The Basic Idea of Bootstrap Methods

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Abstract

Two methods of bootstrap simulation – parametric and nonparametric – are described in this article. Parametric simulation assumes that distribution of random variable X is known. Non-parametric simulation doesn't require this assumption. Concrete examples demonstrate both ways of simulation.

Key words: Parametric and nonparametric bootstrap simulations, resampling, bias, variance estimates.

1. Introduction

The case when the frequency of random sample is very small is very often in research in economics, sciences and technical sciences. Then it is very difficult and untrustworthy to express any conclusions about standard errors, hypotheses testing or confidence intervals. We can use the bootstrap method based on resampled data in this case.

The substantial and frequent problem of statistical data analysis is to determine theoretical properties of some statistics $\hat{\Theta}$. In simple cases it is possible to use classical statistical methods. But when it is difficult to determine theoretical properties of $\hat{\Theta}$ estimate by exact way then it is possible to use the bootstrap method.

The bootstrap method was elaborated by Bradley Efron in 1977. This method makes use advantage of the high-speed power and number-crunching power of computers. The principle is to resample new samples from the original data set with the same rate. This approach involves repeating the original data analysis procedure with many replicated datasets. Very important advantage of this method is that it allows to construct artificial data sets without making any assumptions about bell shaped curves. Problems that can be solved with the help of bootstrap method can be divided into two groups that are later called parametric and nonparametric bootstrap (Efron, Tibshirani,1993).

2. Theoretical assumptions

We assume that $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is random sample from distribution $F(x,\psi)$, belonging to some family of distribution functions $\mathscr{F}$ differentiated with parameter ψ.

Frequent object in view is to determine the properties if statistics $\hat{\Theta} = \hat{\Theta}(X_1, \ldots, X_n)$. When the data are obtained from distribution F, we indicate the distribution function of statistics $\hat{\Theta}$ $G_n(x,F) = P(\hat{\Theta} < x)$. Indication $G_n(x,F)$ expresses that distribution function G of statistics $\hat{\Theta}$ is determined from n randomly selected values of random variable X with distribution function F. Generally function $G_n(x,F)$ depends on parameters of distribution F. In case of pivot statistics $\hat{\Theta}$ the distribution function isn't dependent on these parameters.

Asymptotic theory is one of common classical methods of distribution function $G_n(x,F)$ estimate. The asymptotic approximation enables to substitute the unknown distribution function G_n with known function G_∞. The estimates in econometric applications aren't in principle pivotal. Distribution of them usually depends on one or more unknown parameters. Bootstrap method provides one alternative approximation of distribution of statistics $\hat{\Theta}(X_1, \ldots, X_n)$. Unknown distribution function F is replaced by known estimate that is indicated $\hat{F}$.

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When bootstrap methods are applied, two estimates of distribution function are differentiated and similarly two bootstrap simulation techniques are used. Following paragraph describes how the distribution function was obtained, we use special signification for it.

2.1. The parametric estimate of distribution function F

Let's suppose that for distribution function F of random variable X holds true $F(x) = F(x,\psi)$ for some unknown parameter ψ, that is consistently estimated by statistics $\hat{\psi}$. If the function $F(x,\psi)$ is the continuous function of parameter ψ, then $F(x,\hat{\psi}) \to F(x,\psi)$ for $n \to \infty$. The distribution function $\hat{F} = F(x,\hat{\psi})$ obtained in this way is signified $F_{\hat{\psi}}$ and called the parametric estimate of distribution function F.

2.2. The nonparametric estimate of distribution function F

When the distribution function F is unknown we estimate this distribution with empirical distribution function $\hat{F}$ that is indicated F_n. It holds:

$$F_n = \frac{1}{n}\sum_{i=1}^{n} I(x - x_i),$$

where function I is an indicator, it is $I(x - x_i) = \begin{cases} 0 & x - x_i < 0 \\ 1 & x - x_i \geq 0 \end{cases}$.

