density $p(\mu; d\tau)$ that the increment $d\eta = \int_\tau^{\tau+d\tau} d\tau' \xi(\tau')$ at $d\tau \to 0$ equals μ is given by [14]

$$p(\mu; d\tau) = (1 - \nu d\tau)\delta(\mu) + \nu d\tau q(\mu). \tag{2.6}$$

According to this result, the probability per unit time that $|\mu|$ exceeds any given value is, in general, non-zero in the limit $d\tau \to 0$. As a consequence, the dynamics of the magnetic moment $\mathbf{m}$ described by Eqs. (2.2) is discontinuous. One more feature of these equations is that $d\theta, d\varphi \in (-\infty, \infty)$, while the polar and azimuthal angles $\theta(\tau)$ and $\varphi(\tau)$ must belong to the intervals $[0,\pi)$ and $[0,2\pi)$, respectively, for all τ. Therefore, to find the connection between these angles and the increments $d\theta$ and $d\varphi$, we use the relation $\mathbf{m}(\theta(\tau + d\tau), \varphi(\tau + d\tau)) = \mathbf{m}(\theta(\tau) + d\theta, \varphi(\tau) + d\varphi)$, which yields

$$\begin{aligned} \theta(\tau + d\tau) &= \theta(\tau) + d\theta - 2\pi n, \\ \varphi(\tau + d\tau) &= \varphi(\tau) + d\varphi - 2\pi m \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \theta(\tau + d\tau) &= -\theta(\tau) - d\theta - 2\pi n, \\ \varphi(\tau + d\tau) &= \varphi(\tau) + d\varphi - 2\pi m - \pi. \end{aligned} \tag{2.8}$$

Here, the whole numbers n and m are chosen for given $d\theta$ and $d\varphi$ from the conditions $\theta(\tau + d\tau) \in [0,\pi)$ and $\varphi(\tau + d\tau) \in [0,2\pi)$. These numbers count the number of rotations of $\mathbf{m}$ in the corresponding planes. Thus, Eqs. (2.2) driven by Poisson white noise are now completely defined and we are ready to derive the corresponding generalized FP equation.

3. GENERALIZED FOKKER-PLANCK EQUATION

The probability density $P = P(\theta, \varphi, \tau)$ that $\theta(\tau) = \theta$ and $\varphi(\tau) = \varphi$ is defined in the standard way

$$P = \langle \delta(\theta - \theta(\tau)) \delta(\varphi - \varphi(\tau)) \rangle. \tag{3.1}$$

Denoting $P' = P(\theta, \varphi, \tau + d\tau)$, from the definition (3.1) and conditions (2.7) and (2.8) one obtains

$$\begin{aligned} P' = \Big\langle \sum_{n,m} \big[&\delta(\theta - \theta(\tau) - d\theta + 2\pi n)\delta(\varphi - \varphi(\tau) \\ &- d\varphi + 2\pi m) + \delta(\theta + \theta(\tau) + d\theta + 2\pi n) \\ &\times \delta(\varphi - \varphi(\tau) - d\varphi + 2\pi m + \pi) \big] \Big\rangle. \end{aligned} \tag{3.2}$$

The average in Eq. (3.2) is equivalent to the average over the variables $\theta(\tau)$, $\varphi(\tau)$ and $d\eta_1, d\eta_2$. Taking into account that these two groups of variables are independent of each other and using Eqs. (2.2), we find

$$\begin{aligned} P' = \sum_{n,m} &\int_0^\pi d\theta' \int_0^{2\pi} d\varphi' P(\theta', \varphi', \tau) \iint_{-\infty}^{\infty} d\mu_1 d\mu_2 \\ &\times p(\mu_1; d\tau) p(\mu_2; d\tau) [\delta(\theta - \theta' - f_1' d\tau - \mu_1 + 2\pi n) \\ &\times \delta(\varphi - \varphi' - f_2' d\tau - \mu_2 / \sin\theta' + 2\pi m) \\ &+ \delta(\theta + \theta' + f_1' d\tau + \mu_1 + 2\pi n) \\ &\times \delta(\varphi - \varphi' - f_2' d\tau - \mu_2 / \sin\theta' + 2\pi m + \pi)], \end{aligned} \tag{3.3}$$

where $f_{1,2}' = f_{1,2}(\theta', \varphi', \tau)$. Finally, using Eq. (2.6) and the standard definition $\lim_{d\tau \to 0}(P' - P)/d\tau = \partial P/\partial\tau$, after straightforward calculations we arrive at the following generalized FP equation:

$$\begin{aligned} \frac{\partial}{\partial\tau} P = &-\frac{\partial}{\partial\theta} f_1 P - \frac{\partial}{\partial\varphi} f_2 P - 2\nu P \\ &+ \nu \sin\theta \int_0^{2\pi} d\varphi' P(\theta, \varphi', \tau) \sum_n q(\sin\theta[\varphi - \varphi' + 2\pi n]) \\ &+ \nu \int_0^\pi d\theta' P(\theta', \varphi, \tau) \sum_n q(\theta - \theta' + 2\pi n) \\ &+ \nu \int_0^\pi d\theta' [P(\theta', \varphi + \pi, \tau)|_{\varphi<\pi} + P(\theta', \varphi - \pi, \tau)|_{\varphi>\pi}] \\ &\times \sum_n q(-\theta - \theta' + 2\pi n). \end{aligned} \tag{3.4}$$

The solution of this integro-differential equation must be normalized, i.e., $\int_0^\pi d\theta \int_0^{2\pi} d\varphi P(\theta, \varphi, \tau) = 1$, and satisfy the initial condition $P(\theta, \varphi, 0) = P_0(\theta, \varphi)$. Another form of this equation, which can be useful for studying its properties, follows directly from Eq. (3.4) by using the Poisson summation formula.

4. CONCLUSIONS

We have introduced a toy model for studying the effects of Poisson white noise in the rotational dynamics of the nanoparticle magnetic moment. We have shown that the dynamics is discontinuous and have developed an approach to account for the manifold rotations of the magnetic moment under pulses of the noise. Finally, using this approach, we have derived the corresponding generalized Fokker-Planck equation.

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