Function F_n is called nonparametric estimate of distribution function F.

No matter how the estimate $\hat{F}$ of distribution function F was obtained, the bootstrap estimate of distribution function $G_n(x,F)$ is the function $G_n(x,\hat{F})$. But we often fail to find the function $G_n(x,\hat{F})$ by analytical way. Bootstrap method enables to perform the approximation of distribution function F with function $\hat{F}$ and consequently to estimate the distribution $G_n(x,F)$ of the statistics $\hat{\Theta}$ with function $G_n(x,\hat{F})$. As the functions $\hat{F}$ and F aren't identical then the functions $G_n(x,\hat{F})$ and $G_n(x,F)$ are different except for the case when $\hat{\Theta}$ is the pivotal statistics. That is why the bootstrap estimate $G_n(x,\hat{F})$ is only approximation of distribution function $G_n(x,F)$ of the statistics $\hat{\Theta}$.

3. Parametric bootstrap simulations

3.1. Principle of parametric simulations

Let's assume that X is a random variable with known distribution function F, $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ random sample from this distribution with distribution function F and ψ is some parameter of distribution of random variable X. $\hat{\psi}$ is the estimate of parameter ψ that was obtained from random sample $X_1, X_2, \dots X_n$. Parametric estimate of distribution function F is its parametric estimate $\hat{F} = F_{\hat{\psi}}$. New random samples are obtained when values of random variable with $F_{\hat{\psi}}$ distribution are generated. This way of new samples generation is called parametric simulation. The model based on function $F_{\hat{\psi}}$ is called parametric model.

Technique of parametric simulations:

- Simulate random sample $\boldsymbol{X}^* = (X_1^*, X_2^*, \dots X_n^*)$ of rate n from distribution $F_{\hat{\psi}}$.
- Calculate statistics $\hat{\Theta}^* \equiv \hat{\Theta}^*(X_1^*, X_2^*, \dots X_n^*)$.
- Repeat first and second items and use results for next calculations.

Indication * is used to express the reality that relevant variable relates to the model $F_{\hat{\psi}}$. For example X^* means that distribution of this random variable is equivalent with model $F_{\hat{\psi}}$.

3.2. Moments estimates

When the statistic of interest $\hat{\Theta}$ is calculated from a simulated dataset, we denote it $\hat{\Theta}^*$. From *R* repetitions of the data simulation we obtain $\hat{\Theta}_1^*, \ldots, \hat{\Theta}_R^*$ statistics. Properties of $\hat{\Theta}$ are then estimated from $\hat{\Theta}_1^*, \ldots, \hat{\Theta}_R^*$.

The moment estimates will be realized in following way:

The estimator of bias of $\hat{\Theta}$

$b(F) = E(\hat{\Theta} \mid F) - \Theta$,

is statistics

$$B = b(F_{\hat{\psi}}) = E(\hat{\Theta} \mid F_{\hat{\psi}}) - t = \mathrm{E}^*(\hat{\Theta}^*) - t$$

(Peracchi, 2000)

Distribution function *F* is estimated with function $F_{\hat{\psi}}$, and then estimate of bias is

$$B_R = \frac{1}{R}\sum_{r=1}^{R}\hat{\Theta}_r^* - t = \overline{\hat{\Theta}^*} - t$$

(Peracchi,2000)

where *t* is the concrete value of statistics $\hat{\Theta}$, calculated from original dataset. $\hat{\Theta} - t$ is the simulation analogy of $\hat{\Theta} - \Theta$.

Variance $\sigma^2 = D(\hat{\Theta} \mid F)$ of random variable $\hat{\Theta}$ is estimated with statistics

$$S_R^2 = \frac{1}{R-1}\sum_{r=1}^{R}(\hat{\Theta}_r^* - \overline{\hat{\Theta}^*})^2 .$$

(Peracchi,2000)

Similar estimators can be derived for other moments. These empirical approximations are justified by the law of large numbers.

3.3. Example – simulation from normal distribution

Let's assume that $x_1, x_2, \ldots, x_n$ is some realization of random sample from $N(\mu, \sigma)$ distribution. When parameter μ is estimated with sample average $\overline{x}$ and parameter σ – with sample standard deviation $s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$, then the parametric bootstrap replication of sample $x_1, x_2, \ldots,$ x_n is random sample $x_1^*, x_2^*, \ldots, x_n^*$, that comes from $N(\overline{x}, s)$ distribution. We obtain *R* realizations of bootstrap samples after *R* bootstrap replications.

Let's indicate bootstrap average $\overline{X}^* = \frac{1}{n}\sum_{i=1}^{n} X_i^*$, where $X_1^*, X_2^*, \ldots, X_n^*$ is random sample from $N(\overline{x}, s)$ distribution. Random variable $\overline{X}^*$ is normally distributed with mean $\overline{x}$ and standard deviation $\frac{s}{\sqrt{n}}$. Bootstrap estimate of bias of sample average $\overline{X}^*$ is the statistics $B_R = \frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^* - \overline{x}$ (Davison, Hinkley,1997).

Random variable $\frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^*$ is approximately normally distributed with mean $\overline{x}$ and standard deviation $\sqrt{\frac{s^2}{Rn}}$. Then the bootstrap bias $\frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^*$ - $\overline{x}$ is approximately normally distributed with mean 0 and standard deviation $\sqrt{\frac{s^2}{Rn}}$ as well.

We are interested in bootstrap bias distribution and mean of bias $E^*(B_R)$ distribution, also B_R - $E^*(B_R)$ in the following step. This bootstrap bias is obtained from R bootstrap replications. But $E^*(B_R) = 0$ in our case. Therefore $B_R - E^*(B_R) = B_R = \frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^* - \overline{x}$.

Random variable B_R is approximately normally distributed with mean 0 and standard deviation $\sqrt{\frac{s^2}{Rn}}$. Then random variable B_R - $E^*(B_R)$ is also approximately normally distributed with mean 0 and standard deviation $\sqrt{\frac{s^2}{Rn}}$. This random variable exactly expresses the estimation of an error that is made after finite number of bootstrap replications. After normal transformation $\frac{B_R}{s}\sqrt{Rn}$ we obtain normally distributed random variable with mean 0 and standard deviation 1. Interval estimate of variable $E^*(B_R)$ is then

$$B_R + z_\alpha \frac{s}{\sqrt{Rn}} < E^*(B_R) < B_R + z_{1-\alpha}\frac{s}{\sqrt{Rn}},$$

where z_α is quantile of $N(0,1)$ distribution; $z_\alpha = \Phi^{-1}\left(\frac{\alpha}{2}\right)$. We can estimate by means of this relation extent of the error that we can cause at given number R of bootstrap replications, at given extent of n of original random sample and at chosen value of α.

To be able to compare the results of simulations with real values we assumed, that $X_1, X_2, \ldots, X_n$ is random sample from $N(\mu, \sigma)$ distribution. At fulfilled presumption of normal distribution of variables X_i, $i = 1, \ldots, n$, it is possible to calculate in the exact way the theoretical value of bias and variance of average . These values are 0 and $\frac{\sigma^2}{n}$ by turns.

Table 1 shows the method of parametric simulations at the example of data from $N(100,10)$ distribution. The concrete values of random sample realization are 109, 80, 97, 115, 113, 83, 89, 110, 98, 114, 95, 100, 105, 112, 99. These values are introduced in the first column of the table. Following nine columns show simulated values of random sample, rate of each of these samples is identical to rate of original random sample. These simulated samples come again from normal distribution with estimated parameter $\overline{x}$ =101,267 and s = 10,847. These simulated samples enable us to estimate parameters and results can be used for next calculations.

To compare results of parametric simulations, we used two normally distributed random samples: the first one with mean 100, standard deviation 10 and size 15 and the second one with mean 0, standard deviation 1 and size 15 to illustrate solved problem. 10 000 bootstrap replications were made for each of above samples and basic statistical characteristics were computed.

These characteristics are presented in Table 2. It is visible that results obtained after 10 000 bootstrap parametric replications are “closer” to the real parameters μ and σ of original normal distribution than results obtained on the base of random sample from this normal distribution.

Table 1

Parametric simulation

Original data	1 simulation	2 simulation	3 simulation	4 simulation	5 simulation	6 simulation	7 simulation	8 simulation	9 simulation
109	116,552	111,248	101,449	92,454	102,736	84,239	120,159	104,762	91,297
80	107,456	93,269	112,719	95,816	118,465	86,591	115,768	98,795	84,646
97	97,026	90,843	101,826	100,452	65,183	93,034	93,477	96,981	98,727
115	95,409	99,438	99,603	98,760	63,181	100,595	80,231	100,052	110,125
113	97,782	103,790	95,423	98,941	108,665	100,431	101,741	117,758	77,064
83	96,492	107,183	115,801	94,384	111,088	122,378	112,977	116,095	84,694
89	89,235	98,433	108,453	116,890	82,395	101,000	111,804	105,073	103,552
110	112,429	114,113	117,733	90,943	92,132	97,753	102,115	100,862	92,745
98	99,070	101,942	103,745	88,700	103,963	92,489	104,074	77,667	76,697
114	104,879	104,497	92,209	99,492	91,969	99,681	107,494	119,469	102,898
95	111,479	85,621	103,537	99,726	89,840	103,731	88,210	107,638	100,003
100	120,649	97,936	93,525	102,964	102,539	109,414	101,412	89,768	103,873
105	76,794	97,471	111,277	108,909	111,856	88,535	91,839	103,767	111,609
112	111,465	95,953	115,436	77,358	94,297	100,911	101,214	92,962	107,961
99	126,222	84,854	105,237	93,610	93,622	97,758	80,513	102,767	121,045

Table 2

Estimated parameters – parametric bootstrap

Parameter	N(100, 10) distribution	N(0, 1) distribution
$\overline{x}$	101,267	0,1007
s^2	117,662	1,0614
$\overline{x}_R^*$	101,238	0,0982
s_R^{2*}	109,947	0,9971

Figures 1 and 2 show the results of several simulations, changes of bias with increasing number of bootstrap replications at sampling from *N*(101,267; 10,847) and *N*(0,1007; 1,030) distributions. The problem is demonstrated at 5 repetitions of 2000 replications. Empirical biases were calculated for each value of *R*. We can note how the variability decreases as the simulation size increases and how the simulated values converge to the exact value. To answer the question, how many bootstrap replications are needed, Figures 1 and 2 suggest that *R* = 400 bootstrap replications could be adequate. Values of bias don't markedly change for larger values of *R*.

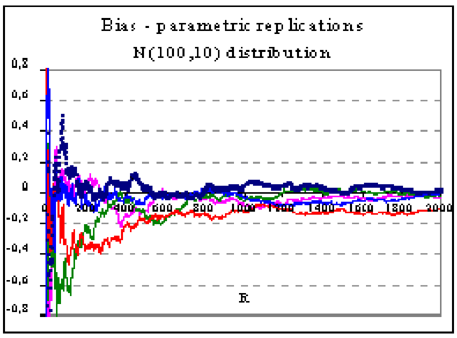

Fig. 1. Sampling from *N*(101,267; 10,847)

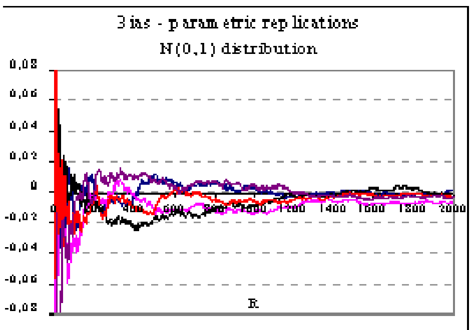

Fig. 2. Sampling from *N*(0,1007; 1,030)

4. Nonparametric bootstrap simulations

4.1. Principle of nonparametric simulations

In case of nonparametric simulation we assume, that $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is random sample from distribution with unknown distribution function F. Empirical distribution function F_n is used for estimate of unknown distribution function F. Application of this distribution function is analogical to parametric model.

Realization of further samples are obtained from digital data (originally measured) $x_1, x_2,\dots, x_n$ that way that we apply random sample with replacement of rate n. That is why random variables X_i^* have really distribution F_n. We use indication X_i^* for simulated variable X_i, n-tuple $X_1^*, \dots, X_n^*$ is random sample from F_n distribution. New concrete samples rise from original data by original data resampling (random reordering and confusion). This technique is called nonparametric simulation (nonparametric bootstrap).

4.2. Moments estimates

Following consideration can be used in connection with theoretical calculation:

The estimate of parameter Θ is random variable $\hat{\Theta}$. We obtain the concrete value t of random variable $\hat{\Theta}$ from concrete realization of random sample. Both value t of statistics $\hat{\Theta}$ and empirical distribution function depend on values of random sample $x_1, x_2,\dots, x_n$. The value t can be considered as function of empirical distribution function F_n. This relation can be expressed $t = g(F_n)$, where g is the relevant function. Relation $t = g(F_n)$ expresses the way how to determine the value of t on base of empirical distribution function F_n. The elementary examples of such functions are terms for mean and variance calculation, that are generally defined in the following way:

$$EX = g(F) = \int_{-\infty}^{\infty} x\, dF(x) \text{ and } DX = g(F) = \int_{-\infty}^{\infty} (x - EX)^2\, dF(x)\,.$$

If we substitute the distribution function F in relation $EX = g(F) = \int_{-\infty}^{\infty} x\, dF(x)$ with empirical distribution function F_n, we obtain estimate for mean

$$g(F_n) = \int_{-\infty}^{\infty} x\, dF_n(x) = \int_{-\infty}^{\infty} x d\left(\frac{1}{n}\sum_{i=1}^{n} I(x - x_i)\right) = \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty} x dI(x - x_i) = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

(because generally holds true $\int_{-\infty}^{\infty} g(y) dI(y-x) g(x)$ for each continuous function g).

The similar relation can be found for variance estimate.

If we substitute in term $DX = g(F) = \int_{-\infty}^{\infty} (x - EX)^2\, dF(x)$ distribution function F with empirical distribution function F_n, we obtain:

$$g(F_n) = \int_{-\infty}^{\infty} (x - EX)^2\, dF_n(x) = \int_{-\infty}^{\infty} (x - EX)^2 d\left(\frac{1}{n}\sum_{i=1}^{n} I(x - x_i)\right) =$$

$$\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\left[x^2 - (EX)^2\right] dI(x - x_i) = \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty} x^2 dI(x - x_i) - \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty} (EX)^2 dI(x - x_i)$$

=

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (EX)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = s^2$$

4.3. Example

To verify theoretical results, we assume, that $x_1, x_{2, \ldots,} x_n$ is some concrete realization of random sample from $N(\mu, \sigma)$ distribution and $\hat{F}$ is the empirical distribution function of this sample. The random sample from distribution $\hat{F}$ is marked $X_1^*, X_2^*, \ldots, X_n^*$. The average of this sample is marked $\overline{X}^* = \frac{1}{n}\sum_{i=1}^{n} X_i^*$. For its mean holds true:

$$E^*(\overline{X}^*) = E^*\left(\frac{1}{n}\sum_{i=1}^{n} X_i^*\right) = \frac{1}{n}\sum_{i=1}^{n} E^* X_i^* = \frac{1}{n} n \cdot E^* X^* = \sum_{i=1}^{n} x_i\, p(x_i) = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}, \quad (1)$$

because all values in random sample have the same probability $\frac{1}{n}$.

Analogous to variance

$$D^*(\overline{X}^*) = D^*\left(\frac{1}{n}\sum_{i=1}^{n} X_i^*\right) = \frac{1}{n^2}\sum_{i=1}^{n} D^* X_i^* = \frac{1}{n^2} n \cdot D^* X^* = \frac{1}{n} E^*(X^* - E^* X^*)^2 =$$

$$= \frac{1}{n}\sum_{i=1}^{n} (X_i - \overline{x})^2 p(X_i) = \frac{1}{n^2}\sum_{i=1}^{n} (X_i - \overline{x})^2 = \frac{s^2}{n}. \quad (2)$$

When we realize R nonparametric bootstrap replications of random sample from distribution $\hat{F}$, we indicate relevant averages $\overline{X}_i^*, \quad i = 1,2, \ldots, n$.

The bootstrap estimate of bias of average $\overline{X}$ is the statistics $B_R = \frac{1}{R}\sum_{r=1}^{R} \overline{X}_r^* - \overline{x}$ (Davison Hinkley,1997).

In case of nonparametric bootstrap we are able to calculate mean and variance. While using terms (1) and (2) we obtain for these variables terms:

$$E^*(B_R) = E^*\left(\frac{1}{R}\sum_{r=1}^{R} \overline{X}_r^* - \overline{x}\right) = E^*\left(\frac{1}{R}\sum_{r=1}^{R} \overline{X}_r^*\right) - E^*(\overline{x}) = \frac{1}{R}\sum_{r=1}^{R} E^* \overline{X}_r^* - \overline{x} =$$

$$= \frac{1}{R} R \cdot E^* \overline{X}^* - \overline{x} = \overline{x} - \overline{x} = 0. \quad (3)$$

$$D^*(B_R) = D^*\left(\frac{1}{R}\sum_{r=1}^{R} \overline{X}_r^* - \overline{x}\right) = D^*\left(\frac{1}{R}\sum_{r=1}^{R} \overline{X}_r^*\right) = \frac{1}{R^2}\sum_{r=1}^{R} D^* \overline{X}_r^* = \frac{1}{R^2} R \cdot D^* \overline{X}_r^* = \frac{1}{R^2} \cdot \frac{Rs^2}{n} = \frac{s^2}{nR}. \quad (4)$$

The term 4 can be used for standard error estimation at given size of random sample and number of bootstrap replications R.

The concrete example of nonparametric simulations is shown in Table 3. The original realization of random sample is in the first column of the table. The same probability n^{-1} of each value is assumed. It is 0,0667 in our concrete case. These values are stated in the second column of the table. Next nine columns show simulated samples with rate n =15 as well. Seeing that sampling with replacement is realized, individual values can repeat in the random sample or they can't be in this sample at all. On the base of samples obtained in a described way we can estimate parameters or express further statistical conclusions.

Table 3

Nonparametric simulation

Original data	$\frac{1}{n}$	1 simulation	2 simulation	3 simulation	4 simulation	5 simulation	6 simulation	7 simulation	8 simulation	9 simulation
109	0,0667	98	95	100	115	89	97	98	113	109
80	0,0667	89	83	112	99	114	97	110	89	113
97	0,0667	113	109	89	89	110	113	89	89	113
115	0,0667	105	112	97	114	114	100	109	110	100
113	0,0667	114	109	98	100	114	95	99	105	115
83	0,0667	113	114	83	112	97	97	105	89	89
89	0,0667	98	100	97	115	112	83	115	89	112
110	0,0667	99	98	98	105	83	83	112	80	113
98	0,0667	80	97	97	105	97	115	113	83	115
114	0,0667	114	114	95	83	80	115	97	110	113
95	0,0667	115	115	95	98	98	89	80	98	115
100	0,0667	110	99	100	98	83	112	95	95	98
105	0,0667	109	99	114	89	80	105	114	109	112
112	0,0667	89	89	105	83	113	113	110	95	99
99	0,0667	83	83	109	110	114	89	112	105	113

To compare results of nonparametric simulations and to verify the features of bootstrap bias estimate, we used two samples of 15 data, where the original samples were generated from normal distribution. The first original sample was generated from $N(100,10)$ distribution and the second one from $N(0,1)$ distribution. The sample average $\overline{x}$ and sample variance s^2 are presented in Table 4. We continued in calculation as though we have never known the distribution of original dataset. Bootstrap estimates of these sample statistics were calculated on base of 10 000 replications of random sample and they are stated in Table 4 as well. We can see similarly to parametric bootstrap that estimated statistics from 10 000 bootstrap replication are "closer" to values of original distribution than values estimated from original sample.

Table 4

Estimated parameters – nonparametric bootstrap

Parameter	*N*(100, 10) distribution	*N*(0, 1) distribution
$\overline{x}$	101,267	0,1007
s^2	117,662	1,0614
$\overline{x}_R^*$	101,233	0,0977
s_R^{2*}	110,156	0,9908

Figures 3 and 4 show relation between nonparametric bootstrap bias estimate and number of R simulated samples. Five repetitions each at 2000 replications [original samples were from $N(100,10)$ distribution and from $N(0,1)$ distribution respectively] were generated. Figures 3 and 4 suggest, if R >600 replications then the values of bias estimate differ in a minimal way.

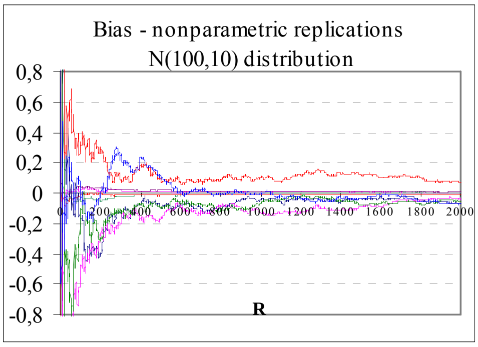

Fig. 3. Nonparametric bootstrap – bias

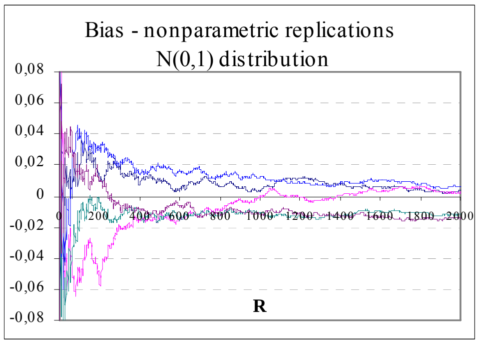

Fig. 4. Nonparametric bootstrap – bias

5. Conclusion

The basic differences of described approaches are as follows:

At parametric bootstrap we know probability model and thereby exact distribution of some important statistics. This knowledge can be used at confidence intervals construction, hypothesis testing and in further statistical analyses.

At nonparametric bootstrap any assumptions about distribution model aren't determined. It is possible to realize a lot of simulations that help to determine the properties of random variable $\hat{\Theta}$ and even to estimate its distribution.

The difference in parameters estimates is the following:

Mean estimate in case of normal distribution – the difference between parametric and nonparametric approach is 0,005 in both cases. It can be considered as insignificant.

Variance estimate in case of normal distribution – the difference between parametric and nonparametric approach is 0,2090 at $N(100, 10)$ distribution and 0,0063 at $N(0,1)$ distribution.

It is possible to state, that estimates of parameters obtained after 10 000 bootstrap replications were in all cases "closer" to values of original parameters than estimates obtained only from 15 values of original sample.

Number of bootstrap replications necessary for bias estimate:

It was found out in presented examples that it is necessary to make 400 parametric bootstrap replications and 600 nonparametric replications to obtain relevant results. The values of estimated bias didn't improve, when more replications were made.

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